Symmetries and reversing symmetries of area-preserving polynomial mappings in generalised standard form

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Abstract

We determine the symmetries and reversing symmetries within $\mathcal{G}$, the group of real planar polynomial automorphisms, of area-preserving nonlinear polynomial maps L in generalised standard form,

$$L: \quad x' = x + p_1(y) \, , \; \; y' = y + p_2(x') \, ,$$

where p_1 and p_2 are polynomials. We do this by using the amalgamated free product structure of $\mathcal{G}$. Our results lead to normal forms for polynomial maps in generalised standard form and to a classification of the group structures of the reversing symmetry groups for such maps.

1 Introduction

In a series of recent articles [1, 3, 19], the symmetries and reversing symmetries of some dynamical systems (automorphisms) have been investigated systematically by means of algebraic methods. An automorphism L of some space is said to have a *symmetry* if there exists an automorphism S satisfying:

$$S \circ L \circ S^{-1} = L\,, \tag{1}$$

and a *reversing symmetry* if there exists an automorphism R satisfying

$$R \circ L \circ R^{-1} = L^{-1}. \tag{2}$$

The set of symmetries is non-empty (it certainly contains all powers of L) and this set is actually a group, the *symmetry group* $\mathcal{S}(L)$. On the other hand, the existence *a priori* of any reversing symmetries for a particular L is unknown. When L has a reversing symmetry, we call it *reversible*, and *irreversible* otherwise.

We denote by $\mathcal{R}(L)$ the set of all symmetries and reversing symmetries of L. It too is a group, called the *reversing symmetry group* [12] of L (see also [9]). If L has no reversing symmetries or if L is an involution (i.e. $L^2 = id$), then $\mathcal{R}(L) = \mathcal{S}(L)$. On the other hand, if R is a reversing symmetry then so is e.g. $S^i \circ R$ and $R \circ S^i$, $i \in \mathbb{Z}$ (in particular, $L^i \circ R = R \circ L^{-i}$ are reversing symmetries), whereas R^{2i} is a symmetry. More technically, if L has reversing symmetries and is not an involution itself, $\mathcal{S}(L)$ is a normal subgroup of $\mathcal{R}(L)$ with $\mathcal{R}(L)/\mathcal{S}(L) \simeq C_2$. If, in addition, L has an involutory reversing symmetry R (i.e. $R^2 = id$) then $\mathcal{R}(L) \simeq \mathcal{S}(L) \times_s C_2$ (it follows from (2) that the reversing symmetries $L^i \circ R$ and $R \circ L^i$ are then also involutions)[1]. In many cases of reversible automorphisms (and also in the analogous continuous-time case of reversible flows), it is in fact found that all reversing symmetries R satisfying (2) are involutions. In this case, the automorphism L can be written as the composition of two involutions, e.g. $L \circ R$ and R, or R and $R \circ L$. The references [18] and [13] include reviews of the properties and applications of reversible automorphisms and flows.

In the papers [1, 19], the structure of $\mathcal{R}(L)$ has been identified when L, and possible symmetries S and reversing symmetries R, belong to the group of linear automorphisms of the two-dimensional torus (or closely related groups). In this case, the solutions of (1) and (2) can be solved exactly using group theoretic and number theoretic techniques. In [3], extensions of some of these results are made for toral automorphisms in higher dimensions.

In this paper, we consider equations (1) and (2) for real polynomial maps L of the plane in so-called *generalised standard form*:

$$L: \quad x' = x + p_1(y)\,, \;\; y' = y + p_2(x')\,, \tag{3}$$

[1] We denote the cyclic group of order n by C_n and use $\times_s$ to denote a semi-direct product.

where p_1 and p_2 are (real) polynomials (and the new x coordinate x' needs to be calculated first so it can be substituted into the second equation). The form of (3) guarantees that the generalised standard form of map is always area-preserving (even with p_1 and p_2 replaced with non-polynomial functions). As such, this form has often been used to create and study models of area-preserving mappings [14, 15, 18]. Two other standard forms of area-preserving maps, known as the McMillan and de Vogelaere forms, can always be transformed to generalised standard form [18, Section 3.2]. For instance, the McMillan map

$$M: \quad x' = y\,, \quad y' = -x + f(y)\,, \tag{4}$$

under the transformation $x \mapsto x,\ y \mapsto x - y$ becomes (3) with $p_1(y) = -y$ and $p_2(x') = 2x' - f(x')$.

The much-studied area-preserving Hénon map in McMillan form is:

$$H_1: \quad x' = y\,, \quad y' = -x + 2y\,(\lambda + y)\,, \tag{5}$$

and in generalised standard form is:

$$H_2: \quad x' = x - y\,, \quad y' = y + 2x'(1 - \lambda - x')\,, \tag{6}$$

where λ is a parameter. The Hénon map (6) is the simplest nonlinear polynomial example of (3) and all area-preserving quadratic polynomial mappings can be reduced to it [10]. A McMillan map (4) is always reversible with e.g. involutory reversing symmetries: $R: x' = y,\ y' = x$ and $M \circ R: x' = x,\ y' = -y + f(x)$ (so that M is the composition of involutions $M \circ R$ and R). Hence H_1 is certainly reversible and, since reversibility is invariant under conjugation, H_2 is reversible with e.g. involutory polynomial reversing symmetries $R: x' = x,\ y' = -y + 2x(1 - \lambda - x)$ or $R \circ H_2: x' = x - y,\ y' = -y$ (so that H_2 is the composition of involutions R and $R \circ H_2$). The reversibility of H_1 or H_2 has been much exploited in numerical studies because it makes it easier to find periodic orbits [14, 15, 18].

For L of the form (3), the inverse map L^{-1} is given by

$$L^{-1}: \quad x' = x - p_1(y')\,, \quad y' = y - p_2(x)\,. \tag{7}$$

Hence, if p_1 and p_2 are polynomial, then L is an example of a polynomial automorphism of $\mathbb{R}^2$ to itself since both L and its inverse are polynomial. The polynomial automorphisms of $\mathbb{R}^2$ to itself form a group $\mathcal{G}$ (often called the general affine group GA_2 or the *affine Cremona group* [7]). In [8], the term *generalised Hénon transformation* is used for the map

$$H_g: \ x' = y\,, \quad y' = -\delta x + p(y), \tag{8}$$

with p a polynomial function satisfying $\deg p \geq 2$ and $\delta \neq 0$ the constant Jacobian determinant of H_g. Moreover, it is shown that any nonlinear element of $\mathcal{G}$ which is dynamically non-trivial is conjugate within $\mathcal{G}$ to a composition

$$H_{g_n} \circ H_{g_{n-1}} \circ \ldots H_{g_2} \circ H_{g_1}, \tag{9}$$

$n \geq 1$, of generalised Hénon transformations with H_{g_i} depending on the constant δ_i and polynomial $p_i(y)$. Some of the dynamical properties of (9) have been studied in [8, 4, 6].

Although the symmetries and reversing symmetries of (9) will be studied more fully elsewhere, the generalised standard form (3) is a particular, and significant, area-preserving case of (9) when $n = 2$, further justifying its study (see [2] and Remark 5 below in Section 5). Therefore, in this paper, we restrict possible symmetries S or reversing symmetries R of L in (3) to also be polynomial automorphisms and, with this restriction, we are able to solve (1) and (2) exactly.

