A note on palindromicity

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Abstract

Two results on palindromicity of bi-infinite words in a finite alphabet are presented. The first is a simple, but efficient criterion to exclude palindromicity of minimal sequences and applies, in particular, to the Rudin-Shapiro sequence. The second provides a constructive method to build palindromic minimal sequences based upon regular, generic model sets with centro-symmetric window. These give rise to diagonal tight-binding models in one dimension with purely singular continuous spectrum.

Introduction

Since the discovery of quasicrystals some 15 years ago, there has been a renewed interest in the spectral properties of non-periodic Schrödinger operators, and in discrete versions of them in particular, see [18] for a review. The best-studied case is the so-called diagonal tight-binding model, written as an operator L acting on the Hilbert space $\ell^2(\mathbb{Z})$ as follows

$$(L_x u)_n \;=\; u_{n+1} + u_{n-1} + x_n u_n \tag{1}$$

where x denotes a bi-infinite sequence of potentials taking finitely many (pairwise different) real values. One key task is now to infer spectral properties of the operator L_x from properties of the sequence x.

Of specific interest is the case where x is linked to the fixed point of a primitive substitution rule, and a lot is known here, see [18, 6, 12, 1] and references therein

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for relevant contributions. One remarkable feature of examples such as the Fibonacci or the Thue-Morse sequence is that the spectrum of the corresponding operator is purely singular continuous, i.e. there is neither a point spectrum nor an absolutely continuous spectrum present, or, in more common terms, the generalized eigenstates are all critical, and neither localized nor extended.

Clearly, this asks for generalizations and for simple criteria to decide upon this property in other examples. This is precisely the starting point of a number of important contributions, and also the starting point of the paper by Hof, Knill and Simon [12] where palindromicity of x was singled out as one efficient tool to decide upon the spectral nature, building on earlier work Jitomirskaya and Simon [13], see also [12, Appendix]. Recall that a word is a palindrome if it reads backwards the same as forwards, such as "level" or "deed" in ordinary language, and that x is called palindromic if it contains palindromes of arbitrary length. With criteria built upon these concepts, the spectral nature of quite a number of popular examples can be understood.

However, one nagging example is, and always has been, the operator L with a potential according to the Rudin-Shapiro sequence. Based upon numerical evidence, Hof et al. conjectured that the Rudin-Shapiro sequence does not contain palindromes of arbitrary length, and hence that this sequence escapes another attempt to be spectrally classified. This conjecture was soon after answered to the affirmative by Allouche [1] who linked the Rudin-Shapiro sequence (and some generalizations of it) to paperfolding sequences. As a result, he could show that the Rudin-Shapiro sequence can only contain palindromes of length 1,2,3,4,5,6,7,8,10,12,14, see the remark at the end of his paper.

It is the first aim of this note to provide an alternative proof of this statement. The justification is mainly that the method is quite different, constructive, and can easily be used for general primitive substitution rules, so it will be helpful in other cases. In a certain sense, the approach shown below is entirely straight-forward and follows easily from standard results in [16], but it seems to be largely unknown, and hence worthwhile being resurfaced. Since this part is mainly combinatorial, we will recollect some notions of ergodic theory only afterwards.

The main aim then is to use palindromicity to extend the known class of diagonal tight-binding models with purely singular continuous spectrum considerably. This is possible by adapting some standard results from the theory of model sets to construct a rather general class of sequences x which lead to strictly ergodic (i.e. minimal and uniquely ergodic) systems X and thus allow the application of the following

Proposition 1 *If X is aperiodic, strictly ergodic and palindromic, there is a generic $Y \subset X$ such that for $x \in Y$ the operator L_x has purely singular continuous spectrum.*

This is essentially [12, Corollary 7.3] except for the requirement of aperiodicity which is missing in [12], though implicitly assumed and, in fact, necessary.

