On Shadows of Intersecting Families

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Abstract

The shadow minimization problem for t–intersecting systems of finite sets is considered. Let $\mathcal{A}$ be a family of k–subsets of $\mathbb{N}$. The ℓ–shadow of $\mathcal{A}$ is the set of all $(k-\ell)$–subsets $\partial_\ell \mathcal{A}$ contained in the members of $\mathcal{A}$. Let $\mathcal{A}$ be a t–intersecting family (any two members have at least t elements in common) with $|\mathcal{A}| = m$. Given k, t, m the problem is to minimize $|\partial_\ell \mathcal{A}|$ (over all choices of $\mathcal{A}$). In this note we solve this problem when m is big enough.

Keywords: combinatorial extremal theory, t–intersecting families, shadows, shifting.

1 Introduction and result

$\mathbb{N}$ denotes the set of positive integers and the set $\{1,\dots,n\}$ is abbreviated as $[n]$. Given $n,k\in\mathbb{N}$ and $X\subset\mathbb{N}$ denote

$$2^{[n]}=\big\{F:F\subset[n]\big\},\ \binom{X}{k}=\{F\subset X:|F|=k\}.$$

A finite family $\mathcal{A}\subset\binom{X}{k}$ is called t–intersecting if $|A\cap B|\geq t$ for all $A,B\in\mathcal{A}$. In the sequel this definition is used for $X=[n]$ resp. $X=\mathbb{N}$. $I(n,k,t)$ resp. $I(\infty,k,t)$ denote the sets of all such families.

The ℓ–shadow of $\mathcal{A}\subset\binom{X}{k}$ is defined by $\partial_\ell\mathcal{A}=\big\{F\in\binom{X}{k-\ell}:\exists A\in\mathcal{A}:F\subset A\big\}$. When $\ell=1$ we write $\partial\mathcal{A}$ and for $|\partial_{k-1}\mathcal{A}|$ we use the notation $\|\mathcal{A}\|$. Define the colex order for the elements $A,B\in\binom{\mathbb{N}}{k}$ as follows:

$$A<B\Leftrightarrow\max\big((A\smallsetminus B)\cup(B\smallsetminus A)\big)\in B.$$

For a family $\mathcal{F}\subset\binom{\mathbb{N}}{k}$ we denote by $L_m\mathcal{F}$ the set of the first m elements of $\mathcal{F}$ ($m\leq|\mathcal{F}|$) in colex order.

Let $\mathcal{A}\subset\binom{[n]}{k}$ (or $\mathcal{A}\subset\binom{\mathbb{N}}{k}$) with $|\mathcal{A}|=m$. How small can $|\partial_\ell\mathcal{A}|$ be?

The well known Kruskal–Katona Theorem (proved by Kruskal [8], by Katona [6] and by Lindström and Zetterström [9]) completely solves the (shadows minimization) problem for any given parameters n,k,m.

Theorem KK.

$$|\partial_\ell\mathcal{A}|\geq\left|\partial_\ell\left(L_m\binom{[n]}{k}\right)\right|. \tag{1.1}$$

Let now $\mathcal{A}\in I(n,k,t)$. What can we say about $|\partial_\ell\mathcal{A}|$? (Can we have a result like Theorem KK?) Note that the case $m\leq\binom{2k-t}{k}$ is out of consideration, because any $\mathcal{A}\subset\binom{[2k-t]}{k}$ is t–intersecting and Theorem KK gives the answer. An important result of Katona [7] is the following

Theorem Ka. *For integers $1\leq\ell\leq t$, $\mathcal{A}\in I(n,k,t)$*

$$|\partial_\ell\mathcal{A}|\geq|\mathcal{A}|\frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}}. \tag{1.2}$$

For extensions and analogues of this inequality see Frankl [4].

However this bound is not tight (see the Lemma below), and one can also easily observe that the statement is not valid for $\ell > t$. Thus in general finding the exact lower bound for $|\partial_\ell \mathcal{A}|$ for any given parameters n, k, t, ℓ, m is an open (and seemingly difficult) problem.

