arXiv:math.CO/0301021 v1 3 Jan 2003

Bravais colourings of planar modules with N-fold symmetry

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Abstract

The first step of the investigation of colour symmetries for periodic and aperiodic systems is the determination of all colouring schemes that are compatible with the symmetry group of the underlying structure, or with a subgroup of it. For an important class of colourings of planar structures, this mainly combinatorial question can be addressed with methods of algebraic number theory. We present the corresponding results for all planar modules with N-fold symmetry that emerge as the rings of integers in cyclotomic fields with class number one. The explicit counting functions are multiplicative and can be encapsulated in Dirichlet series generating functions, which turn out to be the Dedekind zeta functions of the corresponding cyclotomic fields.

Key words: colour symmetry, cyclotomic fields, Dirichlet series, zeta functions
2000 MSC: 05A15, 52C23, 11R99

1 Introduction

Colour symmetries of crystals [15,16] and, more recently, of quasicrystals [12,9] continue to attract a lot of attention, simply because so little is known about their classification, see [9] for a recent review. A first step of this analysis consists in answering the question how many different colourings of an infinite

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point set exist which are compatible with its underlying symmetry. More precisely, one has to determine the possible numbers of colours, and to count the corresponding possibilities to distribute the colours over the point set (up to permutations), in line with all compatible symmetry constraints.

In this generality, the problem has not even been solved for simple lattices. One common restriction is to demand that one colour occupies a subset which is of the same Bravais type as the original set, while the other colours encode the cosets. In this situation, to which we will also restrict ourselves, several results are known and can be given in closed form [1,5,6,9,10,14]. Of particular interest are planar cases because, on the one hand, they show up in quasicrystalline T-phases, and, on the other hand, they are linked to the rather interesting classification of planar Bravais classes with n-fold symmetry [11]. The latter are unique for the following 29 choices of n,

$$\begin{aligned} n \in \{&3,4,5,7,8,9,11,12,13,15,16,17,19,20,21,24,\\ &25,27,28,32,33,35,36,40,44,45,48,60,84\}\,. \end{aligned} \tag{1}$$

The canonical representatives are the sets of cyclotomic integers $\mathcal{M}_n = \mathbb{Z}[\xi_n]$, the ring of polynomials in ξ_n, a primitive nth root of unity. To be explicit (which is not necessary), we choose $\xi_n = \exp(2\pi \mathrm{i}/n)$. Apart from $n = 1$ (where $\mathcal{M}_1 = \mathbb{Z}$ is one-dimensional), the values of n in Eq. (1) correspond to all cases where $\mathbb{Z}[\xi_n]$ is a principal ideal domain and thus has class number one, see [18,6] for details. If n is odd, we have $\mathcal{M}_n = \mathcal{M}_{2n}$. Consequently, $\mathcal{M}_n$ has N-fold symmetry where

$$N \;=\; N(n) \;=\; \begin{cases} n & \text{if } n \text{ is even,}\\ 2n & \text{if } n \text{ is odd.}\end{cases} \tag{2}$$

To avoid duplication of results, values $n \equiv 2 \pmod 4$ do not appear in the above list. There are systematic mathematical reasons to prefer this convention to the notation of [9,11], some of which will appear later on.

It is this very connection to algebraic number theory which results in the Bravais classification [11], and also allows for a solution of the combinatorial part of the colouring problem by means of Dirichlet series generating functions, compare [1,6]. The values of n from our list (1) are naturally grouped according to $\phi(n)$, which is Euler's totient function

$$\phi(n) \;=\; \{1 \le k \le n \mid \gcd(k,n) = 1\}\,. \tag{3}$$

Note that $\phi(n) = 2$ are the crystallographic cases $n = 3$ (triangular lattice) and $n = 4$ (square lattice), while $\phi(n) = 4$ means $n \in \{5, 8, 12\}$ which are the standard symmetries of planar quasicrystals. Again, $n = 10$ is covered implicitly, as explained above.

The methods emerging from this connection allow the full treatment of all 29 cases listed above, which was originally observed in [11,1,6,13]. Here, we will explain the use of algebraic number theory more explicitly, and present detailed results for all 29 cases. Previously, the results for the cases with $\phi(n) \leq 10$ were given in [1,9,3,4], mainly without proofs. Other combinatorial problems of (quasi)crystallography have been addressed by similar methods, see e.g. [2].

2 Number theoretic formulation

Let us consider $\mathcal{M}_n$ with a fixed n from our list (1). Then, $\mathcal{M}_n$ is an Abelian group, and also a $\mathbb{Z}$-module of rank $\phi(n)$. We view it as a subset of the Euclidean plane, identified with $\mathbb{C}$, and hence as a geometric object. In this setting, important subgroups of $\mathcal{M}_n$ are those that are the image of $\mathcal{M}_n$ under a similarity transformation, i.e., a mapping $z \mapsto az$ or $z \mapsto a\bar{z}$ with $a \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Such subgroups are called *similarity submodules*, see [6] for details. Furthermore, we will also present some asymptotic results that can be extracted from the generating functions.

Definition 1 *A* Bravais colouring *of the module $\mathcal{M}_n$ with k colours is a mapping $c\colon \mathcal{M}_n \to \{1, 2, \ldots, k\}$ such that one of the sets $c^{-1}(\ell)$, for $1 \leq \ell \leq k$, is a similarity submodule of $\mathcal{M}_n$ of index k, and the others, one by one, are the corresponding cosets in $\mathcal{M}_n$. The number of Bravais colourings of $\mathcal{M}_n$ with k colours, up to permutations of the colours, is denoted by $a_n(k)$.*

In other words, a Bravais colouring c of $\mathcal{M}_n$ is a partition

$$\mathcal{M}_n \; = \; \dot{\bigcup_{1 \leq \ell \leq k}} c^{-1}(\ell) \tag{4}$$

into k disjoint sets that are translates of each other and carry pairwise different colours. In addition, one of the sets is a similarity submodule of $\mathcal{M}_n$. The interest in this kind of partition originates from its rather important group-theoretical structure, compare [15,16,9] and references therein. Here, we concentrate on the combinatorial aspect, i.e., on the determination of the values of $a_n(k)$, which define an integer-valued arithmetic function.