Our main results, Theorem 1 and its Corollary, are presented in the next Section. The key property that allows us to obtain our results in this paper is that the group $\mathcal{G}$ of polynomial automorphisms has a known structure. This structure is described in Section 3 and its usefulness in deciding conjugacy problems like (1) and (2) is pointed out. In Section 4, we give the details behind the proof of Theorem 1. Finally, in Section 5, we discuss the implications of our results, the possible group structures of $\mathcal{R}(L)$ for L a polynomial map of generalised standard form, and some extensions of the work presented here that will appear elsewhere [2].

2 Statement of results

We restrict ourselves to the dynamically interesting case of (3) when L is nonlinear and consider separately the two cases when both p_1 and p_2 are nonlinear and when just one of p_1 or p_2 is nonlinear. Our main result is:

Theorem 1 *Let L be a nonlinear polynomial mapping of generalised standard form (3). Then L possesses a non-trivial symmetry $S \neq L^k, k \in \mathbb{Z}$, or a reversing symmetry R that is also a polynomial automorphism if and only if p_i satisfy the conditions of Table 1 (if $\deg p_i \geq 2,\ i = 1, 2$) or Table 2 (if $\deg p_1 = 1,\ \deg p_2 \geq 2$). In each case, the corresponding symmetries or reversing symmetries are as indicated. The case when $\deg p_2 = 1,\ \deg p_1 \geq 2$ follows from Table 2 with replacements $x \to y$ and $p_2 \to -p_1$.*

We reiterate from the Introduction that because the symmetries and reversing symmetries form a group, other symmetries and reversing symmetries follow from combining the generating ones presented in Tables 1 and 2 (the groups generated by them are

conditions on p_1 and p_2 of L	*symmetry (S) / reversing symmetry (R)*
S1: p_1 odd around $e/2$ p_2 odd around $c/2$	$S_1 : x' = -x + c, \;\; y' = -y + e \;\; (S_1^2 = id)$
S2: p_1 arbitrary $p_2(y) = \frac{1}{a}\, p_1(\frac{y}{a} + e)$	$S_2 : x' = ay - ea, \;\; y' = \frac{x+p_1(y)}{a} + e \;\; (S_2^2 = L)$
S3: p_1 odd around $\frac{1}{2}(e - \frac{c}{a})$ $p_2(y) = -\frac{1}{a}\, p_1(\frac{y-c}{a})$	$S_3 : x' = ay + c, \;\; y' = -\frac{x+p_1(y)}{a} + e \;\; (S_3^2 = S_1 \circ L)$
R1: p_1 arbitrary p_2 odd around $c/2$	$R_1 : x' = -x - p_1(y) + c, \;\; y' = y \;\; (R_1^2 = id)$ $L \circ R_1 : x' = -x + c, \;\; y' = y - p_2(x)$
R2: p_1 odd around $e/2$ p_2 arbitrary	$R_2 : x' = x + p_1(y), \;\; y' = -y + e \;\; (R_2^2 = id)$ $L \circ R_2 : x' = x, \;\; y' = -y + e + p_2(x)$
R3: p_1 even around $e/2$ p_2 even around $c/2$	$R_3 : x' = -x - p_1(y) + c, \;\; y' = -y + e \;\; (R_3^2 = id)$ $L \circ R_3 : x' = -x + c, \;\; y' = -y + e + p_2(x')$
R4: p_1 arbitrary $p_2(y) = -\frac{1}{a}\, p_1(\frac{y}{a} + e)$	$R_4 : x' = ay - ea, \;\; y' = \frac{x}{a} + e \;\; (R_4^2 = id)$ $L \circ R_4 : x' = a(y - e - p_2(x)), \; y' = \frac{x}{a} + e + p_2(x')$
R5: p_1 odd around $\frac{1}{2}(e - \frac{c}{a})$ $p_2(y) = \frac{1}{a}\, p_1(\frac{y-c}{a})$	$R_5 : x' = ay + c, \;\; y' = -\frac{x}{a} + e \;\; (R_5^4 = id)$

Table 1: Necessary and sufficient conditions for polynomial map L of the form (3) with $\deg p_i \geq 2$ to have a non-trivial symmetry S or reversing symmetry R. The parameters a, c and e are arbitrary. In the case of S3 where $S_3^2 = S_1 \circ L$, the S_1 here is $x' = -x + (c + ae), \; y' = -y + (e - \frac{c}{a})$, since in this case p_1 is odd around $\frac{1}{2}(e - \frac{c}{a})$ and p_2 is odd around $\frac{1}{2}(c + ae)$.

condition on polynomial p_2 of L	*symmetry (S) / reversing symmetry (R)*
S1: p_2 odd around $e/2$	$S_1 : x' = -x + e, \;\; y' = -y - 2B/A \quad (S_1^2 = id)$
R1: p_2 odd around $e/2$	$R_1 : x' = -x + e, \;\; y' = y - p_2(x) \qquad (R_1^2 = id)$ $R_1 \circ L : x' = -x - p_1(y) + e, \;\; y' = y$
R2: p_2 arbitrary	$R_2 : x' = x, \;\; y' = -y + p_2(x) - 2B/A \quad (R_2^2 = id)$ $R_2 \circ L : x' = x + p_1(y) \;\; y' = -y - 2B/A$

Table 2: Necessary and sufficient conditions for polynomial map L of the form (3) with $p_1(y) = Ay + B$, $\deg p_2 \geq 2$ to have a non-trivial symmetry S or reversing symmetry R. The parameter e is arbitrary.

detailed in Section 5, cf. Tables 5 and 6). In particular, in the cases of reversibility R1-R4 of Table 1 and R1-R2 of Table 2, the reversing symmetry R_i in these Tables is easily checked to be an involution and we also list the involutory reversing symmetry $L \circ R_i$ or $R_i \circ L$ for the given maps L (with L being the composition of involutions $L \circ R_i$ with R_i in Table 1 or R_i with $R_i \circ L$ in Table 2). In the case R5 of reversibility in Table 1, the reversing symmetry R_5 has order 4. All possible reversing symmetries of L are orientation-reversing, except in cases R3 and R5 of Table 1 where R is orientation-preserving.

Note that a necessary condition for symmetry possibilities S2 and S3 and reversibility possibilities R4 and R5 of Table 1 is $\deg p_1 = \deg p_2$. In the case of S2 and S3, the non-trivial symmetry S_2 (S_3) is a square root of L ($S_1 \circ L$). More detailed remarks on the results contained in Tables 1 and 2 will be made in Section 5.

Under an affine transformation

$$x \mapsto \alpha x + \beta\,, \quad y \mapsto \gamma y + \delta\,, \tag{10}$$

the mapping L of (3) transforms to L' which is also of generalised standard form:

$$L' : \quad x' = x + \alpha\, p_1\Big(\frac{y-\delta}{\gamma}\Big)\,, \quad y' = y + \gamma\, p_2\Big(\frac{x'-\beta}{\alpha}\Big)\,. \tag{11}$$

Such transformations can be used to bring the cases of L of Tables 1 and 2 into more transparent normal forms. Since possession of a symmetry or reversing symmetry is invariant under conjugation, the symmetries or reversing symmetries of Tables 1 and 2 transform under the same affine transformation used on L.