We state now our result which solves the problem when the ground set is $\mathbb{N}$ and m is big enough. For integers $1 \le r \le s$, $k \ge r$ define

$$\mathcal{B}(k, s, r) = \left\{ B \in \binom{\mathbb{N}}{k} : |B \cap [s]| \ge r \right\}.$$

Theorem. *Let $\mathcal{A} \in I(\infty, k, t)$.*

(a) For $1 \le \ell \le t < k$ and $|\mathcal{A}| = m > m_1(k, t, \ell)$ (suitable) we have

$$|\partial_\ell \mathcal{A}| \ge |\partial_\ell L_m \mathcal{B}(k, 2k - t - 2, k - 1)|.$$

(b) For $1 \le t < \ell < k$ and $|\mathcal{A}| = m > m_2(k, t, \ell)$ (suitable) we have

$$|\partial_\ell \mathcal{A}| \ge |\partial_\ell L_m \mathcal{B}(k, t, t)|.$$

2 An auxiliary result

Lemma. *With the conditions of Theorem Ka the equality in (1.2) holds iff $\mathcal{A} = \binom{[2k-t]}{k}$.*

Proof: Let $\|\mathcal{A}\| = s$.

Case 1: $s \le 2k - t$.

We have (by counting into two ways)

$$|\mathcal{A}| \binom{k}{\ell} \le |\partial_\ell \mathcal{A}| \binom{s - k + \ell}{\ell} \tag{2.1}$$

with the equality iff $\mathcal{A} = \binom{[s-k+\ell]}{k}$.

Hence

$$|\partial_\ell \mathcal{A}| \ge |\mathcal{A}| \frac{\binom{k}{\ell}}{\binom{s-k+\ell}{\ell}} \ge |\mathcal{A}| \frac{\binom{k}{\ell}}{\binom{k-t+\ell}{\ell}} = |\mathcal{A}| \frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}},$$

with the equality in the second relation iff $s = 2k - t$.

Case 2: $s > 2k - t$.

Define

$$\mathcal{A}_1 = \{A \in \mathcal{A} : s \in A\}, \mathcal{A}_0 = \mathcal{A} \setminus \mathcal{A}_1, \quad \mathcal{A}_1' = \{A \setminus \{s\} : A \in \mathcal{A}_1\}.$$

Clearly $|\mathcal{A}'_1| = |\mathcal{A}_1|$ and $|\mathcal{A}| = |\mathcal{A}'_1| + |\mathcal{A}_0|$.

Recall the well known shifting operation S_{ij} defined for any $F \in 2^{[n]}$ and for any family $\mathcal{F} \subset 2^{[n]}$. For integers $1 \le i < j \le n$

$$S_{ij}(F) = \begin{cases} \big((F \smallsetminus \{j\}) \cup \{i\}\big) \text{ if } i \notin F, j \in F, \big((F \smallsetminus \{j\}) \cup \{i\}\big) \notin F \\ F, \text{otherwise.} \end{cases}$$

$$S_{ij}(\mathcal{F}) = \{S_{ij}(F) : F \in \mathcal{F}\}.$$

We say that $\mathcal{F}$ is left–shifted if $S_{ij}(\mathcal{F}) = \mathcal{F}$ for all $1 \le i < j \le n$.

The following properties of $S_{ij}(\mathcal{F})$ are well known (see e.g. [3] or [2])

P_1. $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$

P_2. $\partial_\ell S_{ij}(\mathcal{F}) \subset S_{ij}(\partial_\ell \mathcal{F})$ (hence $|\partial_\ell S_{ij}(\mathcal{F})| \le |\partial_\ell \mathcal{F}|$

P_3. $\mathcal{F} \in I(n,k,t) \Rightarrow S_{ij}(\mathcal{F}) \in I(n,k,t)$

We also need the following important result due to Füredi and Griggs [5], and Mörs [10].

