Similarity submodules and root systems in four dimensions

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Dedicated to H. S. M. Coxeter

Abstract

Lattices and $\mathbb{Z}$-modules in Euclidean space possess an infinitude of subsets that are images of the original set under similarity transformation. We classify such self-similar images according to their indices for certain 4D examples that are related to 4D root systems, both crystallographic and non-crystallographic. We encapsulate their statistics in terms of Dirichlet series generating functions and derive some of their asymptotic properties.

Introduction

This paper begins with the problem of determining the self-similar images of certain lattices and $\mathbb{Z}$-modules in four dimensions and ends in the enchanting garden of Coxeter groups, the arithmetic of several quaternion rings, and the asymptotics of their associated zeta functions. The main results appear in Theorems 2 and 3.

The symmetries of crystals are of fundamental physical importance and, along with the symmetries of lattices, have been studied by mathematicians, crystallographers and physicists for ages. The recent interest in *quasicrystals*, which are non-crystallographic yet still highly ordered structures, has naturally led to speculation about the role of symmetry in this new context. Here, however, it is apparent that a different set of symmetry concepts is appropriate, notably because translational symmetry is either entirely lacking or at least considerably restricted in scope.

[1] Heisenberg-Fellow

One of the most obvious features of quasicrystals is their tendency to have copious inflationary self-similarity. Thus instead of groups of isometries, one is led to semigroups of self-similarities that map a given (infinite) point set into an inflated copy lying within itself. Ordinary point symmetries then show up as a small part of this, namely as the "units", i.e. as the (maximal subgroups of) invertible elements.

The importance of self-similarities is well-known, and has also been used to gain insight into the colour symmetries of crystals [32, 33] and, more recently, of quasicrystals [3, 5, 25]. The focus in the latter cases was on highly symmetric examples in the plane and in 3-space since they are obviously of greatest physical importance.

In this contribution, we extend the investigation of self-similarities to certain exceptional examples in 4-space, namely the hypercubic lattices (of which there are two, represented by the primitive hypercubic lattice $\mathbb{Z}^4$ and the root lattice D_4 or, equivalently, its weight lattice D_4^*) and the icosian ring, seen as the $\mathbb{Z}$-span of the root system of the non-crystallographic Coxeter group H_4, see [10, 19, 13] for notation and background material. The case of lattices has recently also been investigated by Conway, Rains and Sloane [9] who are generally interested in the question under which conditions similarity sublattices of a given index exist. Their methods are complementary to ours, and more general, but do not seem to give direct access to the full combinatorial problem which we can solve here.

It is useful to digress briefly to discuss the role of H_4 in the context of aperiodic order. It is a remarkable fact that the non-crystallographic point symmetries relevant to essentially all known physical quasicrystals are actually Coxeter groups, namely the dihedral groups $I_2(k)$, $k = 5, 8, 10, 12$ and the icosahedral group H_3 [4]. Apart from the remaining infinite series of dihedral groups, the only indecomposable non-crystallographic Coxeter group is H_4 of order 14400, which is the most interesting of them all. In spite of being four-dimensional in nature, there are several good physical reasons to explore the symmetry expressed by this group, and because of its connections with exceptional objects in mathematics, including the root lattice E_8, there are good mathematical reasons, too.

From the point of view of quasicrystals, H_4 may be viewed as the top member of the series $H_2 := I_2(5) \subset H_3 \subset H_4$ of which the first two have been the subject of great attention, see [23] and references therein, while H_4 appears as symmetry group of the Elser-Sloane quasicrystal [17]. Mathematically, this family belongs together and H_4 is the natural parent of the others.

Now, H_4 has a natural quaternionic interpretation which arises as follows. The group of norm 1 units of the real quaternion algebra is easily identifiable with SU(2), see [22] for background material and notation. Using the 2-fold cover of SO(3) by

SU(2) we can find (in many ways) a 2-fold cover of the icosahedral group inside SU(2). This is the binary icosahedral group I of order 120 [14, p. 69]. The point set I is a beautiful object, namely the set of vertices of the exceptional regular 4-polytope called the 600-cell[2] [13, Ch. 22] (under the standard topology of $\mathbb{R}^4$ carried by the quaternion algebra). The $\mathbb{Z}$-span of I is a ring $\mathbb{I}$, dubbed by Conway the ring of *icosians*. This ring is closed under complex conjugation and under left and right multiplications by elements of I. The group of symmetries of $\mathbb{I}$ so obtained is isomorphic to H_4 acting as a reflection group in $\mathbb{R}^4$. The ring $\mathbb{I}$ is itself quite a remarkable object. It is naturally a rank 4-module over $\mathbb{Z}[\tau]$, $\tau := (1+\sqrt{5})/2$, and a rank 8-module over $\mathbb{Z}$ (so it is certainly dense in the ambient space $\mathbb{R}^4$). In fact, as an aside, it has a canonical interpretation as the root lattice of type E_8 with $I \cup \tau I$ making up the 240 roots of E_8. Restricting to the pure quaternions puts us in the 3-dimensional icosahedral case, and by further restriction we can get the H_2 situation.

Now we can state our problem for the icosian case. We have pointed out that the additve group $\mathbb{I}$ has a large finite group of rotational symmetries coming from the left and right multiplications by elements of I – namely $120^2/2 = 7200$ such symmetries. But we have seen that in the study of quasicrystals we have to pay attention to self-similarities, too. So we are now also interested in rotation-inflations of $\mathbb{R}^4$ that map $\mathbb{I}$ into, but not necessarily onto, itself. Each such self-similarity maps $\mathbb{I}$ onto some submodule of finite index, and our question is to determine these images and to count the number of different similarity submodules of a given index. This leads us to introduce a suitable Dirichlet series generating function which encodes the counting information and its asymptotic properties all at once, and indeed determining its exact form is a number theoretical problem that depends heavily on the fact that $\mathbb{I}$ can be interpreted as a maximal order in the split quaternionic algebra over the quadratic field $\mathbb{Q}(\sqrt{5})$.