Palindromes

Let $\mathcal{A} = \{a_1, a_2, \ldots , a_r\}$ be a finite set called *alphabet* and $\mathcal{A}^*$ the set of finite words in elements of $\mathcal{A}$. A *palindrome* is a word that reads backwards the same as forwards. The empty word is considered a palindrome. An element x in $\mathcal{A}^{\mathbb{Z}}$ (bi-infinite sequence) or in $\mathcal{A}^{\mathbb{N}}$ (semi-infinite sequence) is called *palindromic* if it contains palindromes of arbitrary length. If a word w occurs in a sequence x as $w = x_n x_{n+1} \ldots x_m$, we say that w is centred or positioned at $(n+m)/2$.

An element $x \in \mathcal{A}^{\mathbb{Z}}$ is called *strongly palindromic*, with parameter B, if there exists a sequence w_i of palindromes of length ℓ_i centred at m_i such that $|m_i| \to \infty$ together with $e^{B|m_i|}/\ell_i \to 0$ for $i \to \infty$. This means that there is a subsequence of palindromes of diverging length *and* position such that the lengths still grow faster than an exponential of the positions. This somewhat strange property plays a key role in a sufficient criterion for purely singular continuous spectra. If x is strongly palindromic, it is clearly palindromic, but the converse need not be true.

Now, the later treatment of the Rudin-Shapiro sequence rests upon the following, rather trivial observation

Fact 1 *Let x be an infinite sequence of symbols. If there exists an integer $n \geq 1$ such that no palindromes of length n and none of length $n+1$ exist, there are no palindromes of length m for any $m \geq n$, and x is not palindromic.*

PROOF: Let w be a finite string of symbols, of length m. If w is a palindrome, we can chop off one symbol at its beginning and its end, and obtain another palindrome, this time of length $m-2$. By assumption, the sequence x does not contain any substring w of length n or $n+1$ which is a palindrome. If it contained one of length $m > n+1$, we iteratively chop off 2 symbols, one at the beginning and one at the end, until we obtain a palindrome either of length n or of length $n+1$: a contradiction. $\square$

So, this opens a route to establish non-palindromicity by brute force, provided we find an easy and exhaustive way to catalogue all occuring substrings of x of length n, until we eventually find the assumption of the previous Proposition satisfied. This is actually sufficiently easy for substitution sequences, and this is where we deviate from Allouche's approach [1].

Substitution sequences

Consider $\mathcal{A}$ as above, with $|\mathcal{A}| = r$. An r-letter substitution rule is now a mapping σ which attaches to each element of $\mathcal{A}$ a word of finite length in the letters of $\mathcal{A}$,

$$a_j \;\mapsto\; w_j = \sigma(a_j) \tag{2}$$

where we assume that the word w_j has length ℓ_j, i.e. consists of ℓ_j consecutive letters. For several purposes, one needs the corresponding substitution matrix M_σ. Its entries are [16]

$$(M_\sigma)_{ij} \;=\; \text{number of occurrences of } a_i \text{ in the word } w_j\,. \tag{3}$$

This convention leads to $M_{\sigma\circ\varrho} = M_\sigma M_\varrho$. Sometimes the transpose is used, see [3] for details and a formulation in the setting of free groups and their homomorphisms, followed by an abelianization.

There are various interesting subclasses of substitutions. We are interested in *primitive* ones which are characterized by the property that there is an integer k such that M^k has (strictly) positive entries only. We are interested in (bi-infinite) fixed points of a primitive substitution rule σ. After possibly replacing σ by some power of it, such a fixed point always exists [16].

Up to now, things are pretty standard and appear in many texts. What is less well known is the fact that such substitutions also induce substitutions on words of length N, and one standardized way to do this properly is the following. Let $\sigma(w) = a_{i_1} a_{i_2} a_{i_3} \dots a_{i_n}$ be the substitution image of an arbitrary word W of length N. Define

$$\sigma_N(w) \;=\; (a_{i_1} a_{i_2} \dots a_{i_N})(a_{i_2} a_{i_3} \dots a_{i_{N+1}}) \dots (a_{i_m} a_{i_{m+1}} \dots a_{i_{m+N-1}}) \tag{4}$$

where m is the length of $\sigma(a)$ if a is the first letter of w. Consequently, $m+N-1 \leq n$ so that σ_N is well defined. Note that this definition is designed to avoid double counting of strings of length N, see [16] for details.