Corollary 1 *A polynomial automorphism L of generalised standard form* (3) *possesses a symmetry S or reversing symmetry R that is also a polynomial automorphism if and*

conditions on p_1 *and* p_2 *of* L	*symmetry (S) / reversing symmetry (R)*
S1: p_1 odd around 0 p_2 odd around 0	$S_1 : x' = -x,\;\; y' = -y \;\;\; (S_1^2 = id)$
S2: p_1 arbitrary $p_2 = p_1$	$S_2 : x' = y,\;\; y' = x + p_1(y) \;\;\; (S_2^2 = L)$
S3: p_1 odd around 0 $p_2 = -p_1$	$S_3 : x' = y,\;\; y' = -x - p_1(y) \;\;\; (S_3^2 = -L)$
R1: p_1 arbitrary p_2 odd around 0	$R_1 : x' = -x - p_1(y),\;\; y' = y \;\;\; (R_1^2 = id)$ $L \circ R_1 : x' = -x,\;\; y' = y - p_2(x)$
R2: p_1 odd around 0 p_2 arbitrary	$R_2 : x' = x + p_1(y),\;\; y' = -y \;\;\; (R_2^2 = id)$ $L \circ R_2 : x' = x,\;\; y' = -y + p_2(x)$
R3: p_1 even around 0 p_2 even around 0	$R_3 : x' = -x - p_1(y),\;\; y' = -y \;\;\; (R_3^2 = id)$ $L \circ R_3 : x' = -x,\;\; y' = -y + p_2(x)$
R4: p_1 arbitrary $p_2 = -p_1$	$R_4 : x' = y,\;\; y' = x \;\;\; (R_4^2 = id)$ $L \circ R_4 : x' = y + p_1(x),\; y' = x - p_1(x')$
R5: p_1 odd around 0 $p_2 = p_1$	$R_5 : x' = y,\;\; y' = -x \;\;\; (R_5^4 = id)$

Table 3: Normal forms for polynomial map L of the form (3) with $\deg p_i \geq 2$ when L possesses a symmetry S or reversing symmetry R that is a polynomial automorphism.

only if, up to an affine transformation (10), p_i *satisfy the conditions of Table* 3 (*if* $\deg p_i \geq 2$) *or Table* 4 (*if* $\deg p_1 = 1$, $\deg p_2 \geq 2$). *In each case, the corresponding symmetries or reversing symmetries are then as indicated.* $\square$

Explicit affine transformations (10) that convert the cases of Table 1 into the corresponding ones of Table 3 are: S1 and R3 ($\alpha = \gamma = 1$, $\beta = -\frac{c}{2}$, $\delta = -\frac{e}{2}$); S2 ($\alpha = \frac{1}{a}$, $\beta = e$, $\gamma = 1$, $\delta = 0$); S3 and R5 ($\alpha = 1$, $\beta = -\frac{1}{2}(c + ae)$, $\gamma = a$, $\delta = -\frac{a}{2}(e - \frac{c}{a})$); R1 ($\alpha = \gamma = 1$, $\beta = -\frac{c}{2}$, $\delta = 0$); R2 ($\alpha = \gamma = 1$, $\beta = 0$, $\delta = -\frac{e}{2}$); R4 ($\alpha = \frac{1}{a}$, $\beta = e$, $\gamma = 1$, $\delta = 0$). The conversion between Table 2 and Table 4 is achieved by (10) with: S1 and R1 ($\alpha = 1$, $\beta = -\frac{e}{2}$, $\gamma = A$, $\delta = B$); R2 ($\alpha = 1$, $\beta = 0$, $\gamma = A$, $\delta = B$). In all cases, the p_1 and p_2 of Table 3 and Table 4 equal the transformed functions shown in (11).

condition on p_2 of L	*symmetry (S) / reversing symmetry (R)*
S1: p_2 odd around 0	$S_1 : x' = -x, \;\; y' = -y \quad (S_1^2 = id)$
R1: p_2 odd around 0	$R_1 : x' = -x, \;\; y' = y - p_2(x) \quad (R_1^2 = id)$ $R_1 \circ L : x' = -x - y, \;\; y' = y$
R2: p_2 arbitrary	$R_2 : x' = x, \;\; y' = -y + p_2(x) \quad (R_2^2 = id)$ $R_2 \circ L : x' = x + y, \;\; y' = -y$

Table 4: Normal forms for polynomial map L of the form (3) with $\deg p_1 = 1$, $\deg p_2 \geq 2$ when L possesses a symmetry S or reversing symmetry R that is a polynomial automorphism. In each case of the Table, L is given by $x' = x + y, \;\; y' = y + p_2(x')$.

The sufficiency of some of the conditions in Table 3 for allowing a symmetry or reversing symmetry of (3), *irrespective* of p_1 and p_2 being polynomials, has been noted previously in [14, Sections 1.2.4.7, 1.1.4.3] (cases S1, R1 and R2) and [17] (case R4). In fact, it is easy to check the sufficiency of all the (reversing) symmetry conditions of Tables 1 – 4, *irrespective* of p_1 and p_2 being polynomials.

3 Algebraic structure of $\mathcal{G}$

Let g be a polynomial mapping of $\mathbb{R}^2$ (or $\mathbb{C}^2$) of the form:

$$g: \quad x' = P(x,y)\,, \;\; y' = Q(x,y) \tag{12}$$

with $P(x,y)$ and $Q(x,y)$ polynomial in x and y. Suppose g is a bijection (i.e. one-to-one and onto). Then g has a well-defined inverse g^{-1}. Suppose g^{-1} is also a polynomial mapping. Then g is called a *polynomial automorphism* of the real (or complex) plane. The set of polynomial automorphisms of the plane forms a group $\mathcal{G}$ with a known structure, and this is the key ingredient that we exploit here. The mathematical context is the theory of amalgamated free products, convenient sources for background material are [5, Ch. 1.4] and [16, Sec. 4.2]

Specifically, let $\mathcal{A}$ denote the set of *affine* planar maps with $a \in \mathcal{A}$ of the form:

$$a: \quad x' = a_{11}x + a_{12}y + b_1\,, \;\; y' = a_{21}x + a_{22}y + b_2\,, \tag{13}$$

together with the invertibility condition of its linear part,

$$a_{11}a_{22} - a_{12}a_{21} \neq 0\,. \tag{14}$$

It is not hard to see that $\mathcal{A}$ is a (six dimensional) subgroup of $\mathcal{G}$, as we have $\mathcal{A} \simeq K^2 \times_s \mathrm{GL}(2,K)$ with $K = \mathbb{R}$ or $K = \mathbb{C}$.

Another (infinite-dimensional) subgroup $\mathcal{E}$ of $\mathcal{G}$ comprises the so-called *elementary* maps

$$e: \quad x' = \alpha x + p(y)\,, \quad y' = \beta y + \gamma\,, \tag{15}$$

with $\alpha\beta \neq 0$ and $p(y)$ a polynomial in y. Each $e \in \mathcal{E}$ has an inverse of similar form (explicitly, e^{-1} is given by (15) with β replaced by β^{-1} and γ by $-\beta^{-1}\gamma$).