Theorem FGM. *Let $\mathcal{F} \subset \binom{[n]}{k}$ have minimal ℓ–shadow, then its $(\ell+1)$–shadow is minimal as well.* □

In particular the theorem implies that for an optimal family $\mathcal{F}$ the shifting operation does not decrease $\|\mathcal{F}\|$. Combining all these facts we conclude that w.l.o.g. we can assume that $\mathcal{A}$ is left shifted. Then one can easily observe the following

Claim 1. $|\partial_\ell \mathcal{A}| = |\partial_\ell \mathcal{A}_0| + |\partial_\ell \mathcal{A}'_1|$. □

Claim 2. $\mathcal{A}'_1 \subset \binom{[s-1]}{k-1}$ is t–intersecting family. □

We proceed induction on $s \ge 2k - t + 1$.

$s \to s+1$:

If $s = 2k - t + 1$ then $|\mathcal{A}_0| < \binom{2k-t}{k}$, otherwise $\mathcal{A}_1 = \varnothing$ (since $\mathcal{A}$ is t–intersecting), a contradiction. Therefore (by Case 1) we have

$$|\partial_\ell \mathcal{A}_0| > |\mathcal{A}| \frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}}. \tag{2.2}$$

If $s > 2k - t + 1$ then we have (2.2) by induction hypothesis.

Further in view of Claim 2 and (1.2) we have

$$|\partial_\ell \mathcal{A}'_1| \ge |\mathcal{A}'_1| \frac{\binom{2(k-1)-t}{k-1-\ell}}{\binom{2(k-1)-t}{k-1}}. \tag{2.3}$$

Finally by Claim 1, (2.2), (2.3) and the inequality (for $t \geq \ell$)

$$\frac{\binom{2k-t-2}{k-1-\ell}}{\binom{2k-t-2}{k-1}} \geq \frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}} \tag{2.4}$$

we get

$$|\partial_\ell \mathcal{A}| > (|\mathcal{A}_1| + |\mathcal{A}_0|)\frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}} = |\mathcal{A}|\frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}},$$

concluding the result. $\square$

3 Proof of the Theorem

(a) $\boldsymbol{1 \leq \ell \leq t < k}$.

Given $m > \binom{2k-t}{k}$ consider the family

$$\mathcal{A}^* = L_m \mathcal{B}(k, 2k-t-2, k-1),$$

and write $m = |\mathcal{A}^*|$ in the form

$$m = \binom{2(k-1)-t}{k} + n\binom{2(k-1)-t}{k-1} + r, \tag{3.1}$$

where $n \in \mathbb{N}$ and $0 \leq r < \binom{2(k-1)-t}{k-1}$.

Then clearly

$$|\partial_\ell \mathcal{A}^*| = \binom{2(k-1)-t}{k-\ell} + n\binom{2(k-1)-t}{k-1-\ell} + \partial_\ell(r), \tag{3.2}$$

where $\partial_\ell(r) \triangleq \left|\partial_\ell\left(L_r\binom{[2(k-1)-t]}{k-1}\right)\right|$.

From (3.1), (3.2) and the inequality

$$\frac{\binom{2(k-1)-t}{k-1-\ell}}{\binom{2(k-1)-t}{k-1}} > \frac{\binom{2(k-1)-t}{k-\ell}}{\binom{2(k-1)-t}{k}}, \text{ (for } 1 \leq \ell \leq t)$$

we obtain the estimation

$$|\partial_\ell \mathcal{A}^*| \leq m\frac{\binom{2(k-1)-t}{k-1-\ell}}{\binom{2(k-1)-t}{k-1}} + \alpha, \tag{3.3}$$

for some $0 \leq \alpha < \binom{2(k-1)-t}{k-1-\ell}$.

Suppose now $\mathcal{A} \in I(\infty, k, t)$ is an optimal family (has minimal ℓ–shadow) with $|\mathcal{A}| = m > \binom{2k-1}{k}$ and $\|\mathcal{A}\| = u$. Suppose also that $\mathcal{A}$ is left shifted. Partition $\mathcal{A}$ into $s+1$ disjoint classes $\mathcal{A} = \bigcup_{i=0}^{s} \mathcal{A}_i$, where $u - s > 2(k-1) - t$,

$$\begin{aligned}
\mathcal{A}_0 &= \{A \in \mathcal{A} : u \in A\} \\
\mathcal{A}_1 &= \{A \in \mathcal{A} \setminus \mathcal{A}_0 : u - 1 \in A\} \\
&\;\;\cdots\cdots \\
\mathcal{A}_s &= \{A \in \mathcal{A} \setminus (\mathcal{A}_0 \cup \cdots \cup \mathcal{A}_{s-1}) : u - s \in A\}.
\end{aligned}$$