Lemma 2 *Let $\mathcal{M}_n = \mathbb{Z}[\xi_n]$ with $n \geq 3$. A similarity submodule of $\mathcal{M}_n$ is a principal ideal of $\mathbb{Z}[\xi_n]$. Consequently, $a_n(k)$ is the number of principal ideals of $\mathbb{Z}[\xi_n]$ of norm k.*

PROOF. Observe that $\mathbb{Z}[\xi_n]$ is invariant under complex conjugation. Consequently, all similarity submodules of $\mathbb{Z}[\xi_n]$ are of the form $a\mathbb{Z}[\xi_n]$ with $a \in \mathbb{C}^*$.

Table 1
Basic indices for the unramified primes of $\mathbb{Z}[\xi_n]$ with n of Eq. (1). The symbol $\frac{\ell}{k}$ means that primes $p \equiv k \pmod{n}$ contribute via p^ℓ as basic index, with $m = \phi(n)/\ell$.

$\phi(n)$	n	general primes p
2	3	$\frac{1}{1}\ \frac{2}{2}$
	4	$\frac{1}{1}\ \frac{2}{3}$
4	5	$\frac{1}{1}\ \frac{4}{2}\ \frac{4}{3}\ \frac{2}{4}$
	8	$\frac{1}{1}\ \frac{2}{3}\ \frac{2}{5}\ \frac{2}{7}$
	12	$\frac{1}{1}\ \frac{2}{5}\ \frac{2}{7}\ \frac{2}{11}$
6	7	$\frac{1}{1}\ \frac{3}{2}\ \frac{6}{3}\ \frac{3}{4}\ \frac{6}{5}\ \frac{2}{6}$
	9	$\frac{1}{1}\ \frac{6}{2}\ \frac{3}{4}\ \frac{6}{5}\ \frac{3}{7}\ \frac{2}{8}$
8	15	$\frac{1}{1}\ \frac{4}{2}\ \frac{2}{4}\ \frac{4}{7}\ \frac{4}{8}\ \frac{2}{11}\ \frac{4}{13}\ \frac{2}{14}$
	16	$\frac{1}{1}\ \frac{4}{3}\ \frac{4}{5}\ \frac{2}{7}\ \frac{2}{9}\ \frac{4}{11}\ \frac{4}{13}\ \frac{2}{15}$
	20	$\frac{1}{1}\ \frac{4}{3}\ \frac{4}{7}\ \frac{2}{9}\ \frac{2}{11}\ \frac{4}{13}\ \frac{4}{17}\ \frac{2}{19}$
	24	$\frac{1}{1}\ \frac{2}{5}\ \frac{2}{7}\ \frac{2}{11}\ \frac{2}{13}\ \frac{2}{17}\ \frac{2}{19}\ \frac{2}{23}$
10	11	$\frac{1}{1}\ \frac{10}{2}\ \frac{5}{3}\ \frac{5}{4}\ \frac{5}{5}\ \frac{10}{6}\ \frac{10}{7}\ \frac{10}{8}\ \frac{5}{9}\ \frac{2}{10}$
12	13	$\frac{1}{1}\ \frac{12}{2}\ \frac{3}{3}\ \frac{6}{4}\ \frac{4}{5}\ \frac{12}{6}\ \frac{12}{7}\ \frac{4}{8}\ \frac{3}{9}\ \frac{6}{10}\ \frac{12}{11}\ \frac{2}{12}$
	21	$\frac{1}{1}\ \frac{6}{2}\ \frac{3}{4}\ \frac{6}{5}\ \frac{2}{8}\ \frac{6}{10}\ \frac{6}{11}\ \frac{2}{13}\ \frac{3}{16}\ \frac{6}{17}\ \frac{6}{19}\ \frac{2}{20}$
	28	$\frac{1}{1}\ \frac{6}{3}\ \frac{6}{5}\ \frac{3}{9}\ \frac{6}{11}\ \frac{2}{13}\ \frac{2}{15}\ \frac{6}{17}\ \frac{6}{19}\ \frac{6}{23}\ \frac{3}{25}\ \frac{2}{27}$
	36	$\frac{1}{1}\ \frac{6}{5}\ \frac{6}{7}\ \frac{6}{11}\ \frac{3}{13}\ \frac{2}{17}\ \frac{2}{19}\ \frac{6}{23}\ \frac{3}{25}\ \frac{6}{29}\ \frac{6}{31}\ \frac{2}{35}$
16	17	$\frac{1}{1}\ \frac{8}{2}\ \frac{16}{3}\ \frac{4}{4}\ \frac{16}{5}\ \frac{16}{6}\ \frac{16}{7}\ \frac{8}{8}\ \frac{8}{9}\ \frac{16}{10}\ \frac{16}{11}\ \frac{16}{12}\ \frac{4}{13}\ \frac{16}{14}\ \frac{8}{15}\ \frac{2}{16}$
	32	$\frac{1}{1}\ \frac{8}{3}\ \frac{8}{5}\ \frac{4}{7}\ \frac{4}{9}\ \frac{8}{11}\ \frac{8}{13}\ \frac{2}{15}\ \frac{2}{17}\ \frac{8}{19}\ \frac{8}{21}\ \frac{4}{23}\ \frac{4}{25}\ \frac{8}{27}\ \frac{8}{29}\ \frac{2}{31}$
	40	$\frac{1}{1}\ \frac{4}{3}\ \frac{4}{7}\ \frac{2}{9}\ \frac{2}{11}\ \frac{4}{13}\ \frac{4}{17}\ \frac{2}{19}\ \frac{2}{21}\ \frac{4}{23}\ \frac{4}{27}\ \frac{2}{29}\ \frac{2}{31}\ \frac{4}{33}\ \frac{4}{37}\ \frac{2}{39}$