The other situation that we wish to discuss in this paper is crystallographic in origin, but it nevertheless is very much the same problem. It is well known that there are two hypercubic lattices in 4-space [32, 7], namely the primitive and the centred one (face-centred and body-centred are equivalent in 4-space by a similarity tranformation). Let us take $\mathbb{Z}^4$ and the root lattice D_4 as suitable representatives. Note that they have different holohedries, namely one of order 1152 (denoted by 33/16 in [7, Fig. 7]) for D_4, which coincides with the automorphism group of the underlying

[2]The 600-cell and its dual, the 120-cell, are two of the three exceptional regular polytopes in 4-space [12, p. 292]. The remaining one, the 24-cell, also occurs, later in this paper. Beyond 4 dimensions, the only regular polytopes are the simplices, the hypercubes, and their duals, the cross-polytopes (or hyperoctahedra).

root system, and an index 3 subgroup (denoted by 32/21 in [7, Fig. 7]) for $\mathbb{Z}^4$. Due to the previous remark, we may take the weight lattice D_4^* instead of D_4 if we wish, and we will frequently do so. Given any of these cases, we want to determine the Dirichlet series generating function for the sublattices that are self-similar images of it. What makes this situation tractable is that, parallel to the icosian case, there is a highly structured algebraic and arithmetic object in the background, namely Hurwitz' ring of integral quaternions [20, 11]. In our setting, it is $\mathbb{J} = D_4^*$, and it is again a maximal order, this time of the quaternionic algebra over the rationals, $\mathbb{Q}$. The results for the Hurwitzian and icosian cases are striking in their similarity. Another example is that of the maximal order in $\mathbb{H}(\mathbb{Q}(\sqrt{2}))$ the treatment of which equals that of $\mathbb{I}$ whence we only state the results.

The structure of the paper is as follows. In the next Section, we set the scene by collecting some methods and results from algebra, and algebraic number theory in particular. This will be done in slightly greater detail than necessary for a mathematical audience, but since there is also considerable interest in this type of problem from the physics community, we wish to make the article more self-contained and readable this way. The two following Sections give the results, first for lattices, and then for modules. We close with a brief discussion of related aspects and provide an Appendix with material on the asymptotics of arithmetic functions defined through Dirichlet series.

Preliminaries and Recollections

We shall need a number of results from algebraic number theory, both commutative and non-commutative. First of all, we need, of course, the arithmetic of $\mathbb{Z}$, the ring of integers in the field $\mathbb{Q}$. All ideals of $\mathbb{Z}$ are principal, and they are of the form $\mathfrak{a} = m\mathbb{Z}$ with $m \in \mathbb{Z}$. If $\mathfrak{a} \neq 0$, the index is $[\mathbb{Z} : \mathfrak{a}] = |m|$. The corresponding zeta function, which can be seen as the Dirichlet series generating function for the number of non-zero ideals of a given index, is Riemann's zeta function itself [1]

$$\zeta(s) \;=\; \sum_{\mathfrak{a}\subset\mathbb{Z}} \frac{1}{[\mathbb{Z} : \mathfrak{a}]^s} \;=\; \sum_{m=1}^{\infty} \frac{1}{m^s} \;=\; \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}}\,. \tag{1}$$

Here, $\mathcal{P}$ denotes the set of (rational) primes, and the second representation of the Riemann zeta function is its *Euler product expansion*. It is possible because the number of ideals of index m is a multiplicative arithmetic function, a situation that we shall encounter throughout the article.

Next, we need the analogous objects for the real quadratic field $\mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\tau)$. The ring of integers turns out to be

$$\mathbb{Z}[\tau] \; = \; \{m + n\tau \mid m, n \in \mathbb{Z}\} \tag{2}$$

where $\tau = (1 + \sqrt{5})/2$ is the fundamental unit of $\mathbb{Z}[\tau]$, i.e. all units are obtained as $\pm\tau^m$ with $m \in \mathbb{Z}$. Again, $\mathbb{Z}[\tau]$ is a principal ideal domain, and hence a unique factorization domain [18, ch. 15.4]. The zeta function is the Dedekind zeta function [36, §11] defined by

$$\zeta_{\mathbb{Q}(\tau)}(s) \; = \; \sum_{\mathfrak{a} \subset \mathbb{Z}[\tau]} \frac{1}{[\mathbb{Z}[\tau] : \mathfrak{a}]^s} \; = \; \sum_{m=1}^{\infty} \frac{a(m)}{m^s} \tag{3}$$

where $\mathfrak{a}$ runs through the non-zero ideals of $\mathbb{Z}[\tau]$ and $[\mathbb{Z}[\tau] : \mathfrak{a}]$ is the norm of $\mathfrak{a}$. If $\mathfrak{a} = \alpha\mathbb{Z}[\tau]$, it is given by

$$[\mathbb{Z}[\tau] : \mathfrak{a}] \; = \; |\mathrm{N}(\alpha)| \; = \; |\alpha\alpha'| \tag{4}$$

where $'$ denotes algebraic conjugation in $\mathbb{Q}(\tau)$, defined by $\tau \mapsto 1 - \tau$.