Notice that the subgroups $\mathcal{A}$ and $\mathcal{E}$ intersect in those elementary maps that are also affine, which occurs whenever e from (15) has affine p, i.e. $p(y) = c_1 y + c_0$. We define the set $\mathcal{I}$ by

$$\mathcal{I} = \mathcal{A} \cap \mathcal{E} \tag{16}$$

and note that this intersection is again a subgroup of $\mathcal{G}$.

Now, we need the following result of Jung [11], see also [8, p. 68].

Theorem 2 [Jung] *The group $\mathcal{G}$ of polynomial automorphisms of $\mathbb{R}^2$ (or $\mathbb{C}^2$) is the* amalgamated free product *of its subgroups $\mathcal{A}$ and $\mathcal{E}$, understood as groups with real (complex) parameters. More explicitly, $\mathcal{G}$ is the free product of $\mathcal{A}$ and $\mathcal{E}$ amalgamated along their intersection $\mathcal{I}$, i.e. $\mathcal{G} = \mathcal{A} \underset{\mathcal{I}}{*} \mathcal{E}$.* $\square$

For the present purpose, the practical meaning of Theorem 2 is that any $g \in \mathcal{G}\backslash\mathcal{I}$ can be written as a so-called *reduced word*

$$g = g_n \circ g_{n-1} \circ \ldots \circ g_1\,, \tag{17}$$

where $n \geq 1$ and the g_i alternate in belonging to $\mathcal{A}\backslash\mathcal{I}$ or $\mathcal{E}\backslash\mathcal{I}$. The value of $n \geq 1$ is called the *length* of the reduced word. Note that no reduced word is equal to the identity of $\mathcal{G}$, see [8, Cor. 2.2]. Observe that $g \in \mathcal{A}\backslash\mathcal{I}$ implies that also $s \circ g$ and $g \circ s$ are in $\mathcal{A}\backslash\mathcal{I}$, for arbitrary $s \in \mathcal{I}$, and analogously for $g \in \mathcal{E}\backslash\mathcal{I}$. The algebraic property of amalgamated free products then gives us the following structure for reduced words, see [8, Cor. 2.3] or [5, Thms. 25 and 26].

Proposition 1 *The word expression* (17) *for $g \in \mathcal{G}\backslash\mathcal{I}$ is unique up to the modification that we can replace g_i, for any $i > 1$, by $g_i \circ s$, with any $s \in \mathcal{I}$, and simultaneously replace g_{i-1} by $s^{-1} \circ g_{i-1}$. Consequently, if $g \in \mathcal{G}\backslash\mathcal{I}$ can be represented both by $g_n \circ g_{n-1} \circ \ldots \circ g_1$ and $g'_{n'} \circ g'_{n'-1} \circ \ldots \circ g'_1$, then $n = n'$ and g_i and g'_i both belong to either $\mathcal{A}\backslash\mathcal{I}$ or $\mathcal{E}\backslash\mathcal{I}$.* $\square$

As an example, consider the generalised standard form (3). Let the mapping t be given by

$$t: \quad x' = y\,, \quad y' = x\,, \tag{18}$$

and, for $i = 1, 2$, define the following maps:

$$\begin{aligned} e_i &: \quad x' = x + p_i(y)\,, \quad y' = y\,, \\ q_i &: \quad x' = y\,, \quad y' = x + p_i(y)\,, \\ r_i &: \quad x' = x\,, \quad y' = y + p_i(x)\,. \end{aligned} \tag{19}$$

Notice that $q_i = te_i$ (we suppress the symbol $\circ$ from now on). We see the following: $t \in \mathcal{A}\backslash\mathcal{I}$; $e_i \in \mathcal{E}\backslash\mathcal{I}$ if $\deg p_i \geq 2$ and $e_i \in \mathcal{I}$ if $\deg p_i = 1$ or 0; $q_i \in \mathcal{A}\backslash\mathcal{I}$ only if $\deg p_i = 1$ or 0; and $r_i \in \mathcal{A}\backslash\mathcal{I}$ only if $\deg p_i = 1$ and $r_i \in \mathcal{I}$ if $\deg p_i = 0$.

Possible forms of reduced words (17) for (3) depend on the nature of $\deg p_i$ and we classify them as follows:
Type I: If $\deg p_i \geq 2$, then

$$L = te_2te_1. \tag{20}$$

Type II: If $\deg p_2 \geq 2$ and $\deg p_1 \leq 1$, then

$$L = te_2q_1. \tag{21}$$

Type III: If $\deg p_1 \geq 2$ and $\deg p_2 = 1$, then

$$L = r_2e_1. \tag{22}$$

Type IV: If $\deg p_1 \geq 2$ and $\deg p_2 = 0$, then

$$L : x' = x + p_1(y) \;\; y' = y + K, \tag{23}$$

with K a constant, is itself an element of $\mathcal{E}\backslash\mathcal{I}$.

From (1) and (2), existence of a symmetry S (reversing symmetry R) implies a conjugacy between L and itself (its inverse L^{-1}). If we look for possible (reversing) symmetries that are themselves polynomial with polynomial inverse, then the conjugacy question is asked within $\mathcal{G}$. In this respect, the known group structure of $\mathcal{G}$ also helps. We say that the reduced word (17) is *cyclically reduced* if $n = 1$ or if g_n and g_1 do not both belong to $\mathcal{A}\backslash\mathcal{I}$ or $\mathcal{E}\backslash\mathcal{I}$.

Let us recall [16, Thm. 4.6(iii)], compare also [5, Thm. 26] and [5, Prop. 27].

Proposition 2 *Let g and h be cyclically reduced elements of $\mathcal{G}$ that are conjugate in $\mathcal{G}$. If $h = h_nh_{n-1}\ldots h_1$ with $n \geq 2$ and h_i, h_{i+1} as well as h_n and h_1 belong to different factors, then $g = s\{h_nh_{n-1}\ldots h_1\}s^{-1}$ where $s \in \mathcal{I}$ and $\{h_nh_{n-1}\ldots h_1\}$ means some cyclic permutation of the n letters h_i.* □

Finally, we also need the following result (with $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$).

Lemma 1 *Let $p(x)$ be a non-constant real polynomial satisfying the identity:*

$$p(x) = \gamma\, p(\alpha x + \beta), \tag{24}$$

for some γ,$\alpha \in \mathbb{R}^*$ and $\beta \in \mathbb{R}$. Then we have:

(i) *if $\alpha = 1$, then $\gamma = 1$ and $\beta = 0$, while no restrictions apply to $p(x)$ (trivial case);*

(ii) *if* α *is not a root of unity (i.e.* $\alpha \neq \pm 1$*), then* $\gamma\alpha^n = 1$ *for some* $n \geq 1$ *and* $p(x) = c(x + \frac{\beta}{\alpha-1})^n$*, with arbitrary* $\beta \in \mathbb{R}$ *and* $c \in \mathbb{R}^*$*.*

The remaining case is $\alpha = -1$ *which implies* $\gamma^2 = 1$*. In particular:*

(iii) *if* $\alpha = -1$ *and* $\gamma = 1$*, then* β *is arbitrary and* $p(x)$ *is even around* $\beta/2$*, i.e.* $p(x) = p(\beta/2) + \sum_{n=1}^{N} c_{2n}(x - \beta/2)^{2n}$*,* $c_{2n} \in \mathbb{R}$*,* $\deg p = 2N$*;*

(iv) *if* $\alpha = \gamma = -1$*, then* β *is arbitrary and* $p(x)$ *is odd around* $\beta/2$*, i.e.* $p(x) = \sum_{n=0}^{N} c_{2n+1}(x - \beta/2)^{2n+1}$*,* $c_{2n+1} \in \mathbb{R}$*,* $\deg p = 2N + 1$*.*

Proof: Let ξ be a (real or complex) zero of $p(x)$. Let $A(x)$ be the affine map $A : x \mapsto \alpha x + \beta$. The identity (24) implies that $A^n(\xi)$ is also a zero of $p(x)$, for all integer $n \geq 1$. Since $p(x)$ has only finitely many zeros, the orbit $\{A^n(\xi) \mid n \in \mathbb{Z}^+\}$ must be eventually periodic. So we are led to study the finite orbits under iteration of $\mathcal{A}$.