	48	$\frac{1}{1}\ \frac{4}{5}\ \frac{2}{7}\ \frac{4}{11}\ \frac{4}{13}\ \frac{2}{17}\ \frac{4}{19}\ \frac{2}{23}\ \frac{2}{25}\ \frac{4}{29}\ \frac{2}{31}\ \frac{4}{35}\ \frac{4}{37}\ \frac{2}{41}\ \frac{4}{43}\ \frac{2}{47}$
	60	$\frac{1}{1}\ \frac{4}{7}\ \frac{2}{11}\ \frac{4}{13}\ \frac{4}{17}\ \frac{2}{19}\ \frac{4}{23}\ \frac{2}{29}\ \frac{2}{31}\ \frac{4}{37}\ \frac{2}{41}\ \frac{4}{43}\ \frac{4}{47}\ \frac{2}{49}\ \frac{4}{53}\ \frac{2}{59}$
18	19	$\frac{1}{1}\ \frac{18}{2}\ \frac{18}{3}\ \frac{9}{4}\ \frac{9}{5}\ \frac{9}{6}\ \frac{3}{7}\ \frac{6}{8}\ \frac{9}{9}\ \frac{18}{10}\ \frac{3}{11}\ \frac{6}{12}\ \frac{18}{13}\ \frac{18}{14}\ \frac{18}{15}\ \frac{9}{16}\ \frac{9}{17}\ \frac{2}{18}$
	27	$\frac{1}{1}\ \frac{18}{2}\ \frac{9}{4}\ \frac{18}{5}\ \frac{9}{7}\ \frac{6}{8}\ \frac{3}{10}\ \frac{18}{11}\ \frac{9}{13}\ \frac{18}{14}\ \frac{9}{16}\ \frac{6}{17}\ \frac{3}{19}\ \frac{18}{20}\ \frac{9}{22}\ \frac{18}{23}\ \frac{9}{25}\ \frac{2}{26}$
20	25	$\frac{1}{1}\ \frac{20}{2}\ \frac{20}{3}\ \frac{10}{4}\ \frac{5}{6}\ \frac{4}{7}\ \frac{20}{8}\ \frac{10}{9}\ \frac{5}{11}\ \frac{20}{12}\ \frac{20}{13}\ \frac{10}{14}\ \frac{5}{16}\ \frac{20}{17}\ \frac{4}{18}\ \frac{10}{19}\ \frac{5}{21}\ \frac{20}{22}\ \frac{20}{23}\ \frac{2}{24}$
	33	$\frac{1}{1}\ \frac{10}{2}\ \frac{5}{4}\ \frac{10}{5}\ \frac{10}{7}\ \frac{10}{8}\ \frac{2}{10}\ \frac{10}{13}\ \frac{10}{14}\ \frac{5}{16}\ \frac{10}{17}\ \frac{10}{19}\ \frac{10}{20}\ \frac{2}{23}\ \frac{5}{25}\ \frac{10}{26}\ \frac{10}{28}\ \frac{10}{29}\ \frac{5}{31}\ \frac{2}{32}$
	44	$\frac{1}{1}\ \frac{10}{3}\ \frac{5}{5}\ \frac{10}{7}\ \frac{5}{9}\ \frac{10}{13}\ \frac{10}{15}\ \frac{10}{17}\ \frac{10}{19}\ \frac{2}{21}\ \frac{2}{23}\ \frac{5}{25}\ \frac{10}{27}\ \frac{10}{29}\ \frac{10}{31}\ \frac{10}{35}\ \frac{5}{37}\ \frac{10}{39}\ \frac{10}{41}\ \frac{2}{43}$
24	35	$\frac{1}{1}\ \frac{12}{2}\ \frac{12}{3}\ \frac{6}{4}\ \frac{2}{6}\ \frac{4}{8}\ \frac{6}{9}\ \frac{3}{11}\ \frac{12}{12}\ \frac{4}{13}\ \frac{3}{16}\ \frac{12}{17}\ \frac{12}{18}\ \frac{6}{19}\ \frac{4}{22}\ \frac{12}{23}\ \frac{6}{24}\ \frac{6}{26}\ \frac{4}{27}\ \frac{2}{29}\ \frac{6}{31}\ \frac{12}{32}\ \frac{12}{33}\ \frac{2}{34}$
	45	$\frac{1}{1}\ \frac{12}{2}\ \frac{6}{4}\ \frac{12}{7}\ \frac{4}{8}\ \frac{6}{11}\ \frac{12}{13}\ \frac{6}{14}\ \frac{3}{16}\ \frac{4}{17}\ \frac{2}{19}\ \frac{12}{22}\ \frac{12}{23}\ \frac{2}{26}\ \frac{4}{28}\ \frac{6}{29}\ \frac{3}{31}\ \frac{12}{32}\ \frac{6}{34}\ \frac{4}{37}\ \frac{12}{38}\ \frac{6}{41}\ \frac{12}{43}\ \frac{2}{44}$
	84	$\frac{1}{1}\ \frac{6}{5}\ \frac{6}{11}\ \frac{2}{13}\ \frac{6}{17}\ \frac{6}{19}\ \frac{6}{23}\ \frac{3}{25}\ \frac{2}{29}\ \frac{6}{31}\ \frac{3}{37}\ \frac{2}{41}\ \frac{2}{43}\ \frac{6}{47}\ \frac{6}{53}\ \frac{2}{55}\ \frac{6}{59}\ \frac{6}{61}\ \frac{6}{65}\ \frac{6}{67}\ \frac{2}{71}\ \frac{6}{73}\ \frac{6}{79}\ \frac{2}{83}$