Explicitly, the zeta function reads (see the Appendix for details):

$$\begin{aligned}\zeta_{\mathbb{Q}(\tau)}(s) &= \frac{1}{1 - 5^{-s}} \cdot \prod_{p \equiv \pm 1 \; (5)} \frac{1}{(1 - p^{-s})^2} \cdot \prod_{p \equiv \pm 2 \; (5)} \frac{1}{1 - p^{-2s}} \\ &= 1 + \tfrac{1}{4^s} + \tfrac{1}{5^s} + \tfrac{1}{9^s} + \tfrac{2}{11^s} + \tfrac{1}{16^s} + \tfrac{2}{19^s} + \tfrac{1}{20^s} + \tfrac{1}{25^s} + \tfrac{2}{29^s} + \tfrac{2}{31^s} + \tfrac{1}{36^s} + \tfrac{2}{41^s} + \cdots\end{aligned} \tag{5}$$

As before, $a(m)$ is a multiplicative arithmetic function, i.e. $a(mn) = a(m)a(n)$ for coprime m, n. It is thus completely specified by its value for m being a prime power, and from the Euler product in (5) one quickly derives that $a(5^r) = 1$ (for $r \geq 0$). Then, for primes $p \equiv \pm 2$ (5), one obtains $a(p^{2r+1}) = 0$ and $a(p^{2r}) = 1$, while for primes $p \equiv \pm 1$ (5), the result is $a(p^r) = r + 1$.

One benefit of relating the numbers $a(m)$ to zeta functions with well-defined analytic behaviour is that one can rather easily determine the asymptotic behaviour of $a(m)$ from the poles of the zeta function, see the Appendix for a summary. In this case, the function $a(m)$ is constant on average, the constant being the residue of $\zeta_{\mathbb{Q}(\tau)}(s)$ at its right-most pole, $s = 1$. Explicitly, we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} a(m) \; = \; \mathrm{res}_{s=1}\, \zeta_{\mathbb{Q}(\tau)}(s) \; = \; \frac{2\log(\tau)}{\sqrt{5}} \; \simeq \; 0.430409\,. \tag{6}$$

We shall also need the zeta function of the quadratic field $\mathbb{Q}(\sqrt{2})$, where $\mathbb{Z}[\sqrt{2}]$ is the corresponding ring of integers, and $1+\sqrt{2}$ its fundamental unit [18]. The zeta function reads

$$\begin{aligned}\zeta_{\mathbb{Q}(\sqrt{2})}(s) &= \frac{1}{1-2^{-s}} \cdot \prod_{p \equiv \pm 1\ (8)} \frac{1}{(1-p^{-s})^2} \cdot \prod_{p \equiv \pm 3\ (8)} \frac{1}{1-p^{-2s}} \\ &= 1 + \tfrac{1}{2^s} + \tfrac{1}{4^s} + \tfrac{2}{7^s} + \tfrac{1}{8^s} + \tfrac{1}{9^s} + \tfrac{2}{14^s} + \tfrac{1}{16^s} + \tfrac{2}{17^s} + \tfrac{1}{18^s} + \tfrac{2}{23^s} + \tfrac{1}{25^s} + \tfrac{2}{28^s} + \cdots\end{aligned} \tag{7}$$

The coefficients are $a(2^r) = 1$, $a(p^r) = r+1$ for $p \equiv \pm 1$ (8) and $a(p^r) = 0$ resp. 1 for $p \equiv \pm 3$ (8) and r odd resp. even. The asymptotic behaviour is given by $\lim_{N\to\infty} \frac{1}{N} \sum_{m \le N} a(m) = \log(1+\sqrt{2})/\sqrt{2} \simeq 0.623225$.

Let us now move to the non-commutative results we shall need. We will be concerned with the quaternionic algebra $\mathbb{H}(K)$, mainly over the field $K = \mathbb{Q}$ or over $K = \mathbb{Q}(\tau)$. The case of $K = \mathbb{Q}(\sqrt{2})$ is treated more as an aside. In all cases, we are interested in the corresponding ring of integers, these being the *Hurwitzian ring* $\mathbb{J}$, the *icosian ring* $\mathbb{I}$, and the *cubian ring* $\mathbb{K}$. These are maximal orders in their respective quaternionic algebras [20, 31].

Let us first consider $\mathbb{J} = D_4^*$. In terms of the standard basis[3] $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of $\mathbb{H}(\mathbb{Q})$, $\mathbb{J}$ consists of the points (x_0, x_1, x_2, x_3) whose coordinates x_i either all lie in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$. Though non-commutative, $\mathbb{J}$ is still a principal ideal domain, i.e. all left-ideals (and also all right-ideals) are principal [20]. Consequently, we have unique factorization up to units, the units being the 24 elements obtained from $(\pm 1, 0, 0, 0)$ plus permutations and from $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$. These 24 units form a group, $\mathbb{J}^\times$, that is isomorphic to the binary tetrahedral group [14, p. 69]. The number of non-zero left-ideals (or, equivalently, of non-zero right-ideals) can now be counted explicitly, which results in the corresponding zeta function $\zeta_{\mathbb{J}}(s) = \sum_{\mathfrak{a}} \frac{1}{[\mathbb{J}:\mathfrak{a}]^s}$, where $\mathfrak{a}$ runs over the non-zero left ideals of $\mathbb{J}$, see [15, Sec. VII, § 8 and § 9] for details on zeta functions of quaternionic algebras. The result is [28, § 63, A. 15]:

- The zeta function of Hurwitz' ring of integer quaternions, $\mathbb{J}$, is given by[4]

$$\zeta_{\mathbb{J}}(s) \;=\; (1 - 2^{1-2s}) \cdot \zeta(2s)\,\zeta(2s-1)\,. \tag{8}$$

Using (1), one can easily determine the first few terms

$$\zeta_{\mathbb{J}}(s) \;=\; 1 + \tfrac{1}{4^s} + \tfrac{4}{9^s} + \tfrac{1}{16^s} + \tfrac{6}{25^s} + \tfrac{4}{36^s} + \tfrac{8}{49^s} + \tfrac{1}{64^s} + \tfrac{13}{81^s} + \tfrac{6}{100^s} + \tfrac{12}{121^s} + \tfrac{4}{144^s} + \cdots \tag{9}$$

[3]The defining relations [11] are: $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$.

[4]Note that the formula given in the first line after [2, Eq. 3.17] contains a misprint in the prefactor.