Let $\alpha = 1$ in $A(x)$. Then $A^n(\xi) = \xi + n\beta$. All orbits of A^n are infinite and aperiodic unless $\beta = 0$, in which case $A(x)$ is the identity. Substituting $\alpha = 1$ and $\beta = 0$ in (24) gives $\gamma = 1$ since $p(x)$ is not the zero polynomial. This gives the trivial case (i).

Suppose $\alpha \neq 1$. Then the dynamics of $A(x)$ is conjugate to that of the linear mapping $L : z \mapsto \alpha z$ via the coordinate transformation $z = x + \frac{\beta}{\alpha-1}$. Since $L^n(z) = \alpha^n z$, we see that, for α real, there are two types of behaviour. The first occurs if $\alpha \neq -1$, in which case L has only one periodic orbit, the fixed point $z = 0$, whence A has only one periodic orbit, the fixed point $\xi = -\frac{\beta}{\alpha-1}$. It follows that if $\alpha \neq \pm 1$, $p(x)$ has only one repeated zero over $\mathbb{C}$, the real zero $\xi = -\frac{\beta}{\alpha-1}$, so necessarily $p(x)$ takes the form $c(x + \frac{\beta}{\alpha-1})^n$, where $n \geq 1$ (since p is non-constant by assumption) and $c \in \mathbb{R}^*$ is arbitrary. Substituting this form into (24) reveals that we must have $\gamma\alpha^n = 1$ and that no restrictions apply to β. This gives case (ii).

If $\alpha = -1$, we obtain the second possibility. Then $A(x) = -x + \beta$ is an involution, so every point has period 2 except for the fixed point at $\beta/2$. Eq. (24) can now be written as $p(\frac{\beta}{2} + x) = \gamma\, p(\frac{\beta}{2} - x)$. Evaluating this at $x = 0$ we see that either $\gamma = 1$ or $p(\beta/2) = 0$. If $\gamma = 1$, p must be even around $\beta/2$, but no further restrictions apply. This gives case (iii). Finally, let $\gamma \neq 1$, so $\xi = \beta/2$ is a root of p. If we write p as a polynomial in powers of $(x - \beta/2)$ we see, by comparison of coefficients, that only $\gamma = -1$ remains, and that p must then be odd around $\beta/2$. This gives case (iv). $\square$

Remark: Lemma 1 has a natural complex counterpart. Cases (i) and (ii) are the same, except that one now has to exclude all roots of unity in (ii). Next, if $\alpha \neq 1$ is a root of unity, i.e. $\alpha^n = 1$ for some $n \geq 2$, then $\gamma^m = 1$ for some $m|n$ because iteration of (24) leads to the equation

$$p\Big(\frac{\beta}{1-\alpha} + x\Big) = \gamma^k\, p\Big(\frac{\beta}{1-\alpha} + \alpha^k x\Big)$$

for all $k \geq 1$. Consequently, cases (iii) and (iv) are to be replaced by appropriate cyclic symmetry requirements around $\frac{\beta}{1-\alpha}$.

4 Proof of main result

To prove our Theorem 1 of Section 2, we consider the above-mentioned Type I (20) and Type II (21) word forms for L in polynomial generalised standard form and use the conjugacy result Proposition 2. In this Section, we will write our general element of $\mathcal{I}$ as

$$s: \quad x' = ax + by + c\,, \quad y' = dy + e\,, \tag{25}$$

with $ad \neq 0$.

Type I words

Let us first consider the reversibility of Type I words (20), when both p_1 and p_2 are nonlinear. We have

$$L = te_2te_1 \quad \Rightarrow \quad L^{-1} = e_1^{-1}te_2^{-1}t, \tag{26}$$

noting that t is an involution so that $t^{-1} = t$. The word for L, and hence that for L^{-1}, is cyclically reduced. Assume that L is reversible, so that L and L^{-1} are conjugate in $\mathcal{G}$. Using Proposition 2, we must have:

$$L = te_2te_1 = ste_2^{-1}te_1^{-1}s^{-1} \tag{27}$$

or

$$L = te_2te_1 = ste_1^{-1}te_2^{-1}s^{-1}. \tag{28}$$

Notice that $te_2^{-1}te_1^{-1}$ and $te_1^{-1}te_2^{-1}$ are the two cyclic permutations of L^{-1} that begin with an element from $\mathcal{A}\backslash\mathcal{I}$ (by Proposition 1 we must reject the cyclic permutations that begin with an element from $\mathcal{E}\backslash\mathcal{I}$ because this is preserved on adding an s from the left and disagrees with the word for L which starts with an element of $\mathcal{A}\backslash\mathcal{I}$).

If (27) holds, a reversing symmetry is:

$$R = se_1 \tag{29}$$

since (27) can be rewritten

$$L = te_2te_1 = ste_2^{-1}te_1^{-1}s^{-1} = se_1(e_1^{-1}te_2^{-1}t)e_1^{-1}s^{-1} = RL^{-1}R^{-1}, \tag{30}$$

and (2) with R replaced by R^{-1} is an equivalent definition of a reversing symmetry. Similarly, if (28) holds, a reversing symmetry is:

$$R = st \tag{31}$$

since (28) can be rewritten

$$L = te_2te_1 = ste_1^{-1}te_2^{-1}s^{-1} = st(e_1^{-1}te_2^{-1}t)ts^{-1} = RL^{-1}R^{-1}. \tag{32}$$

In the case (29), R is nonlinear; in the case (31), R is affine.

The leading element from $\mathcal{A}\backslash\mathcal{I}$ on the left hand side of (27) is t, whereas the leading element from $A\backslash\mathcal{I}$ on the right hand side is st (the same holds for (28)). Since the left and right hand sides are different reduced words for the same element L, it follows from Proposition 1 that:

$$t = sts', \tag{33}$$

where $s' \in \mathcal{I}$. Writing out (33) readily shows that the coefficient of y in the x' part of s and s' vanishes. Consequently, we can take $b = 0$ in (25) for s in both (27) and (28).