As $a\mathbb{Z}[\xi_n] \subset \mathbb{Z}[\xi_n]$, we must have $0 \neq a \in \mathbb{Z}[\xi_n]$. So, a similarity submodule is a principal ideal, and vice versa. Each principal ideal gives rise to precisely one Bravais colouring c of $\mathbb{Z}[\xi_n]$, up to permutations of the colours, where, without loss of generality, $c^{-1}(1)$ is the principal ideal. The norm of an ideal equals the subgroup index $k = \big[\mathbb{Z}[\xi_n] : a\mathbb{Z}[\xi_n]\big]$, hence counts the number of cosets, resp. colours. In view of Eq. (4), and the meaning of the sets $c^{-1}(\ell)$, our assertion follows. $\square$

On the other hand, for the n from the list (1), *all* ideals of $\mathbb{Z}[\xi_n]$ are principal, because the corresponding cyclotomic fields $\mathbb{Q}(\xi_n)$ have class number one, see [18, Thm. 11.1]. This list is exhaustive except for the case $n = 1$ which corresponds to the field $\mathbb{Q}$ itself. For these cases, our combinatorial problem thus amounts to counting all (non-zero) ideals of a given index. Consequently, the arithmetic function $a_n(k)$ is *multiplicative*, i.e., $a_n(k\ell) = a_n(k)a_n(\ell)$ for k, ℓ coprime. A succinct way to encapsulate the numbers $a_n(k)$ is the use of a Dirichlet series generating function [19],

$$F_n(s) \; := \; \sum_{k=1}^{\infty} \frac{a_n(k)}{k^s}\,. \tag{5}$$

As we will see, this function converges for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The connection to the cyclotomic fields admits the direct calculation of this Dirichlet series by means of algebraic number theory.

Proposition 3 *For all n in the list* (1)*, the Dirichlet series $F_n(s)$ of Eq.* (5) *equals the Dedekind zeta function of the cyclotomic field $\mathbb{Q}(\xi_n)$, i.e.,*

$$F_n(s) \; = \; \zeta_{\mathbb{Q}(\xi_n)}(s) \; := \; \sum_{\mathfrak{a}} \frac{1}{\mathrm{norm}(\mathfrak{a})^s}\,, \quad \mathrm{Re}(s) > 1\,,$$

where $\mathfrak{a}$ runs over all non-zero ideals of $\mathbb{Z}[\xi_n]$.

PROOF. The Dedekind zeta function of $\mathbb{Q}(\xi_n)$ is, by definition, the Dirichlet series generating function of the number of ideals of a given index in the maximal order $\mathbb{Z}[\xi_n]$. Since n is from our list (1), we are in the class number one case, hence all ideals are principal, and the claim follows from Lemma 2. The convergence result is standard [18]. $\square$

The multiplicativity of the arithmetic function $a_n(k)$ for n in the list (1) implies that it is sufficient to know the values $a_n(p^r)$ for all primes p and powers $r \geq 0$. On the level of the generating function, this corresponds to the Euler product

Table 2
List of all ramified primes for $\mathbb{Z}[\xi_n]$ with n of Eq. (1).

$\phi(n)$	n	p	ℓ	m	$\phi(n)$	n	p	ℓ	m
2	3	3	1	1	16	17	17	1	1
	4	2	1	1		32	2	1	1
4	5	5	1	1		40	2	4	1
	8	2	1	1			5	2	2
	12	2	2	1		48	2	2	1
		3	2	1			3	4	2
6	7	7	1	1		60	2	4	2
	9	3	1	1			3	4	2
8	15	3	4	1			5	2	2
		5	2	1	18	19	19	1	1
	16	2	1	1		27	3	1	1
	20	2	4	1	20	25	5	1	1
		5	1	2		33	3	5	2
	24	2	2	1			11	2	1
		3	2	2		44	2	10	1
10	11	11	1	1			11	2	1
12	13	13	1	1	24	35	5	6	1
	21	3	6	1			7	4	1
		7	1	2		45	3	4	1
	28	2	3	2			5	6	1
		7	2	1		84	2	6	2
	36	2	6	1			3	6	2
		3	2	1			7	2	2

expansion, again for $\{\mathrm{Re}(s) > 1\}$,

$$F_n(s) \;:=\; \sum_{k=1}^{\infty} \frac{a_n(k)}{k^s} \;=\; \prod_p E_n(p^{-s}) \tag{6}$$

where p runs over the primes of $\mathbb{Z}$.

As we will see in the following section, each Euler factor is of the form

$$E_n(p^{-s}) \;=\; \frac{1}{(1-p^{-\ell s})^m} \;=\; \sum_{i=0}^{\infty} \binom{i+m-1}{m-1} \frac{1}{(p^s)^{\ell i}} \tag{7}$$

from which one extracts the values of $a_n(p^r)$ for $r \geq 0$. The integers ℓ and m are characteristic quantities that depend on n and on the prime p.

If $p \equiv k \pmod{n}$ with k and n coprime, then $\ell \cdot m = \phi(n)$. Such primes are simply listed as $\llcorner\frac{\ell}{k}\urcorner$ in Table 1, because m is determined. In addition, for each n, there are finitely many primes p which divide n, the so-called ramified primes. They are listed in Table 2, together with the corresponding values of ℓ and m. With this information, one can easily calculate the possible numbers of colours and the generating functions by inserting (7) into (6) and expanding the Euler product.

3 Dirichlet characters and zeta functions

The purpose of this section is to summarise some basic properties of cyclotomic fields that are needed to calculate the Dedekind zeta functions explicitly. This is done by means of the theory of Dirichlet characters, which we will now explain.

The first object we need is the *Galois group* G_n of the (cyclotomic) field extension $\mathbb{Q}(\xi_n)/\mathbb{Q}$. Here, the Galois group simply consists of all automorphisms σ of $\mathbb{Q}(\xi_n)$ that fix all rational elements, i.e., $\sigma(x) = x$ for all $x \in \mathbb{Q}$. It is an Abelian group of order $\phi(n)$. The following result is standard [18, Thm. 2.5].