The possible indices of ideals are the squares of integers. It is thus convenient to write the Dirichlet series as

$$\zeta_{\mathbb{J}}(s) \;=\; \sum_{m=1}^{\infty} \frac{a_{\mathbb{J}}(m)}{m^{2s}} \tag{10}$$

so that $a_{\mathbb{J}}(m)$ is actually the number of left-ideals of index m^2 (rather than m). This then results in $a_{\mathbb{J}}(2^r) = 1$ (for $r \geq 0$) and in $a_{\mathbb{J}}(p^r) = (p^{r+1}-1)/(p-1)$ for odd primes. Let us add that $a_{\mathbb{J}}(m)$ is also the sum of the odd divisors of m, see [30, sequence M 3197] or [29, sequence A 000593].

Let us again briefly comment on the asymptotic behaviour. In this case, the average of $a_{\mathbb{J}}(m)$ grows linearly with m, i.e.

$$\frac{1}{N}\sum_{m=1}^{N} a_{\mathbb{J}}(m) \;\sim\; \frac{\pi^2}{24}N \qquad\qquad (N\to\infty) \tag{11}$$

where the coefficient is half the residue of $\zeta_{\mathbb{J}}(s)$ at its right-most pole, $s = 1$. This can easily be calculated from the details provided in the Appendix.

Note that the linear growth of the average of $a_{\mathbb{J}}(m)$ stems from the definition used in (10). With the usual definition (i.e. with denominators m^s rather than m^{2s}), the average would tend to a constant, as in (6).

Now, let us consider the analogous situation, with $\mathbb{I}$ being a maximal order in the algebra $\mathbb{H}(\mathbb{Q}(\tau))$. Again, all left-ideals (and all right-ideals) of $\mathbb{I}$ are principal, and we also get unique factorization up to units again [31]. The unit group $\mathbb{I}^{\times}$ of $\mathbb{I}$ consists of the 120 elements of the binary icosahedral group I inside $\mathbb{I}$. Taking the unit quaternions $(1,0,0,0)$, $\frac{1}{2}(1,1,1,1)$, $\frac{1}{2}(\tau,1,-1/\tau,0)$ together with all even permutations and arbitrary sign flips results in an explicit choice of the group I, and hence of $\mathbb{I}$. Again defining the zeta function for one-sided ideals of $\mathbb{I}$ we have

- The zeta function of the icosian ring, $\mathbb{I}$, is

$$\zeta_{\mathbb{I}}(s) \;=\; \zeta_{\mathbb{Q}(\tau)}(2s)\,\zeta_{\mathbb{Q}(\tau)}(2s-1)\,. \tag{12}$$

This result follows from [31, Ch. III, Prop. 2.1]. The first few terms of this series read

$$\zeta_{\mathbb{I}}(s) \;=\; 1 + \tfrac{5}{16^s} + \tfrac{6}{25^s} + \tfrac{10}{81^s} + \tfrac{24}{121^s} + \tfrac{21}{256^s} + \tfrac{40}{361^s} + \tfrac{30}{400^s} + \tfrac{31}{625^s} + \tfrac{60}{841^s} + \tfrac{64}{961^s} + \cdots \tag{13}$$

The possible indices (= denominators) are the squares of integers that are representable by the quadratic form $x^2 + xy - y^2$, i.e. of integers all of whose prime factors congruent to 2 or 3 (mod 5) occur with even exponent only. Using a definition analogous to (10) above, the coefficient $a_{\mathbb{I}}(m)$ is again a multiplicative arithmetic function.

It is given by $a_{\mathbb{I}}(5^r) = (5^{r+1}-1)/4$ (for $r \geq 0$), and, for primes $p \equiv \pm 2$ (5), by $a_{\mathbb{I}}(p^{2r+1}) = 0$ and $a_{\mathbb{I}}(p^{2r}) = (p^{2r+2}-1)/(p^2-1)$. Finally, for $p \equiv \pm 1$ (5), one finds $a_{\mathbb{I}}(p^r) = \sum_{l=0}^{r}(l+1)(r-l+1)p^l$. It is now listed as [29, sequence A 035282].

The asymptotic behaviour of $a_{\mathbb{I}}(m)$ is similar to that of $a_{\mathbb{J}}(m)$ above, and we obtain

$$\frac{1}{N}\sum_{m=1}^{N} a_{\mathbb{I}}(m) \;\sim\; \frac{2\pi^4 \log(\tau)}{375} N \;\simeq\; 0.249997 \cdot N \qquad (N \to \infty) \tag{14}$$

where the slope is again half the residue of $\zeta_{\mathbb{I}}(s)$ at its right-most pole, $s = 1$, see the Appendix for details.

Very similar is the situation of the ring $\mathbb{K}$ in $\mathbb{H}(\mathbb{Q}(\sqrt{2}))$, generated as the $\mathbb{Z}[\sqrt{2}]$-span of the basis $\{1, (1+\boldsymbol{i})/\sqrt{2}, (1+\boldsymbol{j})/\sqrt{2}, (1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})/2\}$. The unit group $\mathbb{K}^\times$ is the binary octahedral group of order 48 [14, p. 69], and the symmetry group of $\mathbb{K}$ contains that of $\mathbb{J}$ as an index 2 subgroup. $\mathbb{K}$ is again a maximal order and a principal ideal domain [31]. The zeta function of $\mathbb{K}$ reads [31, Ch. III, Prop. 2.1]

$$\begin{aligned}\zeta_{\mathbb{K}}(s) &= \zeta_{\mathbb{Q}(\sqrt{2})}(2s)\,\zeta_{\mathbb{Q}(\sqrt{2})}(2s-1) \\ &= 1 + \tfrac{3}{4^s} + \tfrac{7}{16^s} + \tfrac{16}{49^s} + \tfrac{15}{64^s} + \tfrac{10}{81^s} + \tfrac{48}{196^s} + \tfrac{31}{256^s} + \tfrac{36}{289^s} + \tfrac{30}{324^s} + \tfrac{48}{529^s} + \tfrac{26}{625^s} + \cdots\end{aligned} \tag{15}$$

and further details can be worked out in complete analogy to the icosian case.