Define also

$$\mathcal{A}'_i = \{A \setminus \{u - i\} : A \in \mathcal{A}_i\},\ i = 0, \ldots, s.$$

It is not hard to see that

$$|\partial_\ell \mathcal{A}| = |\partial_\ell \mathcal{A}_s| + |\partial_\ell \mathcal{A}'_{s-1}| + \cdots + |\partial_\ell \mathcal{A}'_0|.$$

We distinguish now two cases.

Case 1: $\mathcal{A}'_i \neq \binom{[2(k-1)-t]}{k-1}$, $i = 0, 1, \ldots, s$.

Then by the Lemma we have

$$|\partial_\ell \mathcal{A}_i| \geq |\mathcal{A}_i| \frac{\binom{2(k-1)-t}{k-1-\ell}}{\binom{2(k-1)-t}{k-1}} + 1,\ i = 0, \ldots, s.$$

Hence

$$|\partial_\ell \mathcal{A}| \geq |\mathcal{A}| \frac{\binom{2(k-1)-t}{k-1-\ell}}{\binom{2(k-1)-t}{k-1}} + s = m \frac{\binom{2(k-1)-t}{k-1-\ell}}{\binom{2(k-1)-t}{k-1}} + s.$$

If now m is big enough, and consequently s is big (say $s \geq \binom{2(k-1)-t}{k-1-\ell}$), then in view of (3.3) we get $|\partial_\ell \mathcal{A}| > |\partial_\ell \mathcal{A}^*|$, a contradiction with the optimality of $\mathcal{A}$.

Case 2: There exists a family $\mathcal{A}_i$, $i \in \{0, \ldots, s\}$ such that

$$\mathcal{A}'_i = \binom{[2(k-1)-t]}{k-1}.$$

Then observe that for $j = 0, \ldots, s$

$$\mathcal{A}'_j \subset \binom{[2(k-1)-t]}{k-1}.$$

This is clear because the opposite assumption (remember that $\mathcal{A}$ is left shifted) would violate the t–intersecting condition.

Thus we have shown that for $m > m_1(k,t)$

$$\mathcal{A} \subset \mathcal{B}(k, 2k-t-2, k-1).$$

To complete the proof of part (a) we use a special case of the result in [1].

Theorem AAK. *For $s, r, k \in \mathbb{N}$, $1 \le r \le s, \ell \le k$ let $\mathcal{F} \subset \mathcal{B}(k,s,r)$ with $|\mathcal{F}| = m$. Then*

$$|\partial_\ell \mathcal{F}| \ge |\partial_\ell \big(L_m \mathcal{B}(k,s,r)\big)|.$$

$\square$

(b) $\boldsymbol{1 \le t < \ell < k}$.

Claim. It is sufficient to consider the case $\ell = t+1$.

Suppose we have proved the statement for some $\ell \ge t+1$. Then for a family $\mathcal{A} \in I(\infty, k, t)$ with $|\mathcal{A}| = m$ and $|\partial_\ell \mathcal{A}| = M$ we have

$$|\partial_{\ell+1}\mathcal{A}| = |\partial(\partial_\ell \mathcal{A})| \ge \left| \partial L_M \binom{\mathbb{N}}{k-\ell} \right|.$$

But

$$\partial_\ell \big(L_m \mathcal{B}(k,t,t)\big) = L_{M^*} \binom{\mathbb{N}}{k-\ell},$$

where $M^* = |\partial_\ell \big(L_m \mathcal{B}(k,t,t)\big)| \le M$, concluding the claim. $\square$

Thus let $\ell = t+1$ and let $\mathcal{A} \in I(\infty, k, t)$ be an optimal, left shifted family with $|\mathcal{A}| = m$.