Proposition 4 *The Galois group G_n of $\mathbb{Q}(\xi_n)/\mathbb{Q}$ is Abelian and of order $\phi(n)$. It is isomorphic to $\{1 \leq k \leq n \mid \gcd(k,n) = 1\}$ with multiplication modulo n as the group operation, and k representing the automorphism defined by the map $\xi_n \mapsto \xi_n^k$. For the cases with class number one, the group structure and a possible set of generators are as given in Table* 3. $\square$

Next, we need the Dirichlet characters, which can be seen as extensions of the characters χ of G_n in the sense of representation theory of Abelian groups. So, each χ is a homomorphism from G_n to the unit circle in $\mathbb{C}$. With G_n as given in Proposition 4, χ is defined on all elements of $\{1 \leq k \leq n \mid \gcd(k,n) = 1\}$ modulo n. To obtain a primitive Dirichlet character [18, Ch. 3], of which there are precisely $\phi(n)$ different ones, we have to extend χ to all integers. Clearly, χ is periodic with period n, so it suffices to define χ on the missing positive integers $\leq n$. This has to be done in such a way that the resulting character, which we still denote by χ, is totally multiplicative, i.e., $\chi(k\ell) = \chi(k)\chi(\ell)$ for all k, ℓ, and that the period of the resulting character is minimal. This minimal period, which is always a divisor of n, is called the *conductor* of χ, denoted by f_χ. In this process, one has $\chi(k) = 0$ whenever $\gcd(k, f_\chi) \neq 1$. An illustrative example ($n = 20$) is shown in Table 4.

Table 3
Galois groups G_n of the non-trivial cyclotomic fields with class number one.

n	$\phi(n)$	Galois group	generators	n	$\phi(n)$	Galois group	generators
3	2	C_2	(2)	24	8	$C_2 \times C_2 \times C_2$	(5),(7),(13)
4	2	C_2	(3)	25	20	C_{20}	(2)
5	4	C_4	(2)	27	18	C_{18}	(2)
7	6	C_6	(3)	28	12	$C_6 \times C_2$	(3),(13)
8	4	$C_2 \times C_2$	(3),(5)	32	16	$C_8 \times C_2$	(3),(15)
9	6	C_6	(2)	33	20	$C_{10} \times C_2$	(2),(10)
11	10	C_{10}	(2)	35	24	$C_{12} \times C_2$	(2),(6)
12	4	$C_2 \times C_2$	(5),(7)	36	12	$C_6 \times C_2$	(5),(19)
13	12	C_{12}	(2)	40	16	$C_4 \times C_2 \times C_2$	(3),(11),(21)
15	8	$C_4 \times C_2$	(2),(11)	44	20	$C_{10} \times C_2$	(3),(21)
16	8	$C_4 \times C_2$	(3),(7)	45	24	$C_{12} \times C_2$	(2),(26)
17	16	C_{16}	(3)	48	16	$C_4 \times C_2 \times C_2$	(5),(17),(23)
19	18	C_{18}	(2)	60	16	$C_4 \times C_2 \times C_2$	(7),(11),(19)
20	8	$C_4 \times C_2$	(3),(11)	84	24	$C_6 \times C_2 \times C_2$	(5),(13),(43)
21	12	$C_6 \times C_2$	(2),(13)				

Table 4
Construction of all primitive Dirichlet characters for $n = 20$. The generators of $G_{20} \simeq C_4 \times C_2$ are $g = (3)$ and $h = (11)$. Each character χ originates from a product of characters of C_4 and C_2, and is thus labelled by a pair (i, j).

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
χ	f_χ	e		g				g^3		g^2		h		gh				g^3h		g^2h	
$(0,0)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$(1,0)$	5	1	-i	i	-1	0	1	-i	i	-1	0	1	-i	i	-1	0	1	-i	i	-1	0
$(2,0)$	5	1	-1	-1	1	0	1	-1	-1	1	0	1	-1	-1	1	0	1	-1	-1	1	0
$(3,0)$	5	1	i	-i	-1	0	1	i	-i	-1	0	1	i	-i	-1	0	1	i	-i	-1	0
$(0,1)$	20	1	0	1	0	0	0	1	0	1	0	-1	0	-1	0	0	0	-1	0	-1	0
$(1,1)$	20	1	0	i	0	0	0	-i	0	-1	0	-1	0	-i	0	0	0	i	0	1	0
$(2,1)$	4	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0
$(3,1)$	20	1	0	-i	0	0	0	i	0	-1	0	-1	0	i	0	0	0	-i	0	1	0
	ℓ	1	4	4		1		4		2		2		4				4		2	
	m	8	1	2		2		2		4		4		2				2		4	

Consider a primitive Dirichlet character χ. Its L-function is defined as

$$L(s,\chi) \;:=\; \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s} \;=\; \sum_{k=1}^{f_\chi} \chi(k) \sum_{j=0}^{\infty} \frac{1}{(k+jf_\chi)^s} \tag{8}$$

which has nice analytic properties. In particular, it converges for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, which follows by expressing the last sum in terms of the Hurwitz zeta function [18, Ch. 4]. Due to the total multiplicativity of χ, its L-series has a particularly simple Euler product expansion [18, p. 31], namely

$$L(s,\chi) \;=\; \prod_p \frac{1}{1-\chi(p)\,p^{-s}}\,, \qquad \mathrm{Re}(s) > 1\,. \tag{9}$$

The following result is standard [18, Thm. 4.3].