The relevant question is how to obtain a complete list $\mathcal{A}_N$ of all words of length N that occur in the (by assumption existing !) fixed point u of the primitive substitution, $\sigma(u) = u$. Since σ is primitive, σ_N is also primitive when viewed as a substitution on the new "alphabet" $\mathcal{A}_N$, see [16, Lemma V.12]. There are now basically two straightforward methods to determine $\mathcal{A}_N$, one based on the repetitivity of u, and one on the nature of σ_N. Let us start with a description of the latter.

The possible words of length N certainly form a subset of those words that are obtained from possible words of length $N-1$ by adding to them any of the letters of our alphabet $\mathcal{A} = \mathcal{A}_1$. Let us denote this superset of potential words by $\mathcal{B}_N$. The induced substitution σ_N can now be used to determine $\mathcal{A}_N$ from $\mathcal{B}_N$. If w is a word of length N, let $\tilde{\sigma}_N(w)$ denote the *set* of words of length N in $\sigma_N(w)$ as they appear on the right hand side of Eq. (4). Clearly, $\tilde{\sigma}_N(\mathcal{B}_N) \subset \mathcal{B}_N$, and after finitely many iterations of $\tilde{\sigma}_N$ we arrive at a stable set C, so that $\tilde{\sigma}_N(C) = C$. This means that the substitution σ_N itself is irreducible on C, hence primitive by the previous remarks, and $C = \mathcal{A}_N$. Usually, one actually finds $C = \tilde{\sigma}_N(\mathcal{B}_N)$, so only one iteration is required in this process.

Starting with $\mathcal{A}_1$, we may use this procedure to determine $\mathcal{A}_N$ iteratively up to any (finite) N we wish – a simple recursive program will do. In fact, for typical examples, it is no problem to explicitly do it up to $N = 100$ say, and this is often sufficient to find the conditions to apply Fact 1 and rule out palindromicity.

The other method mentioned to determine $\mathcal{A}_N$ relies on a special property of fixed points of primitive substitutions. If $\sigma(u) = u$, any word of length ℓ occurs in every substring of u of length $L = L(\ell)$, and L grows only linearly in ℓ. So, $L(\ell) \leq c\ell$ and one can estimate c from the explicit form of σ. Then, to obtain $\mathcal{A}_N$, one cuts out of u an arbitrary subword of length cN and collects the different substrings of length N. This is usually even quicker than the previous method, but requires the determination of c.

But whatever method one prefers, the exclusion of palindromicity is usually very efficient. Let us demonstrate this with an example.

The Rudin-Shapiro sequence

Let $a(n)$ be the number of (possibly overlapping) blocks of type 11 in the binary expansion of n, for $n \geq 0$. If we define

$$x_n \;:=\; \frac{1}{2}(1 - (-1)^{a(n)})\,, \tag{5}$$

we obtain the Rudin-Shapiro sequence as the half-infinite sequence $(x_0, x_1, \ldots)$, i.e. as

$$0\,0\,0\,1\,0\,0\,1\,0\,0\,0\,0\,1\,1\,1\;\ldots \tag{6}$$

One well-known alternative way is through a primitive[2] 4-letter substitution rule,

$$\sigma\;:\;\begin{pmatrix} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & db \\ d & \mapsto & dc \end{pmatrix}, \tag{7}$$

[2]The 3rd power of the substitution matrix has all entries positive.

n	$\#^{(4)}(n)$	P?	$\#^{(2)}(n)$	P?
1	4	yes	2	yes
2	8	no	4	yes
3	16	yes	8	yes
4	24	no	16	yes
5	32	yes	24	yes
6	40	no	36	yes
7	48	yes	46	yes
8	56	**no**	56	yes
9	64	**no**	64	no
10	72		72	yes
11	80		80	no
12	88		88	yes
13	96		96	no
14	104		104	yes
15	112		112	**no**
16	120		120	**no**
17	128		128	
18	136		136	
19	144		144	
20	152		152	