Consider now the identity (27) in more detail. Using (18), (19) and (25) with $b = 0$, the x component of the right hand side of (27) equals

$$x - a\,p_1\Big(\frac{y-e}{d}\Big), \tag{34}$$

and the y component is

$$y - d\,p_2\left(\frac{x-c}{a} - p_1\Big(\frac{y-e}{d}\Big)\right). \tag{35}$$

Comparing (34) to the x component of L in (3) shows that $p_1(y)$ satisfies:

$$p_1(y) = -a\,p_1\Big(\frac{y-e}{d}\Big). \tag{36}$$

Comparing (35) to the y component of L in (3) and using (36) shows that $p_2(y)$ satisfies:

$$p_2\big(x + p_1(y)\big) = -d\,p_2\Big(\frac{x + p_1(y) - c}{a}\Big),$$

or, equivalently,

$$p_2(x) = -d\,p_2\Big(\frac{x-c}{a}\Big). \tag{37}$$

The analysis of the second type of reversibility for a Type I map, described by (28), follows close to that above. Writing out both sides of (28), again with $b = 0$ in s, and equating now leads to the conditions:

$$p_1(y) = -a\,p_2\Big(\frac{y-e}{d}\Big), \tag{38}$$

and

$$p_2(y) = -d\,p_1\Big(\frac{y-c}{a}\Big). \tag{39}$$

These show that a necessary condition for this type of reversibility is that $\deg p_1 = \deg p_2$. This can also be seen algebraically from (28) and Proposition 1 since corresponding elements belonging to $\mathcal{E}\backslash\mathcal{I}$ on the left and right hand sides of (28) must have the same degree (and $\deg e_i^{-1} = \deg e_i$).

It remains to study the conditions (36) and (37) and (38) and (39). Conditions (36) and (37) are both examples of the identity (24), and Lemma 1 can be used to work through the possible values for a,c,d and e in s of (25), together with the conditions on p_1 and p_2. For example, taking case (i) of Lemma 1 for $p_1(y)$ in (36) imposes $a = -1, d = 1$ and $e = 0$, which implies in (37) that $p_2(x) = -p_2(-x + c)$. Hence p_2 is odd around $c/2$ for arbitrary c, $s : x' = -x + c, y' = y$ from (25), and from (29) we obtain the reversing symmetry R_1 shown in R1 of Table 1. In a similar way, taking case (iii) or (iv) of Lemma 1 for $p_1(y)$ gives, respectively, cases R3 and R2 of Table 1. Taking case (ii) of Lemma 1 for $p_1(y)$ in (36) does not produce a further example of reversibility. It implies taking $d \neq \pm 1$, $ad^{-n} = -1$ and e arbitrary. Substituting into (37) shows that for p_2, only case (ii) of Lemma 1 is possible, and only if $a \neq \pm 1$ and $da^{-n'} = -1$. The two conditions $ad^{-n} = -1$ and $da^{-n'} = -1$ imply $(-1)^{n'} d^{nn'-1} = -1$ which cannot occur since $d \neq \pm 1$ and $n \geq 2$, $n' \geq 2$ by the assumption $\deg p_i \geq 2$.

The conditions (38) and (39) imply

$$p_1(y) = ad\, p_1\Big(\frac{y - (e + dc)}{ad}\Big), \tag{40}$$

another case of the identity (24) in which it is readily seen case (i) and case (iv) of Lemma 1 are the only possibilities. Taking case (i) for p_1 in (40) implies $ad = 1$, $e + dc = 0$ and p_1 arbitrary. Substituting in (39) gives p_2 defined as in R4 of Table 1. The reversing symmetry R_4 in this instance follows from (31) with $s : x' = ax - ea, y' = y/a + e$. Similarly, taking case (iv) of Lemma 1 for (40) gives R5 of Table 1.

Finally, consider the symmetries of L when it is a Type I word (20). We now need to investigate the necessary and sufficient conditions for $L = te_2te_1$ to be conjugate to itself in $\mathcal{G}$. Using Proposition 2, we must either have:

$$L = te_2te_1 = ste_2te_1s^{-1}, \tag{41}$$

in which case a symmetry is obviously $S = s$, or

$$L = te_2te_1 = ste_1te_2s^{-1}, \tag{42}$$

in which case a symmetry is

$$S = ste_1\,. \tag{43}$$

The analysis in each case is identical to the corresponding conditions for reversibility modulo a sign change (stemming from the fact that e_i^{-1} is obtained from e_i by $p_i \to$

$-p_i$). For (41) to hold, the conditions are

$$p_1(y) = a\, p_1\Big(\frac{y-e}{d}\Big)\,, \quad p_2(y) = d\, p_2\Big(\frac{y-c}{a}\Big). \tag{44}$$

For (42) to hold, the conditions are

$$p_1(y) = a\, p_2\Big(\frac{y-e}{d}\Big)\,, \quad p_2(y) = d\, p_1\Big(\frac{y-c}{a}\Big). \tag{45}$$

The analysis of (44) and (45) is identical to the corresponding conditions for reversibility and uses Lemma 1. Conditions (44) yield S1 of Table 1 (obtained from taking case (iv) of Lemma 1 for p_1) and conditions (45) yield S2 and S3 of Table 1.

Type II words

We consider the (reversing) symmetries of a Type II word (21) in the case that $\deg p_2 \geq 2$ and $\deg p_1 = 1$. Such a word is not cyclically reduced since t and q_1 both belong to $A \backslash \mathcal{I}$. Consequently, we work with such a word after conjugation with t (and also call this transformed word L for convenience) :

$$L = t(te_2q_1)t = e_2r_1 : \quad x' = x + p_2(y')\,, \;\; y' = y + p_1(x)\,, \tag{46}$$

which is now cyclically reduced. Conjugation by t involves interchange of x and y in the original map. Using Proposition 2 and Proposition 1, we must have:

$$L = e_2 r_1 = s e_2^{\pm 1} r_1^{\pm 1} s^{-1}, \tag{47}$$

with the $+$ sign taken when considering symmetries and the $-$ sign for reversibility. Clearly, a symmetry is given by the affine transformation s whereas a reversing symmetry is given by

$$R = s e_2^{-1}. \tag{48}$$

Writing out (47) with s given by (25) and taking $p_1 = Ax + B$, $A \neq 0$, for the affine function p_1, shows that $b = 0$ in s and:

$$p_1(ax + c) = \pm d\, p_1(x) \tag{49}$$

together with

$$p_2(dy + e) = \pm a\, p_2(y)\,. \tag{50}$$

Substituting $p_1(x) = Ax + B$ into (49) gives information about the parameters of the mapping s:

$$a = \pm d\,, \;\; c = (a-1)B/A\,. \tag{51}$$

Consequently, (50) becomes

$$p_2(y) = \frac{1}{d}\, p_2(dy + e)\,, \tag{52}$$

which is an example of (24) with $\alpha = d = 1/\gamma$ and $\beta = e$. Only case (i) ($d = 1$, $e = 0$ and p_2 arbitrary) and case (iv) ($d = -1$, e arbitrary and p_2 odd around $e/2$) of Lemma 1 apply. Case (i) gives R2 of Table 2, using (51) and (48) to write out R_2, and similarly, case (iv) gives S1 and R1 of Table 2 (remembering to interchange x and y in order to get the results for te_2q_1). This solves the problem of finding symmetries and reversing symmetries for Type II words with $\deg p_1 = 1$.