Let us now briefly describe how the quaternions enter our (mainly geometric) picture, and how they provide a parametrization of (S)O(4) = (S)O(4, $\mathbb{R}$), see [22] for details. The key is that pairs of quaternions in $\mathbb{H}(\mathbb{R})$, i.e. quaternions $\boldsymbol{q} = (q_0, q_1, q_2, q_3)$ as written in the standard basis $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of the quaternion algebra over $\mathbb{R}$, induce an action on vectors of $\mathbb{R}^4$ via

$$M(\boldsymbol{q}_1, \boldsymbol{q}_2)\boldsymbol{x}^t \;=\; \boldsymbol{q}_1 \boldsymbol{x}\, \overline{\boldsymbol{q}}_2 \tag{16}$$

where $M(\boldsymbol{q}_1, \boldsymbol{q}_2) \in \mathrm{Mat}(4, \mathbb{R})$ and $\boldsymbol{x}^t$ is $\boldsymbol{x}$ written as a column vector in $\mathbb{R}^4$. Evidently, for nonzero quaternions $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_1, \boldsymbol{r}_2$ we have $M(\boldsymbol{q}_1, \boldsymbol{q}_2) = M(\boldsymbol{r}_1, \boldsymbol{r}_2)$ if and only if $\boldsymbol{r}_1 = a\boldsymbol{q}_1$ and $\boldsymbol{r}_2 = a^{-1}\boldsymbol{q}_2$ for some $a \in \mathbb{R}\backslash\{0\}$. With $\boldsymbol{q}_1 = (a, b, c, d)$ and $\boldsymbol{q}_2 = (t, u, v, w)$, the matrix $M = M(\boldsymbol{q}_1, \boldsymbol{q}_2)$ reads explicitly

$$\begin{pmatrix} at+bu+cv+dw & -bt+au+dv-cw & -ct-du+av+bw & -dt+cu-bv+aw \\ bt-au+dv-cw & at+bu-cv-dw & -dt+cu+bv-aw & ct+du+av+bw \\ ct-du-av+bw & dt+cu+bv+aw & at-bu+cv-dw & -bt-au+dv+cw \\ dt+cu-bv-aw & -ct+du-av+bw & bt+au+dv+cw & at-bu-cv+dw \end{pmatrix} . \tag{17}$$

It has determinant

$$\begin{aligned}\det(M(\boldsymbol{q}_1, \boldsymbol{q}_2)) &= (a^2+b^2+c^2+d^2)^2 \cdot (t^2+u^2+v^2+w^2)^2 \\ &= \{(\boldsymbol{q}_1)^2 \cdot (\boldsymbol{q}_2)^2\}^2\end{aligned} \tag{18}$$

and also fulfils

$$MM^t \;=\; \sqrt{\det M}\cdot \mathbf{1}_4\,. \tag{19}$$

Consequently, when $|\boldsymbol{q}_1| = |\boldsymbol{q}_2| = 1$, we obtain a 4D rotation matrix and the homomorphism $M : S^3 \times S^3 \longrightarrow \mathrm{SO}(4)$ provides the standard double cover of the rotation group SO(4) [22], with $M(\boldsymbol{q}_1, \boldsymbol{q}_2) = M(-\boldsymbol{q}_1, -\boldsymbol{q}_2)$. The orientation reversing transformations, i.e. the elements of $\mathrm{O}(4)\backslash\mathrm{SO}(4)$, are obtained by the mapping $\boldsymbol{x} \mapsto \boldsymbol{q}_1\, \overline{\boldsymbol{x}}\, \overline{\boldsymbol{q}}_2$ with unit quaternions $\boldsymbol{q}_1, \boldsymbol{q}_2$. Let us finally note that, for non-zero quaternions,

$$R(\boldsymbol{q}_1, \boldsymbol{q}_2) \;:=\; M\Big(\frac{\boldsymbol{q}_1}{|\boldsymbol{q}_1|}, \frac{\boldsymbol{q}_2}{|\boldsymbol{q}_2|}\Big) \;=\; \frac{1}{|\boldsymbol{q}_1 \boldsymbol{q}_2|}\, M(\boldsymbol{q}_1, \boldsymbol{q}_2) \tag{20}$$

always gives a rotation matrix, which is handy for finding suitable parametrizations of groups such as $\mathrm{SO}(4,\mathbb{Q})$ or $\mathrm{SO}(4,\mathbb{Q}(\tau))$ in closely related problems, see [2] for details.

Arguments in common

In this Section, we focus on $\mathbb{I}$ versus $\mathbb{J}$ and carry the arguments as far as possible without having to separate the two rings $\mathbb{I}$ and $\mathbb{J}$ too seriously[5]. Thus we introduce the following notation to cover both situations simultaneously:

$$\begin{array}{ccccc} K & := & \mathbb{Q} & \text{or} & \mathbb{Q}(\sqrt{5}) \\ \scriptstyle\mathcal{O} & := & \mathbb{Z} & \text{or} & \mathbb{Z}[\tau] \\ \mathcal{O} & := & \mathbb{J} & \text{or} & \mathbb{I} \\ \mathcal{L} & := & \mathbb{Z}^4 & \text{or} & \mathbb{Z}[\tau]^4 \end{array} \tag{21}$$