Table 1: Number of words of length n (complexity) in quaternary and binary Rudin-Shapiro sequence for $1 \leq n \leq 20$, and status of palindromicity.

which gives the sequence $a\,b\,a\,c\,a\,b\,d\,b\,a\,b\,a\,c\,d\,c...$, followed by the mapping φ which sends a and b to 0 and c and d to 1. This brings us back to (6). The 4-letter version may be called *quaternary* or 4-letter RS-sequence, while (6) is the *binary* or "classic" RS-sequence.

So, we can use the methods from substitution rules, and determine the atlas of substrings of length n. If we subject this atlas to the mapping φ, we obtain the atlas of the Rudin-Shapiro sequence and can then, eventually, apply Fact 1. We summarize the result in Table 1.

If $n \geq 8$, the complexity of both versions is the same. It is known [2] to be $\#(n) = 8n - 8$. The 4-letter version has only palindromes of length 1,3,5 and 7, while the reduction to the binary version creates some new palindromes, whence we get some of length 1,2,3,4,5,6,7,8,10,12 and 14, but not beyond. Consequently, neither version is palindromic.

Further preliminaries and recollections

For the sequel, we need some notions from symbolic dynamics and ergodic theory, see [16] and [15] for details. Two elements x, y of $\mathcal{A}^{\mathbb{Z}}$ are *locally indistinguishable* (LI) if each substring of x also occurs in y and vice versa. This is an equivalence relation and $\mathrm{LI}(x)$ denotes the class represented by x, called LI-class. In view of this, we call an LI-class palindromic if one (and hence any) representative is palindromic. Also, $\mathrm{LI}(x)$ is called *aperiodic* if x is aperiodic. Consequently, an aperiodic LI-class does not contain any periodic member.

A sequence x is called *repetitive* if each substring of x re-occurs in x in a relatively dense way, i.e. the distance between any two consecutive occurences is bounded[3]. Examples of repetitive LI-classes are obtained by fixed points of primitive substitution rules (see above), but also by generic model sets (as we shall see later).

The (two-sided) *shift* S on $\mathcal{A}^{\mathbb{Z}}$ is defined by $(S(x))_n := x_{n+1}$. The *orbit* of $x \in \mathcal{A}^{\mathbb{Z}}$ under the shift is $\mathcal{O}(x) = \{S^n x \mid n \in \mathbb{Z}\}$. Its closure in the product topology is called the *orbit closure* of x and denoted by $\overline{\mathcal{O}(x)}$. A compact shift-invariant subset X of $\mathcal{A}^{\mathbb{Z}}$ is called *minimal* if $\overline{\mathcal{O}(x)} = X$ for all $x \in X$. The following result is standard and follows directly from Gottschalk's Theorem [15, Thm. 4.1.2]:

Proposition 2 *If x is in $\mathcal{A}^{\mathbb{Z}}$, the following statements are equivalent:*

(1) *x is repetitive.*

(2) *$\overline{\mathcal{O}(x)}$ is minimal.*

(3) *$\overline{\mathcal{O}(x)} = \mathrm{LI}(x)$.*

With this result, we can rephrase Prop. 2.1 of [12] as follows.

Proposition 3 *Let $x \in \mathcal{A}^{\mathbb{Z}}$ be repetitive and palindromic. If x is periodic, all elements of* $\mathrm{LI}(x)$ *are strongly palindromic. If x is aperiodic,* $\mathrm{LI}(x)$ *contains uncountably many strongly palindromic sequences.*

[3] Note that some authors call this property "minimal" (which will show up in a different meaning shortly) or "almost-periodic" (which has even more other meanings), wherefore I prefer "repetitive" in this context.

Strong palindromicity of periodic sequences is trivial. The proof of the existence of uncountably many strongly palindromic sequences in [12] is constructive, but the statement is weaker than it appears, because this could still be a thin set in $\mathrm{LI}(x)$. For some recent work on extensions we refer to [7, 8].