Note that the remaining case of nonlinear maps (3) after Type I and II words is when p_2 is affine and p_1 is nonlinear. This is a Type III word (22) $L = r_2e_1$ whose inverse is $L^{-1} = e_1^{-1}r_2^{-1}$. The (reversing) symmetries of the latter follow from those of e_2r_1, just considered, by the replacements $p_2 \to -p_1$, $p_1 \to -p_2$ and are precisely the (reversing) symmetries of $L = r_2e_1$ (because L and L^{-1}, for arbitrary L, always have the same symmetries, and have the same reversing symmetries when these are involutions). This completes the proof of Theorem 1.

5 Remarks on results

In this paper, we have calculated explicitly the symmetries and reversing symmetries within $\mathcal{G}$, the group of planar polynomial automorphisms, of a real nonlinear polynomial map L in generalised standard form (3). The conditions for, and nature of, possible symmetries S and reversing symmetries R are given in Tables 1 and 2. These results lead to normal forms for polynomial generalised standard maps with the corresponding (reversing) symmetries in Tables 3 and 4. Some observations about the results in Tables 1-4 were already made following Theorem 1 and Corollary 1. We conclude with the following additional remarks:

Remark 1: Suppose L is an element of $\mathcal{G}$ with a kth root M also belonging to $\mathcal{G}$, so $M^k = L$. Then, obviously, M is a symmetry of L in $\mathcal{G}$. It follows from our results that when L is nonlinear and of generalised standard form, it can only have a square root in $\mathcal{G}$ and only when p_1 and p_2 are *both* nonlinear (because the only non-trivial symmetry S_1 of Tables 2 or 4 is a linear involution and $(L \circ S_1)^2 = L^2$). Case S2 of Table 3 makes clear the only case of a square root that can occur at the level of the normal form (the conditions at Case S2 of Table 1 are the general necessary and sufficient conditions for this to occur). When the conditions in S3 of Table 1 or Table 3 are satisfied, S1 is also satisfied and it is seen that S_3 is a square root of $S_1 \circ L$ and a fourth root of L^2 (since S_3 commutes with S_1) but neither of the polynomial automorphisms $S_1 \circ L$ or L^2 are in generalised standard form.

Remark 2: Whenever L is nonlinear and one of p_1 or p_2 is affine, it follows from R2 of Table 2 that L is reversible in $\mathcal{G}$ since it always has an involutory reversing symmetry (e.g. the Hénon map in the form H_2 of (6)). When both p_1 and p_2 are nonlinear, we need to examine the conditions of Table 1 to see whether L is reversible but it is obvious that irreversibility is the generic case. In the special case $\deg p_1 = \deg p_2 = 2$, L is *always* reversible since a quadratic polynomial is always even about some point and case R3 is satisfied (i.e. $p(x) = ax^2 + bx + c = a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a})$ so $p(x - \frac{b}{2a}) = p(-x - \frac{b}{2a})$). It turns out then that the lowest degree combination of p_1 and p_2 for which the conditions Ri, $i = 1, \ldots, 5$ of Table 1 can be avoided is $\deg p_1 = 2$, $\deg p_2 = 3$ or vice versa. After [8], define the degree of a polynomial map (12) by $\deg g = \max(\deg x', \deg y') = \max(\deg P(x, y), \deg Q(x, y))$. It is seen that for a map of the form (3), $\deg L = \deg p_1 \cdot \deg p_2$. Hence the least degree in which irreversibility in $\mathcal{G}$ occurs for area-preserving maps of generalized standard form is $\deg L = 6$ (in [17], using the idea of local reversibility, an example is given of a map (3) of degree 9 that has no C^3 involutory reversing symmetry). More generally, for area-preserving maps g in $\mathcal{G}$ that are not in the form (3), degree four is the minimum degree at which irreversibility occurs (see Remark 5 below).

Remark 3: Tables 3 and 4 help to make transparent the various possible structures of the reversing symmetry group $\mathcal{R}(L) \subset \mathcal{G}$.

Firstly, consider the case that p_1 in L is affine. From R2 of Table 4, the reversible L can be taken to be in the form $x' = x + y$, $y' = y + p_2(x')$. If p_2 in the latter is not odd around some value $e/2$, then $\mathcal{S}(L) = \langle L \rangle \simeq C_\infty$ since S1 of Table 2 is not satisfied and L has no root in $\mathcal{G}$ by Remark 1 above. Because $R_2 : x' = x$, $y' = -y + p_2(x)$ of Table 4 is an involution, $\mathcal{R}(L)$ is a semidirect product of $\mathcal{S}(L)$ and the C_2 group generated by R_2 [1] (recall the dihedral group $D_\infty \simeq C_\infty \times_s C_2$). When p_2 is odd around some value $e/2$, the symmetry group increases to accommodate the involutory symmetry S_1 of Table 2. Note that R_1 of Table 2 now becomes an additional reversing symmetry of L and is the composition of S_1 with R_2. In fact, in this case, R_2 and S_1 commute and we can simplify the structure of $\mathcal{R}(L)$ to that of a direct product of its D_∞ subgroup $(\langle L \rangle \times_s \langle R_2 \rangle)$ and its C_2 subgroup $\langle S_1 \rangle$. The situation is summarized in Table 5.

Secondly, consider the case that p_1 and p_2 are both nonlinear. Table 6 summarizes the possible distinct symmetry group and reversing symmetry group structures and their generators. The top case of Table 6 corresponds to the absence of reversing symmetries *and* nontrivial symmetries. Whenever S2 or S3 of Table 1 hold in Table 6, the C_∞ part of $\mathcal{S}(L)$ is generated by the corresponding S_2 or S_3 of Table 1 – otherwise the C_∞ part is generated by L itself. Table 6 illustrates various ways that the cases of Table 1 or Table 3 interrelate. For example, it is seen from Table 1 that whenever the conditions for S1 hold, then R1 and R2 follow automatically and the reversing

$\mathcal{S}(L)$	$\mathcal{R}(L)$	*Possibilities*
C_∞ $\langle L\rangle$	$C_\infty \times_s C_2 \simeq D_\infty$ $\langle L\rangle \times_s \langle R_2\rangle$	not S1
$C_\infty \times C_2$ $\langle L\rangle \times \langle S_1\rangle$	$(C_\infty \times C_2) \times_s C_2 \simeq D_\infty \times C_2$ $(\langle L\rangle \times_s \langle R_2\rangle) \times \langle S_1\rangle$	S1

Table 5: Possible group structures of $\mathcal{S}(L)$ and $\mathcal{R}(L)$ with their generators when L of (3) has one of p_1 or p_2 affine. The term 'S1' or 'not S1' refers to satisfaction or non-satisfaction of the condition listed for S1 in Table 2.

symmetry R_2 is seen to be the composition of the R_1 together with the symmetry S_1. Also, whenever the conditions for R5 hold in Table 1, then S1, S2, R1 and R2 immediately follow. Furthermore, since a non-zero polynomial cannot be even about some point and odd about some (other) point, it follows that reversibility possibility R3 of Table 1 or Table 3 can only co-exist with R4 or S2, with any two of R3, R4 and S2 implying the third.