Define

$$\mathcal{A}_1 = \{A \in \mathcal{A} : 1 \in A\}, \mathcal{A}_1' = \{A \smallsetminus \{1\} : A \in \mathcal{A}_1\}, \mathcal{A}_0 = \mathcal{A} \smallsetminus \mathcal{A}_1,$$

and let $|\mathcal{A}_0| = m_0$, $|\mathcal{A}_1| = m_1 = m - m_0$.

We consider two cases.

Case 1: $\mathcal{A}_0 = \varnothing$

Then clearly

$$|\partial_{t+1}\mathcal{A}| = |\partial_t \mathcal{A}_1'| + |\partial_{t+1}\mathcal{A}_1'|.$$

Since $\mathcal{A}_1' \in I(\infty, k-1, t-1)$ we can proceed induction on $t \ge 2$ (the case $t = 1$ can be easily derived). That is we have

$$|\partial_{t+1}\mathcal{A}| \ge |\partial_t L_m \mathcal{B}(k-1,t-1,t-1)| + |\partial_{t+1} L_m \mathcal{B}(k-1,t-1,t-1)| = |\partial_{t+1} L_m \mathcal{B}(k,t,t)|.$$

Case 2: $\mathcal{A}_0 \neq \varnothing$.

Note first that $\mathcal{A}_0 \in I(\infty, k, t+1)$. This easily follows from the left–shiftedness of $\mathcal{A}$.

Hence by (1.2) we have

$$|\partial_{t+1}\mathcal{A}_0| \geq |\mathcal{A}_0|. \tag{3.4}$$

Since $\mathcal{A}$ is left shifted and $\mathcal{A}_0 \neq \varnothing$ then clearly $\{2, \ldots, k+1\} \in \mathcal{A}$. This implies that $\mathcal{A}_1' \subset \mathcal{F} =: \left\{F \subset \binom{\mathbb{N} \setminus \{1\}}{k-1} : |F \cap \{2, \ldots, k+1\}| \geq t\right\}$.

We apply now Theorem AAK to $\mathcal{A}_1'$ identifying $\mathcal{F}$ with $\mathcal{B}(k-1, k, t)$. We get

$$|\partial_{t+1}\mathcal{A}_1'| \geq |\partial_{t+1}L_{m_1}\mathcal{B}(k-1,k,t)|, \quad |\partial_t\mathcal{A}_1'| \geq |\partial_t L_{m_1}\mathcal{B}(k-1,k,t)|. \tag{3.5}$$

Obviously we also have

$$\sum_{i=0}^{k-t-1} \binom{k}{t+i}\binom{x-k}{k-t-1-i} \leq |\mathcal{A}_1'| = m_1 < \sum_{i=0}^{k-t+1} \binom{k}{t+i}\binom{x-k+1}{k-t-1-i}, \tag{3.6}$$

where $x = \|L_{m_1}\mathcal{B}(k-1,k,t)\| - 1$.

On the other hand

$$|\partial_{t+1}\mathcal{A}_1| = |\partial_{t+1}\mathcal{A}_1'| + |\partial_t\mathcal{A}_1'|.$$

This with (3.5) and (3.6) gives the estimation

$$|\partial_{t+1}\mathcal{A}_1| \geq |\partial_t\mathcal{A}_1'| \geq \binom{x}{k-t-1} \geq c_1 m_1 \tag{3.7}$$

for some constant $c_1 > 0$.

Let now $\mathcal{A}^* = L_m\mathcal{B}(k,t,t)$ and observe that for any $\varepsilon > 0$ $|\partial_{t+1}\mathcal{A}^*| \setminus |\mathcal{A}| < \varepsilon$, if m is big enough.