Proposition 5 *The Dedekind zeta function of the cyclotomic field $\mathbb{Q}(\xi_n)$ is given by*

$$\zeta_{\mathbb{Q}\,(\xi_n)}(s) \;=\; \prod_{\chi\in\hat{G}_n} L(s,\chi)$$

where $\hat{G}_n$ is the set of primitive Dirichlet characters of $\mathbb{Q}(\xi_n)/\mathbb{Q}$. □

The Euler product expansion of $\zeta_{\mathbb{Q}\,(\xi_n)}(s)$ is

$$\zeta_{\mathbb{Q}\,(\xi_n)}(s) \;=\; \prod_{\chi\in\hat{G}_n}\prod_p \frac{1}{1-\chi(p)\,p^{-s}} \;=\; \prod_p \prod_{\chi\in\hat{G}_n} \frac{1}{1-\chi(p)\,p^{-s}}\,, \tag{10}$$

so our Euler factors of Eq. (6) are $E_n(p^{-s}) = \prod_{\chi\in\hat{G}_n}(1-\chi(p)\,p^{-s})^{-1}$, using Proposition 3. Since χ is f_χ-periodic, and $f_\chi|n$, all primitive Dirichlet characters in $\hat{G}_n$ are n-periodic. Therefore, the structure of the Euler factors only depends on the residue class of p modulo n.

Calculating the Euler factors $E_n(p^{-s})$, one finds that they are always of the form $(1-p^{-\ell s})^{-m}$ with integers ℓ and m. If the prime p is not ramified, which happens if and only if $\chi(p)\neq 0$ for all $\chi\in\hat{G}_n$, these integers satisfy $\ell m = \phi(n)$. For all other primes, which are precisely those dividing n, ℓm is a true divisor of $\phi(n)$. In the example of Table 4, these are the primes $p=2$ and $p=5$, while all other primes are unramified. Further details, together with the meaning of ℓ and m for the splitting of primes in the algebraic field extension, can be found in [18, Ch. 3 and Ch. 4]. An explicit calculation can easily be done with an algebraic program package. We summarise the result as follows.

Theorem 6 *The Euler factors of the Dedekind zeta function of $\mathbb{Q}(\xi_n)$ are of the form given in Eq.* (7) *with characteristic indices ℓ and m that depend on p and n. Using the explicit representation via the L-series, the results for the class number one cases are those given in Tables* 1 *and* 2. □

Table 5
First terms of Dirichlet series for the class number one cases of Eq. (1).