Recall that a necessary condition for symmetry possibilities S2 and S3 and reversibility possibilities R4 and R5 of Table 1 is $\deg p_1 = \deg p_2$. It is also clear that symmetry possibilities S2 and S3 cannot co-exist. As found in the S1 possibility of Table 5, it is possible to simplify the structure of $\mathcal{R}(L)$ in Table 6 to a direct product for possibility S1;not S2 and possibility S3. This follows because R_1 and S_1 commute in these cases and, significantly in the case S3, because R_1 is a reversing symmetry of S_3 (as well as being a reversing symmetry of $L = S_1 S_3^2$). Note in possibility S3, the reversing symmetry $R_4 = S_3 \circ R_1$. Interestingly, in possibility S1 and S2 of Table 6 we *cannot* achieve $\mathcal{R}(L)$ as a direct product because R_1 is a reversing symmetry of $L = S_2^2$ but not of S_2 (in fact, $R_1 \circ S_2 \circ R_1 = S_1 \circ S_2^{-1}$). In other words, $\langle S_2\rangle$ is not a normal subgroup of $\langle S_2, R_1\rangle$. The map S_2 is said to have R_1 as a reversing 2-symmetry because R_1 conjugates S_2^2 to its inverse but not S_2 to its inverse. The reversing symmetry R_5 in possibility S1 and S2 is obtained from $R_5 = S_2 \circ R_1$.

Remark 4: Veselov [20, 21] has shown that an area-preserving map L in $\mathcal{G}$ possesses a nonconstant polynomial or rational integral $I(x,y) = I(x',y')$ if and only if L has a symmetry S with $\langle L, S\rangle \simeq C_\infty \times C_\infty$. ¿From Tables 5 and 6, this symmetry group structure does not occur for the polynomial generalised standard maps considered here so integrable cases are absent.

Remark 5: In future work [2], we will consider $\mathcal{R}(L)$ for arbitrary $L \in \mathcal{G}$, i.e. L not necessarily in generalised standard form. There we can show that some of the results

$\mathcal{S}(L)$	$\mathcal{R}(L)$	*Possibilities*
C_∞	$\mathcal{S}(L)$	
$\langle L\rangle$	$\langle L\rangle$	not Si, $i=1,2,3$; not Ri, $i=1,\ldots,5$
$\langle S_2\rangle$	$\langle S_2\rangle$	S2; not Ri, $i=1,\ldots,5$
C_∞	$C_\infty \times_s C_2 \simeq D_\infty$	
$\langle L\rangle$	$\langle L\rangle \times_s \langle R_i\rangle$	not Si, $i=1,2,3$; one Ri, $i=1,\ldots,4$
$\langle S_2\rangle$	$\langle S_2\rangle \times_s \langle R_3\rangle$	S2 and R3 ($\Leftrightarrow$ S2 and R4)
$C_\infty \times C_2$	$(C_\infty \times C_2) \times_s C_2$ $\simeq D_\infty \times C_2$	
$\langle L\rangle \times \langle S_1\rangle$	$(\langle L\rangle \times_s \langle R_1\rangle) \times \langle S_1\rangle$	S1; not S2 ($\Rightarrow$ R1 and R2)
$\langle S_3\rangle \times \langle S_1\rangle$	$(\langle S_3\rangle \times_s \langle R_1\rangle) \times \langle S_1\rangle$	S3 ($\Rightarrow$ S1, R1, R2 and R4)
$C_\infty \times C_2$	$(C_\infty \times C_2) \times_s C_2$	
$\langle S_2\rangle \times \langle S_1\rangle$	$(\langle S_2\rangle \times \langle S_1\rangle) \times_s \langle R_1\rangle$	S1 and S2 ($\Leftrightarrow$ R5; $\Rightarrow$ R1 and R2)

Table 6: Possible group structures of $\mathcal{S}(L)$ and $\mathcal{R}(L)$ with their generators when L of (3) has both p_1 and p_2 nonlinear. The term 'S1' or 'not S1' etc. refers to the satisfaction or non-satisfaction of the condition listed for S1 in Table 1.

found here hold more generally. Furthermore, we can obtain these results using just algebraic properties of the amalgamated free product structure (rather than by means of constructive calculations for the standard form of L, as we did here). For instance, if $L \in \mathcal{G}$ is not conjugate to an affine map or to an elementary map, it can be shown that $\mathcal{S}(L) \subset \mathcal{G}$ is always an Abelian group of the form C_∞ or $C_\infty \times C_2$. Furthermore, any reversing symmetries $R \in \mathcal{G}$ are involutions or of order 4.

A necessary condition for $L \in \mathcal{G}$ to be reversible is that L is area-preserving or anti area-preserving (i.e. $\det dL = -1$). To study reversibility, one can use from [8] the normal form (9) for a nonlinear (non-elementary) $L \in \mathcal{G}$. The degree of the normal form (9) can be shown [8] to be $\prod_i \deg p_i$, with $\deg p_i \geq 2$. When (9) has $n = 1$ and $\delta_1 = 1$, the area-preserving mapping H_{g_1} is precisely of McMillan form (4) and so is reversible. In particular, this implies whenever $\deg L$ is prime and L is area-preserving, then L is reversible in $\mathcal{G}$. If we consider the area-preserving case of (9) with $n = 2$ (i.e. $\delta_1\delta_2 = 1$), we find that $H_{g_2} \circ H_{g_1}$ is reversible when $\deg p_1 \neq \deg p_2$ if and only if: (i) $\delta_1 = \delta_2 = 1$ (in which case H_{g_i} are both (reversible) McMillan maps of the form (4) and $H_{g_2} \circ H_{g_1}$ always has e.g. the involutory reversing symmetry $R : x' = -x + p_1(y),\ y' = y)$; or (ii) $\delta_1 = \delta_2 = -1$ (i.e. $H_{g_2} \circ H_{g_1}$ is in the generalised standard form (3)) and at least one of the conditions R1, R2 or R3 of Table 1 is satisfied. In particular, with respect to (ii), if $\deg p_1$ is odd and $\deg p_2$ is even or vice versa, Table 1 shows that there are no non-trivial symmetries and the only possibility for reversibility is one of R1 and R2.

When $\deg p_1 = \deg p_2$, an area-preserving $H_{g_2} \circ H_{g_1}$ has, in addition to (i) and (ii), other possibilities to be reversible but these are not generically satisfied. This is true even for the least possible case of $\deg(H_{g_2} \circ H_{g_1}) = 4$ (i.e. $\deg p_1 = \deg p_2 = 2$) when the additional reversibility condition for $p_1(y) = y^2 + C_1$, $p_2(y) = y^2 + C_2$ in the (δ_1, C_1, C_2)-space is: (iii) $C_1 = \delta_1^{2/3} C_2$ (this form for p_1 and p_2 in $H_{g_2} \circ H_{g_1}$ can be taken without loss of generality from [8, Theorem 2.6]). Thus, considering quartic polynomial maps in $\mathcal{G}$ that are area-preserving, these are always reversible if they are derived from (9) with $n = 1$ or are derived from (9) with $n = 2$ and satisfy $\delta_1 = \delta_2 = \pm 1$ (noting, from Remark 2 above, that (3) is always reversible when $\deg L = 4$). Otherwise, if they are derived from (9) with $n = 2$, $\delta_1\delta_2 = 1$ and $\delta_1 \neq \pm 1$, they are generically irreversible in $\mathcal{G}$.

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