By $\mathcal{L}$ we really mean the ring $\scriptstyle\mathcal{O}\displaystyle 1 + \scriptstyle\mathcal{O}\displaystyle\boldsymbol{i} + \scriptstyle\mathcal{O}\displaystyle\boldsymbol{j} + \scriptstyle\mathcal{O}\displaystyle\boldsymbol{k} \subset \mathcal{O}$, and we observe that

$$2\mathcal{O} \;\subset\; \mathcal{L} \;\subset\; \mathcal{O}\,. \tag{22}$$

Let us first note that $\mathcal{O}$ is a maximal order in $\mathbb{H}(K)$ [31]. As such, each prime ideal $\mathfrak{P}$ of $\mathcal{O}$ corresponds to one prime ideal of $\scriptstyle\mathcal{O}$, namely to $\mathfrak{p} := \mathfrak{P} \cap \scriptstyle\mathcal{O}$, and this sets up a one-to-one correspondence between their prime ideals [27, Thm. 22.4]. Furthermore, we have unique factorization of each 2-sided ideal $\mathfrak{A}$ of $\mathcal{O}$:

$$\mathfrak{A} \;=\; \mathfrak{P}_1^{k_1} \cdot \ldots \cdot \mathfrak{P}_r^{k_r} \tag{23}$$

where $\mathfrak{P}_1^{k_1}, \ldots, \mathfrak{P}_r^{k_r}$ are prime ideals and all $k_i \geq 0$.

[5] The case $\mathcal{O} = \mathbb{K}$ is entirely parallel to that of $\mathcal{O} = \mathbb{I}$ and need not be spelled out here. We shall mention details later when we need them.

The prime 2 (which is prime both in $\mathbb{Z}$ and in $\mathbb{Z}[\tau]$) plays a special role in this paper. In the case of $\mathbb{I}$, $2\mathbb{I}$ is the prime ideal of $\mathbb{I}$ lying over $2\mathbb{Z}[\tau]$. The case of $\mathbb{J}$, however, is more complicated. Here, $(1+\boldsymbol{i})\mathbb{J} = \mathbb{J}(1+\boldsymbol{i})$ is the prime ideal lying over $2\mathbb{Z}$ and $(1+\boldsymbol{i})^2\mathbb{J} = 2\mathbb{J}$ [20]. It is this ramification of $2\mathbb{Z}$ in $\mathbb{J}$ that accounts for the stray factor in Eq. (8) and, later on, in Eq. (34). To cope with this, we shall call $\boldsymbol{a} \in \mathbb{J}$ an *odd* element of $\mathbb{J}$ if $\boldsymbol{a} \notin (1+\boldsymbol{i})\mathbb{J}$. It is useful to note that $\boldsymbol{a} \in \mathbb{J}$ is odd if and only if $|\boldsymbol{a}|^2 \in \mathbb{Z}$ (its quaternionic norm) is odd.

Let $\boldsymbol{a} \in \mathcal{O}$. We define

$$A(\boldsymbol{a}) \;:=\; \{r \in K \mid r\boldsymbol{a} \in \mathcal{O}\}\,. \tag{24}$$

For $\boldsymbol{a} \neq 0$, $\mathcal{o} \subset A(\boldsymbol{a}) \subset \mathcal{O}\boldsymbol{a}^{-1} \cap K$. Thus $A(\boldsymbol{a})$ is a finitely generated $\mathcal{o}$-module and hence is a fractional ideal of $\mathcal{o}$ [21, Ch. I.4]. It follows that $A(\boldsymbol{a})^{-1}$ is an ordinary ideal of $\mathcal{o}$ and, consequently, $A(\boldsymbol{a})^{-1} = \mathcal{o}c$ for some $c \in \mathcal{o}$. We call $c = c(\boldsymbol{a}) \in \mathcal{o}$ (which is determined up to a unit of $\mathcal{o}$) the *content* of $\boldsymbol{a}$: $A(\boldsymbol{a}) = \mathcal{o}c(\boldsymbol{a})^{-1}$.

Definition 1 *We say that $\boldsymbol{a} \in \mathcal{O}$ is* $\mathcal{O}$-primitive *if $A(\boldsymbol{a}) = \mathcal{o}$.*

We have just seen that $\boldsymbol{a}$ is $\mathcal{O}$-primitive if and only if $c(\boldsymbol{a})$ is a unit in $\mathcal{o}$. So, we have

Lemma 1 *For any $\boldsymbol{a} \in \mathcal{O}\backslash\{0\}$, $c(\boldsymbol{a})^{-1}\boldsymbol{a}$ is a primitive element of $\mathcal{O}$.* $\square$

Let us now come to the link between submodules and similarities.

Definition 2 *Let L be a $\mathbb{Z}$-module in $\mathbb{R}^4$ that spans $\mathbb{R}^4$. $\mathcal{M} \subset L$ is called a* similarity submodule *(SSM) of L if there is an $\alpha \in \mathbb{R}\backslash\{0\}$ and an $R \in O(4)$ so that $\mathcal{M} = \alpha R(\mathcal{O}) \subset L$. If L is a* lattice, *we call $\mathcal{M}$ a* similarity sublattice *(SSL) of L.*

Consider the case when $\mathcal{M}$ is an SSM of $\mathcal{O}$. It is immediate that such an $\mathcal{M}$ is an $\mathcal{o}$-submodule of $\mathcal{O}$. Since $\mathcal{o}$ is a principal ideal domain and $\mathcal{O}$ is a free $\mathcal{o}$-module of rank 4, we see that $\mathcal{M}$ is also a free $\mathcal{o}$-module of rank 4. Consequently, the index $[\mathcal{O} : \mathcal{M}]$ of $\mathcal{M}$ in $\mathcal{O}$ is finite.

We now come to the first crucial assertion in our classification of the similarity submodules according to their indices.

Proposition 1 *Let $\mathcal{M} \subset \mathcal{O}$ be a similarity submodule. Then there exist $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{O}$, with $\boldsymbol{a}$ primitive, such that $\mathcal{M} = \boldsymbol{a}\mathcal{O}\boldsymbol{b}$. In addition, in the case $\mathcal{O} = \mathbb{J}$, we can arrange for $\boldsymbol{a}$ to be odd.*

Proof: By assumption, $\mathcal{M} = \alpha R(\mathcal{O})$ with $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and $R \in \mathrm{O}(4) = \mathrm{O}(4,\mathbb{R})$. Using (16), and noting that $\overline{\mathcal{O}} = \mathcal{O}$, we can write $\mathcal{M} = \boldsymbol{a}\mathcal{O}\boldsymbol{b}$ where $\boldsymbol{a} = (a_0, a_1, a_2, a_3)$, $\boldsymbol{b} = (b_0, b_1, b_2, b_3)$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{H}(\mathbb{R})$.