n	$F_n(s)$
3	$1+\frac{1}{3^s}+\frac{1}{4^s}+\frac{2}{7^s}+\frac{1}{9^s}+\frac{1}{12^s}+\frac{2}{13^s}+\frac{1}{16^s}+\frac{2}{19^s}+\frac{2}{21^s}+\frac{1}{25^s}+\frac{1}{27^s}+\frac{2}{28^s}+\dots$
4	$1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{2}{5^s}+\frac{1}{8^s}+\frac{1}{9^s}+\frac{2}{10^s}+\frac{2}{13^s}+\frac{1}{16^s}+\frac{2}{17^s}+\frac{1}{18^s}+\frac{2}{20^s}+\frac{3}{25^s}+\dots$
5	$1+\frac{1}{5^s}+\frac{4}{11^s}+\frac{1}{16^s}+\frac{1}{25^s}+\frac{4}{31^s}+\frac{4}{41^s}+\frac{4}{55^s}+\frac{4}{61^s}+\frac{4}{71^s}+\frac{1}{80^s}+\frac{1}{81^s}+\frac{4}{101^s}+\dots$
7	$1+\frac{1}{7^s}+\frac{2}{8^s}+\frac{6}{29^s}+\frac{6}{43^s}+\frac{1}{49^s}+\frac{2}{56^s}+\frac{3}{64^s}+\frac{6}{71^s}+\frac{6}{113^s}+\frac{6}{127^s}+\frac{3}{169^s}+\dots$
8	$1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\frac{2}{9^s}+\frac{1}{16^s}+\frac{4}{17^s}+\frac{2}{18^s}+\frac{2}{25^s}+\frac{1}{32^s}+\frac{4}{34^s}+\frac{2}{36^s}+\frac{4}{41^s}+\dots$
9	$1+\frac{1}{3^s}+\frac{1}{9^s}+\frac{6}{19^s}+\frac{1}{27^s}+\frac{6}{37^s}+\frac{6}{57^s}+\frac{1}{64^s}+\frac{6}{73^s}+\frac{1}{81^s}+\frac{6}{109^s}+\frac{6}{111^s}+\dots$
11	$1+\frac{1}{11^s}+\frac{10}{23^s}+\frac{10}{67^s}+\frac{10}{89^s}+\frac{1}{121^s}+\frac{10}{199^s}+\frac{2}{243^s}+\frac{10}{253^s}+\frac{10}{331^s}+\frac{10}{353^s}+\frac{10}{397^s}+\dots$
12	$1+\frac{1}{4^s}+\frac{1}{9^s}+\frac{4}{13^s}+\frac{1}{16^s}+\frac{2}{25^s}+\frac{1}{36^s}+\frac{4}{37^s}+\frac{2}{49^s}+\frac{4}{52^s}+\frac{4}{61^s}+\frac{1}{64^s}+\frac{4}{73^s}+\dots$
13	$1+\frac{1}{13^s}+\frac{4}{27^s}+\frac{12}{53^s}+\frac{12}{79^s}+\frac{12}{131^s}+\frac{12}{157^s}+\frac{1}{169^s}+\frac{12}{313^s}+\frac{4}{351^s}+\frac{12}{443^s}+\frac{12}{521^s}+\dots$
15	$1+\frac{2}{16^s}+\frac{1}{25^s}+\frac{8}{31^s}+\frac{8}{61^s}+\frac{1}{81^s}+\frac{4}{121^s}+\frac{8}{151^s}+\frac{8}{181^s}+\frac{8}{211^s}+\frac{8}{241^s}+\frac{3}{256^s}+\dots$
16	$1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\frac{1}{16^s}+\frac{8}{17^s}+\frac{1}{32^s}+\frac{8}{34^s}+\frac{4}{49^s}+\frac{1}{64^s}+\frac{8}{68^s}+\frac{2}{81^s}+\frac{8}{97^s}+\dots$
17	$1+\frac{1}{17^s}+\frac{16}{103^s}+\frac{16}{137^s}+\frac{16}{239^s}+\frac{2}{256^s}+\frac{1}{289^s}+\frac{16}{307^s}+\frac{16}{409^s}+\frac{16}{443^s}+\frac{16}{613^s}+\dots$
19	$1+\frac{1}{19^s}+\frac{18}{191^s}+\frac{18}{229^s}+\frac{6}{343^s}+\frac{1}{361^s}+\frac{18}{419^s}+\frac{18}{457^s}+\frac{18}{571^s}+\frac{18}{647^s}+\frac{18}{761^s}+\dots$
20	$1+\frac{2}{5^s}+\frac{1}{16^s}+\frac{3}{25^s}+\frac{8}{41^s}+\frac{8}{61^s}+\frac{2}{80^s}+\frac{2}{81^s}+\frac{8}{101^s}+\frac{4}{121^s}+\frac{4}{125^s}+\frac{8}{181^s}+\dots$
21	$1+\frac{2}{7^s}+\frac{12}{43^s}+\frac{3}{49^s}+\frac{2}{64^s}+\frac{12}{127^s}+\frac{6}{169^s}+\frac{12}{211^s}+\frac{24}{301^s}+\frac{12}{337^s}+\frac{4}{343^s}+\frac{12}{379^s}+\dots$
24	$1+\frac{1}{4^s}+\frac{2}{9^s}+\frac{1}{16^s}+\frac{4}{25^s}+\frac{2}{36^s}+\frac{4}{49^s}+\frac{1}{64^s}+\frac{8}{73^s}+\frac{3}{81^s}+\frac{8}{97^s}+\frac{4}{100^s}+\dots$
25	$1+\frac{1}{5^s}+\frac{1}{25^s}+\frac{20}{101^s}+\frac{1}{125^s}+\frac{20}{151^s}+\frac{20}{251^s}+\frac{20}{401^s}+\frac{20}{505^s}+\frac{20}{601^s}+\frac{1}{625^s}+\dots$
27	$1+\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{27^s}+\frac{1}{81^s}+\frac{18}{109^s}+\frac{18}{163^s}+\frac{1}{243^s}+\frac{18}{271^s}+\frac{18}{327^s}+\frac{18}{379^s}+\frac{18}{433^s}+\dots$
28	$1+\frac{2}{8^s}+\frac{12}{29^s}+\frac{1}{49^s}+\frac{3}{64^s}+\frac{12}{113^s}+\frac{6}{169^s}+\frac{12}{197^s}+\frac{24}{232^s}+\frac{12}{281^s}+\frac{12}{337^s}+\frac{2}{392^s}+\dots$
32	$1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\frac{1}{16^s}+\frac{1}{32^s}+\frac{1}{64^s}+\frac{16}{97^s}+\frac{1}{128^s}+\frac{16}{193^s}+\frac{16}{194^s}+\frac{1}{256^s}+\dots$
33	$1+\frac{20}{67^s}+\frac{1}{121^s}+\frac{20}{199^s}+\frac{2}{243^s}+\frac{20}{331^s}+\frac{20}{397^s}+\frac{20}{463^s}+\frac{10}{529^s}+\frac{20}{661^s}+\frac{20}{727^s}+\dots$
35	$1+\frac{24}{71^s}+\frac{24}{211^s}+\frac{24}{281^s}+\frac{24}{421^s}+\frac{24}{491^s}+\frac{24}{631^s}+\frac{24}{701^s}+\frac{12}{841^s}+\frac{24}{911^s}+\frac{24}{1051^s}+\dots$
36	$1+\frac{1}{9^s}+\frac{12}{37^s}+\frac{1}{64^s}+\frac{12}{73^s}+\frac{1}{81^s}+\frac{12}{109^s}+\frac{12}{181^s}+\frac{6}{289^s}+\frac{12}{333^s}+\frac{6}{361^s}+\frac{12}{397^s}+\dots$
40	$1+\frac{1}{16^s}+\frac{2}{25^s}+\frac{16}{41^s}+\frac{4}{81^s}+\frac{8}{121^s}+\frac{16}{241^s}+\frac{1}{256^s}+\frac{16}{281^s}+\frac{8}{361^s}+\frac{2}{400^s}+\dots$
44	$1+\frac{20}{89^s}+\frac{1}{121^s}+\frac{20}{353^s}+\frac{20}{397^s}+\frac{10}{529^s}+\frac{20}{617^s}+\frac{20}{661^s}+\frac{20}{881^s}+\frac{20}{1013^s}+\frac{1}{1024^s}+\dots$
45	$1+\frac{1}{81^s}+\frac{24}{181^s}+\frac{24}{271^s}+\frac{12}{361^s}+\frac{24}{541^s}+\frac{24}{631^s}+\frac{24}{811^s}+\frac{24}{991^s}+\frac{24}{1171^s}+\frac{24}{1531^s}+\dots$
48	$1+\frac{1}{4^s}+\frac{1}{16^s}+\frac{8}{49^s}+\frac{1}{64^s}+\frac{2}{81^s}+\frac{16}{97^s}+\frac{16}{193^s}+\frac{8}{196^s}+\frac{16}{241^s}+\frac{1}{256^s}+\frac{8}{289^s}+\dots$
60	$1+\frac{2}{16^s}+\frac{2}{25^s}+\frac{16}{61^s}+\frac{2}{81^s}+\frac{8}{121^s}+\frac{16}{181^s}+\frac{16}{241^s}+\frac{3}{256^s}+\frac{8}{361^s}+\frac{4}{400^s}+\dots$
84	$1+\frac{2}{49^s}+\frac{2}{64^s}+\frac{12}{169^s}+\frac{24}{337^s}+\frac{24}{421^s}+\frac{24}{673^s}+\frac{2}{729^s}+\frac{24}{757^s}+\frac{12}{841^s}+\frac{24}{1009^s}+\dots$

Since these results have concrete applications in crystallography and materials science, we spell out the first few terms of the generating functions in Table 5.