1D model sets and derived sequences

To keep things simple, we restrict to the following cut and project scheme for a model set in one dimension. By definition, this consists of a collection of spaces and mappings:

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R}\times\mathbb{R}^n & \xrightarrow{\pi_2} & \mathbb{R}^n \\ & & \cup & & \\ & & \tilde{L} & & \end{array} \tag{8}$$

where $\mathbb{R}$ and $\mathbb{R}^n$ are two real spaces, π_1 and π_2 are the canonical projection maps onto them, and $\tilde{L} \subset \mathbb{R}\times\mathbb{R}^n$ is a lattice. We assume that $\pi_1|_{\tilde{L}}$ is injective and that $\pi_2(\tilde{L})$ is dense in $\mathbb{R}^n$. We call $\mathbb{R}$ (resp. $\mathbb{R}^n$) the physical (resp. internal) space. We will assume that $\mathbb{R}$ and $\mathbb{R}^n$ are equipped with Euclidean metrics and that $\mathbb{R}\times\mathbb{R}^n$ is the orthogonal sum of the two spaces.

A cut and project scheme involves, then, the projection of a lattice into a space of smaller dimension, but a lattice that is transversally located with respect to the projection maps involved.

Let $L := \pi_1(\tilde{L})$ and let

$$(\,)^* : \quad L \longrightarrow \mathbb{R}^n \tag{9}$$

be the mapping $\pi_2 \circ (\pi_1|_{\tilde{L}})^{-1}$. This mapping extends naturally to a mapping on the rational span $\mathbb{Q}L$ of L, also denoted by $(\,)^*$. The lattice $\tilde{L}$ can now also be written as

$$\tilde{L} \;=\; \{(z, z^*) \mid z \in L\}. \tag{10}$$

Now, let $\Omega \subset \mathbb{R}^n$. Define

$$\Lambda \;=\; \Lambda(\Omega) \;:=\; \{z \in L \mid z^* \in \Omega\}\,. \tag{11}$$

We call such a set Λ a *model set* (or *cut and project set*) if the following two conditions are fulfilled,

W1 $\Omega \subset \mathbb{R}^n$ is compact.

W2 $\Omega = \overline{\mathrm{int}(\Omega)} \neq \emptyset$.

In addition, Λ is called *regular*, if Ω is Riemann measurable, i.e. if

W3 The boundary of Ω has Lebesgue measure 0,

and it is called non-singular or *generic*, if

W4 $L^* \cap \partial\Omega = \emptyset$.

Let us just mention that there is no need to restrict to Euclidean internal spaces, and one can, with little extra complication, extend the setup to locally compact Abelian groups instead, which widens the class of structures covered considerably [14, 5, 17]. In any case, the extension beyond the Euclidean codimension one situation is very natural, both physically and mathematically. This is certainly needed for the model set description of more general Pisot substitution (e.g. on alphabets of more than two letters), i.e. substitution rules whose inflation multiplier is a Pisot-Vijayaraghavan number [5]. Also, it quite naturally provides a huge class of other model sets that cannot be described by a local substitution, but gives rise to interesting tight binding models, as we shall see.

The key aspect of interest here is that generic, regular model sets are aperiodic and repetitive [17], and any finite patch that occurs in the set does so with a positive uniform frequency [11, 17]. Also, the model set is a Delone set of finite type, and this means in particular that only finitely many different distances between two consecutive points exist, r say. If we attach r different letters to these intervals, we have mapped the model set to a bi-infinite sequence x in r letters. Any finite word in it then occurs with a positive, uniform frequency, and $\mathrm{LI}(x)$ is strictly ergodic w.r.t. the action of $\mathbb{Z}$. So we have

Proposition 4 *Let Λ be a generic, regular model set in one dimension. Then, mapping different intervals between consecutive points of Λ to different letters gives rise to a bi-infinite sequence $x = x_\Lambda$ in a finite alphabet that is strictly ergodic w.r.t. the action of $\mathbb{Z}$.*