This fact with (3.4) and optimality of $\mathcal{A}$ enforces that for any $\varepsilon > 0$ and $m > m_2(k,t)$ (suitable) one has

$$\varepsilon > \frac{|\partial_{t+1}\mathcal{A}|}{|\mathcal{A}|} \geq \frac{|\partial_{t+1}\mathcal{A}_0|}{|\mathcal{A}|} \geq \frac{|\mathcal{A}_0|}{|\mathcal{A}|} = \frac{m_0}{m} = \frac{m_0}{m_0+m_1}. \tag{3.8}$$

We also have (by (3.7))

$$\frac{|\partial_{t+1}\mathcal{A}|}{|\mathcal{A}|} \geq \frac{|\partial_{t+1}\mathcal{A}_1|}{|\mathcal{A}|} \geq \frac{c_1 m_1}{m}.$$

On the other hand (3.8) implies that $\frac{m_1}{m} > c$, for some constant c.

This contradiction completes the proof of the Theorem. $\square$

4 Remarks

1. One can easily obtain the numerical version of our Theorem. Let $m > \binom{2k-t}{k}$ be an integer. Clearly m can be uniquely represented in the form (3.1). We can also uniquely represent r in the $(k-1)$–cascade form (see [2] or [3])

$$r = \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s},$$

where $a_{k-1} > a_{k-2} > \cdots > a_s \geq s \geq 1$.

Then in view of our Theorem and the Kruskal–Katona Theorem we have the following: For $1 \leq \ell \leq t < k$ and $m > m_1(k, t, \ell)$ holds

$$|\partial_\ell \mathcal{A}| \geq \binom{2(k-1)-t}{k-\ell} + s\binom{2(k-1)-t}{k-\ell-1} + \binom{a_{k-1}}{k-1-\ell} + \cdots + \binom{a_s}{s-\ell}. \tag{4.1}$$

From (3.1) and (4.1) we obtain an asymptotic version of our Theorem given in the form of Theorem Ka.

Namely, for $\mathcal{A} \in I(\infty, k, t)$, $1 \leq \ell < t < k$ and $|\mathcal{A}| = m \to \infty$ one has

$$|\partial_\ell \mathcal{A}| \geq |\mathcal{A}| \frac{\binom{2(k-1)-t}{k-\ell-1}}{\binom{2(k-1)-t}{k-1}} + o(1).$$

It can be seen from our construction that this bound is best possible.

2. As an improvement of the Theorem it would be interesting to find minimal values of $m_1(k, \ell, t)$ and $m_2(k, \ell, t)$ for which the result holds.

In fact for the case $1 \leq \ell \leq t$ the proof gives also an upper bound for $m_1(k, \ell, t)$. However this estimation seems to be rough.

On the other hand our Theorem is not valid for all $m > \binom{2k-t}{k}$. Here are examples for cases (a) and (b).

(a) $1 \leq \ell < t$.

Let $k > 3$, $t = \ell = 1$, $m = \binom{2k-1}{k} + 1$. Then we can write

$$m = \binom{2k-3}{k} + 3\binom{2k-3}{k-1} + 1.$$

Hence

$$\Delta_1 \triangleq |\partial(L_m \mathcal{B}(k, 2k-3, k-1))| = \binom{2k-3}{k-1} + 3\binom{2k-3}{k-2} + k - 1 = 4\binom{2k-3}{k-1} + k - 1.$$

Define now the following intersecting family

$$\mathcal{F} = \left(\binom{[2k-1]}{k} \setminus \{k, \ldots, 2k-1\}\right) \cup \{1, \ldots, k-1, 2k\} \cup \{1, \ldots, k-1, 2k+1\}.$$

Clearly $|\mathcal{F}| = \binom{2k-1}{k} + 1$ and $|\partial\mathcal{F}| = \binom{2k-1}{k-1} + 2(k-2) = 3\binom{2k-3}{k-1} + \binom{2k-3}{k-3} + 2(k-2)$. Thus $|\partial\mathcal{F}| < \Delta_1$.

(b) $\boldsymbol{1 \leq t < \ell}$.

For the same m, k and $t = 1$, let now $\ell = 2$. Then

$$\Delta_2 \triangleq |\partial_2\big(L_m\mathcal{B}(k,1,1)\big)| = \binom{2k}{k-2} + \binom{k-1}{2},$$

while

$$|\partial_2\mathcal{F}| = \binom{2k-1}{k-2} + 2\binom{k-1}{2} < \Delta_2.$$

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