4 Asymptotic properties

Generating functions are an efficient way to encapsulate an entire series of numbers into one object. If the generating function is not only a formal object, but also has well-defined analytic properties, an important advantage is the possibility to extract asymptotic properties of the sequence from the generating function. In the case of Dirichlet series, this requires some relatively advanced techniques from complex analysis. Fortunately, in the cases that emerged here, we are still in a rather simple situation, fully covered by the following special case of Delange's theorem, see [17, Ch. II.7, Thm. 15].

Proposition 7 *Let $F(s) = \sum_{k=1}^{\infty} a(k)\, k^{-s}$ be a Dirichlet series with non-negative coefficients which converges for $\mathrm{Re}(s) > 1$. Suppose that $F(s)$ is holomorphic at all points of the line $\{\mathrm{Re}(s) = 1\}$ except at $s = 1$. Approaching $s = 1$ from the half-plane $\{\mathrm{Re}(s) > 1\}$, let $F(s)$ have a singularity of the form $F(s) = g(s) + h(s)/(s-1)$, where both $g(s)$ and $h(s)$ are holomorphic at $s = 1$. Then, as $x \to \infty$, we have*

$$A(x) \;:=\; \sum_{k \le x} a(k) \;\sim\; h(1)\, x\,.$$

In other words, $h(1)$ is the asymptotic average value of $a(k)$ as $k \to \infty$. □

Our Dirichlet series $F_n(s)$ are Dedekind zeta functions of cyclotomic fields, according to Proposition 3. Proposition 7 leads to the following explicit result.

Theorem 8 *Let n be a number from the list* (1)*. The average number of Bravais colourings of $\mathcal{M}_n$, with a given number k of colours, is asymptotically, as $k \to \infty$, given by the residue of the Dedekind zeta function of $\mathbb{Q}(\xi_n)$ at $s = 1$. Consequently, for $x \to \infty$, one has*

$$A_n(x) \;:=\; \sum_{k \le x} a_n(k) \;\sim\; \alpha_n\, x\,, \quad \text{with} \quad \alpha_n \;=\; \prod_{1 \not\equiv \chi \in \hat{G}_n} L(1, \chi)\,.$$

Some values of α_n are given in Table 6*.*

PROOF. The Dedekind zeta functions of the cyclotomic fields (resp. their analytic continuations) always possess an isolated pole of first order at $s = 1$, and no other singularity in the entire complex plane, see [18, Ch. 4]. Consequently, by Proposition 7, $A_n(x)$ has linear growth as $x \to \infty$. The growth rate is then the residue of the zeta function at $s = 1$. The L-series $L(s, \chi)$ have analytic continuations to entire functions for all Dirichlet characters χ except for $\chi_0 \equiv 1$. In the latter case, one has $L(s, \chi_0) = \zeta(s)$. This is Riemann's zeta function which has a simple pole at $s = 1$ with residue 1. For all $\chi \ne \chi_0$, one

Table 6
Residues of Dedekind zeta functions.

$\phi(n)$	n	R_n	residue	numerical
2	3	1	$\frac{\pi\sqrt{3}}{3^2}$	0.604 600
	4	1	$\frac{\pi}{2^2}$	0.785 398
4	5	$2\log(\frac{1+\sqrt{5}}{2})$	$\frac{2\pi^2\sqrt{5}}{5^3}R_4$	0.339 837
	8	$2\log(1+\sqrt{2})$	$\frac{\pi^2}{2^5}R_8$	0.543 676
	12	$\log(2+\sqrt{3})$	$\frac{\pi^2}{2^2 3^2}R_{12}$	0.361 051
6	7	2.101 819	$\frac{2^2\pi^3\sqrt{7}}{7^4}R_7$	0.287 251
	9	3.397 150	$\frac{2^2\pi^3\sqrt{3}}{3^7}R_9$	0.333 685
8	15	4.661 821	$\frac{2^3\pi^4}{3^3 5^4}R_{15}$	0.215 279
	16	19.534 360	$\frac{\pi^4}{2^{12}}R_{16}$	0.464 557
	20	7.411 242	$\frac{\pi^4}{2^2 5^4}R_{20}$	0.288 769
	24	10.643 594	$\frac{\pi^4}{2^7 3^3}R_{24}$	0.299 995

has $L(1,\chi) \neq 0$, see [18, Cor. 4.4], and the expression for α_n is then immediate from Proposition 3. □

To apply this Theorem, we need to know how to calculate the values $L(1,\chi)$ for the non-trivial characters. This is described explicitly on p. 36 and in Thm. 4.9 of [18]. The result is

$$\alpha_n = \frac{R_n}{N(n)} \left(\frac{2\pi \prod_{p|n} p^{1/(p-1)}}{n} \right)^{\phi(n)/2} , \tag{11}$$

where p in the product runs over all prime divisors of n, hence precisely over the ramified primes, $N(n)$ is the function of Eq. (2), and R_n is the so-called regulator of the cyclotomic field $\mathbb{Q}(\xi_n)$. Its calculation, which is a bit technical, is explained in [18]. We show the first few cases in Table 6. The numerical values for $\phi(n) = 6$ and $\phi(n) = 8$ were calculated by means of the program package KANT [7,8].

Acknowledgements

It is a pleasure to thank Robert V. Moody for cooperation and helpful discussions. We express our gratitude to the Erwin Schrödinger International Institute for Mathematical Physics in Vienna for support during a stay in 2002/2003, where this work was completed.

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