Let $x \in \mathcal{A}^{\mathbb{Z}}$. We say that $\mathrm{LI}(x)$ has *generalized inversion symmetry* if also Rx is in $\mathrm{LI}(x)$, where $(Rx)_i := x_{-i}$, and *strict inversion symmetry* if there is some $y \in \mathrm{LI}(x)$ so that $Ry = S^m y$ for some $m \in \mathbb{Z}$, where $(Sx)_i = x_{i+1}$ is the shift. Clearly, if $\mathrm{LI}(x)$ is strictly inversion symmetric, then there must be a y with either $Ry = y$ or $Ry = Sy$, and $\mathrm{LI}(x)$ is palindromic. The converse is also true if x is repetitive.

Theorem 1 *Let $x \in \mathcal{A}^{\mathbb{Z}}$ be repetitive. Then x, and hence* $\mathrm{LI}(x)$*, is palindromic if and only if* $\mathrm{LI}(x)$ *is strictly inversion symmetric.*

PROOF: In view of the previous remark, we only have to show that palindromicity of x implies strict inversion symmetry of $\mathrm{LI}(x)$. If x is palindromic, it contains palindromes w_i of length ℓ_i, centred at m_i, with $\ell_{i+1} > \ell_i$. Define $n_i = m_i$ if m_i integer and $n_i = m_i + \frac{1}{2}$ if not. Now, consider the sequence $S^{n_i}x$ where the ith element has a palindrome of length ℓ_i centred at 0 or at 1/2. Compactness of $\mathcal{A}^{\mathbb{Z}}$ guarantees that there is a subsequence that converges to an infinite palindrome, i.e. to a y with either $Ry = y$ or $Ry = Sy$. Since x is repetitive, $\mathrm{LI}(x)= \overline{\mathcal{O}(x)}$ by Prop. (2), and y lies in $\mathrm{LI}(x)$. □

Now, if we start with a generic regular model Λ set and attach the corresponding letter sequence x_Λ to it, we obtain a palindromic LI-class if the original model set was strictly inversion symmetric (which is defined by the requirement that $-\Lambda = \Lambda + t$ for some $t \in \mathbb{R}$). This can easily be achieved by choosing a window Ω that is inversion symmetric and has the property [**W4**] that $\partial\Omega \cap L^* = \emptyset$. For a given cut and project setup with inversion symmetric window, this is clearly the generic case. But we can still extend the situation in two ways.

First of all, the window clearly need not be inversion symmetric with respect to the (completely artificial) origin of internal space, it is sufficient if it is centro-symmetric, i.e. if $\Omega = -\Omega + c$ for some $c \in \mathbb{R}^n$. Also, the condition [**W4**] can be replaced by a weaker one. Assume that [**W4**] is violated. Then, some shifted version $\Omega + c$ will observe it again and produce a generic, regular model set. Now consider the corresponding LI-class defined by it. It follows from the so-called *torus parametrization* [4, App. A.1] that this LI-class contains at least 2^{n+1} inversion symmetric members, with equality holding if and only if all of them are generic. So, even if the original condition [**W4**] fails for Ω, we can still constructively check the other possibilities which correspond to specific shifts of the window.

This situation is actually met in the standard example of the Fibonacci chain: here, 4 fixed points on the torus exist, three of which correspond to generic members and the fourth to a pair of singular members, see [4] for details. To summarize, we apply Propositions 4, 3 and 1 and Theorem 1 to obtain

Theorem 2 *Let Λ be a regular, generic model set that is strictly inversion symmetric, and let $x = x_\Lambda$ be the corresponding aperiodic bi-infinite letter sequence. Then, LI(x) contains an uncountable (and even generic) subset Y with the property that the tight-binding operators L_y on $\ell^2(\mathbb{Z})$ in the sense of Eq. (1) has purely singular continuous spectrum for all $y \in Y$.*

Concluding remarks

There are many repetitive, palindromic LI-classes to which the original argument by Hof, Knill and Simon can be applied, and examples obtained by substitution rules or by the codimension one projection method with the "standard" window constitute only a thin set in comparison. So, the appearance of purely singular continuous spectra in the 1D diagonal tight binding model is also more common than originally anticipated.