Since $2\boldsymbol{a}\{1,\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}\boldsymbol{b} \subset 2\boldsymbol{a}\mathcal{O}\boldsymbol{b} \subset \mathcal{L}$, we can infer from the explicit matrix form (17) that all the matrix entries of $2M(\boldsymbol{a},\boldsymbol{b})$ lie in $\mathcal{o}$. Combining suitable entries, four at a time, we can obtain that

$$8a_i b_j \in \mathcal{o}\,, \qquad \text{for all } i,j \in \{0,1,2,3\}\,. \tag{25}$$

Since $\boldsymbol{a}\mathcal{O}\boldsymbol{b} = \boldsymbol{a}r\mathcal{O}r^{-1}\boldsymbol{b}$ for all $r \in \mathbb{R}\backslash\{0\}$, we can arrange that some $a_i \in K\backslash\{0\}$, whereupon we see, via (25), that we can choose $\boldsymbol{a},\boldsymbol{b} \in \mathbb{H}(K)$ without loss of generality. Clearing denominators and using Lemma 1, we may further assume that $\mathcal{M} = \boldsymbol{a}\mathcal{O}\boldsymbol{b}$ with $\boldsymbol{a} \in \mathcal{O}$ and $\boldsymbol{a}$ primitive. From (25), we now get $8b_j\boldsymbol{a} \in \mathcal{O}$, whence $8b_j \in A(\boldsymbol{a}) = \mathcal{o}$ for all $j \in \{0,1,2,3\}$. We conclude that $8\boldsymbol{b} \in \mathcal{O}$.

We now have to dispose of the factor 8 in order to prove the Proposition. Consider the 2-sided ideal $\mathcal{O}\boldsymbol{a}\mathcal{O}\boldsymbol{b}\mathcal{O}$ of $\mathcal{O}$. Since $\mathcal{O}$ is a maximal order [20, 27, 31] in $\mathbb{H}(K)$, we have a unique factorization

$$\mathcal{O}\boldsymbol{a}\mathcal{O}\boldsymbol{b}\mathcal{O} \;=\; \mathfrak{P}_1^{k_1}\cdot \ldots \cdot \mathfrak{P}_r^{k_r}\,, \quad k_1,\ldots,k_r \in \mathbb{N}_0\,, \tag{26}$$

where $\mathfrak{P}_1^{k_1},\ldots,\mathfrak{P}_r^{k_r}$ are prime ideals of $\mathcal{O}$ and $\mathfrak{P}_i \cap \mathcal{o} = \mathfrak{p}_i$ are in one-to-one correspondence with distinct prime ideals of $\mathcal{o}$ [27]. Similarly, we have $\mathcal{O}\boldsymbol{a}\mathcal{O} = \prod \mathfrak{P}_i^{m_i}$ and $\mathcal{O}\boldsymbol{b}\mathcal{O} = \prod \mathfrak{P}_i^{n_i}$, with $k_i = m_i + n_i$ for all $i \in \{1,\ldots,r\}$, $m_i \in \mathbb{N}_0$, $n_i \in \mathbb{Z}$.

Since $8\mathcal{O}\boldsymbol{b}\mathcal{O} \subset \mathcal{O}$, the only primes for which $n_i \leq 0$ is possible are those lying over $2 \in \mathcal{o}$. Recall that there is exactly one prime ideal of $\mathcal{O}$ that corresponds to 2, and this is $\mathfrak{P}_1 = \boldsymbol{x}\mathcal{O} = \mathcal{O}\boldsymbol{x}$, where $\boldsymbol{x} = (1+\boldsymbol{i})$ in the case $\mathcal{O} = \mathbb{J}$ and $\boldsymbol{x} = 2$ in the case $\mathcal{O} = \mathbb{I}$.

Thus we have $n_i \geq 0$ for all $i > 1$. If we can now show that $n_1 = 0$, then $\boldsymbol{b} \in \mathcal{O}$ and our assertion follows.

Suppose $n_1 < 0$. Then $m_1 \geq |n_1| > 0$, and we can write

$$\boldsymbol{a}\mathcal{O}\boldsymbol{b} \;=\; \boldsymbol{a}\boldsymbol{x}^{-|n_1|}\mathcal{O}\boldsymbol{x}^{|n_1|}\boldsymbol{b} \tag{27}$$

and $\mathcal{O}\boldsymbol{a}\boldsymbol{x}^{-|n_1|}\mathcal{O} \subset \mathfrak{P}_1^{m_1-|n_1|}\mathfrak{P}_2^{m_2}\cdot\ldots\cdot\mathfrak{P}_r^{m_r} \subset \mathcal{O}$. Similarly, $\mathcal{O}\boldsymbol{x}^{|n_1|}\boldsymbol{b}\mathcal{O} \subset \mathcal{O}$.