An open question, however, still is to what extent this spectral type is shared by other members of the same LI-class of potentials, see [9] for a survey. As can be seen from recent results on the complexity of palindromes in substitution generated sequences [10], it is unlikely that the method of strong palindromicity is able to settle this in general, and other ideas seem to be needed here.

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References

[1] J.-P. Allouche, "Schrödinger operators with Rudin-Shapiro potentials are not palindromic", *J. Math. Phys.* **38** (1997) 1843–8.

[2] J.-P. Allouche and J. Shallit, "Complexité des suites de Rudin-Shapiro généralisées", *Bull. Belg. Math. Soc.* **1** (1994) 133–43.

[3] M. Baake, U. Grimm and D. Joseph, "Trace maps, invariants, and some of their applications", *Int. J. Mod. Phys.* **B 7** (1993) 1527–50.

[4] M. Baake, J. Hermisson and P. A. B. Pleasants, "The torus parametrization of quasiperiodic LI-classes", *J. Phys.* **A 30** (1997) 3029–56.

[5] M. Baake, R. V. Moody and M. Schlottmann, "Limit-(quasi)periodic point sets as quasicrystals with p-adic internal spaces", *J. Phys.* **A 31** (1998) 5755–65.

[6] A. Bovier and J.-M. Ghez, "Spectral properties of one-dimensional Schrödinger operators with potentials generated by substitutions", *Commun. Math. Phys.* **158** (1993) 45–66, and *Commun. Math. Phys.* **166** (1994) 431–2 (Erratum).

[7] D. Damanik, "Singular continuous spectrum for the period doubling Hamiltonian on a set of full measure", *Commun. Math. Phys.* **196** (1998) 477-83.

[8] D. Damanik, "Singular continuous spectrum for a class of substitution Hamiltonians", *Lett. Math. Phys.* **46** (1998) 303–11.

[9] D. Damanik, "Gordon-type arguments in the spectral theory of one-dimensional quasicrystals", to appear in: *Directions in Mathematical Quasicrystals*, eds. M. Baake and R. V. Moody, CRM Monograph Series, AMS, Providence (1999), in preparation.

[10] D. Damanik and D. Zare, "Palindrome complexity bounds for primitive substitution sequences", preprint (1999).

[11] A. Hof, "Uniform distribution and the projection method", in: *Quasicrystals and Discrete Geometry*, ed. J. Patera, Fields Institute Monographs, vol. 10, AMS, Providence (1998), pp. 201–6.

[12] A. Hof, O. Knill and B. Simon, "Singular continuous spectrum for palindromic Schrödinger operators", *Commun. Math. Phys.* **174** (1995) 149–59.

[13] S. Jitomirskaya and B. Simon, "Operators with singular continuous spectrum: III. Almost periodic Schrödinger operators", *Commun. Math. Phys.* **165** (1994) 201–5.

[14] R. V. Moody, "Meyer sets and their duals", in: *The Mathematics of Long-Range Aperiodic Order*, ed. R. V. Moody, NATO ASI Series C 489, Kluwer, Dordrecht (1997), pp. 403–41.

[15] K. Petersen, *Ergodic Theory*, 2nd corr. printing, Cambridge University Press, Cambridge (1989).

[16] M. Queffélec, *Substitution Dynamical Systems – Spectral Analysis*, Lecture Notes in Mathematics, vol. 1294, Springer, Berlin (1987).

[17] M. Schlottmann, "Cut-and-project sets in locally compact Abelian groups", in: *Quasicrystals and Discrete Geometry*, ed. J. Patera, Fields Institute Monographs, vol. 10, AMS, Providence (1998), pp. 247–64.

[18] A. Sütő, "Schrödinger difference equation with deterministic ergodic potentials", in: *Beyond Quasicrystals*, eds. F. Axel and D. Gratias, Springer, Berlin (1995), pp. 481–549.