In the icosian case (where $\boldsymbol{x} = 2 \in \mathcal{o}$), the primitivity of $\boldsymbol{a}$ rules out that $\boldsymbol{a}\boldsymbol{x}^{-|n_1|} \in \mathcal{O}$, so $n_1 < 0$ is impossible here. In the Hurwitzian case, since $(1+\boldsymbol{i})^2 = 2$ (up to units), $n_1 = -1$ is still possible. Then, $\boldsymbol{a}\boldsymbol{x}^{-1}, \boldsymbol{x}\boldsymbol{b} \in \mathcal{O}$ and $\boldsymbol{a}\boldsymbol{x}^{-1}$ is still primitive. We take these as the new $\boldsymbol{a},\boldsymbol{b}$. This achieves the correct form, $\mathcal{M} = \boldsymbol{a}\mathcal{O}\boldsymbol{b}$, with $\boldsymbol{a},\boldsymbol{b} \in \mathcal{O}$ and $\boldsymbol{a}$ $\mathcal{O}$-primitive. When $\mathcal{O} = \mathbb{J}$ and $\boldsymbol{a}$ is even, we have $\boldsymbol{a} \in (1+\boldsymbol{i})\mathcal{O}\backslash 2\mathcal{O}$, and we may replace $\boldsymbol{a}\mathcal{O}\boldsymbol{b}$ by $\boldsymbol{a}\boldsymbol{x}^{-1}\mathcal{O}\boldsymbol{x}\boldsymbol{b}$. □

Remark 1: Given $\mathcal{M} = \boldsymbol{a}_1\mathcal{O}\boldsymbol{b}_1 \subset \mathcal{O}$ where $\boldsymbol{a}_1,\boldsymbol{b}_1 \in \mathbb{H}(K)$, the above argument shows that we can constructively adjust $\boldsymbol{a}_1,\boldsymbol{b}_1$ to $\boldsymbol{a} = \boldsymbol{a}_1 r^{-1}$, $\boldsymbol{b} = r\boldsymbol{b}_1$, where $r \in K$

for $\mathcal{O} = \mathbb{I}$ and $r \in K \cup K(1+\boldsymbol{i})$ for $\mathcal{O} = \mathbb{J}$, so as to have $\mathcal{M} = \boldsymbol{a}\mathcal{O}\boldsymbol{b}$ with $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{O}$ and $\boldsymbol{a}$ primitive (and odd for $\mathcal{O} = \mathbb{J}$). The argument also shows that once $\boldsymbol{a}$ is adjusted to be primitive (and odd if $\mathcal{O} = \mathbb{J}$) then $\boldsymbol{b}$ necessarily lies in $\mathcal{O}$.

Since we are ultimately interested in the similarity submodules, and not so much in the actual self-similarities themselves, we have to draw the attention now to the symmetries of our maximal orders. The group of units $\mathcal{O}^{\times}$ of $\mathcal{O}$ is, geometrically, a finite root system Δ of type[6] D_4 (resp. H_4) for $\mathbb{J}$ (resp. $\mathbb{I}$). Any self-similarity of $\mathcal{O}$ that is *surjective* is necessarily an isometry (norm preserving) and so must map Δ onto itself. Conversely, any isometry which stabilizes Δ will also stabilize its $\mathcal{o}$-span which is $\mathcal{O}$. Thus

$$\mathrm{stab}_{\mathrm{O}(4)}\mathcal{O} \; = \; \mathrm{Aut}(\Delta)\,. \tag{28}$$

For $\mathcal{O} = \mathbb{I}$, $\mathrm{Aut}(\Delta)$ is the Weyl group $W(\Delta)$ of Δ, which is H_4, and consequently all elements of $\mathrm{Aut}(\Delta)^{+}$ (the orientation preserving part of $\mathrm{Aut}(\Delta)$) are realized by mappings $M(\boldsymbol{u}, \boldsymbol{v})$ with $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{I}^{\times}$, i.e. units.

For $\mathcal{O} = \mathbb{J}$, $[\mathrm{Aut}(\Delta) : W(\Delta)] = 6$, the additional symmetry being due to the diagram automorphisms of D_4. In fact, $\mathrm{Aut}(\Delta)$ is the Weyl group of F_4, and the root system of type F_4 can be realized explicitly as

$$\tilde{\Delta} \; := \; \Delta \cup \{\pm\boldsymbol{u} \pm \boldsymbol{v} \mid \boldsymbol{u}, \boldsymbol{v} \in \{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}, \; \boldsymbol{u} \neq \boldsymbol{v}\}\,, \tag{29}$$

i.e. by adjoining to Δ the elements of $\mathbb{J}$ of square length 2. This time, $\mathrm{Aut}(\Delta)^{+}$ is realized as the set of mappings $M(\boldsymbol{u}, \boldsymbol{v})$ with either $\boldsymbol{u}, \boldsymbol{v} \in \Delta$ or $\boldsymbol{u}, \boldsymbol{v} \in (\tilde{\Delta}\backslash\Delta)/\sqrt{2}$. It will be observed that all the elements of $\tilde{\Delta}\backslash\Delta$ lie in the ideal $(1+\boldsymbol{i})\mathbb{J}$ (in fact they are all the generators of this ideal). So, we have proved

Proposition 2 *The orientation preserving self-similarities of $\mathcal{O}$ onto itself are precisely the maps $M(\boldsymbol{u}, \boldsymbol{v})$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{O}$ with $|\boldsymbol{u}| = |\boldsymbol{v}| = 1$ and, in the case $\mathcal{O} = \mathbb{J}$, also the maps $\frac{1}{2}M(\boldsymbol{u}, \boldsymbol{v})$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{O}$ with $|\boldsymbol{u}|^2 = |\boldsymbol{v}|^2 = 2$.* □

We say that an SSM $\boldsymbol{a}\mathcal{O}\boldsymbol{b}$ is given in *canonical form* if $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{O}$ with $\boldsymbol{a}$ being $\mathcal{O}$-primitive (and with $\boldsymbol{a}$ odd if $\mathcal{O} = \mathbb{J}$).

Proposition 3 *Similarity submodules $\boldsymbol{a}_1\mathcal{O}\boldsymbol{b}_1$ and $\boldsymbol{a}_2\mathcal{O}\boldsymbol{b}_2$ written in canonical form are equal if and only if both $\boldsymbol{a}_1^{-1}\boldsymbol{a}_2$ and $\boldsymbol{b}_2\boldsymbol{b}_1^{-1}$ are units in $\mathcal{O}$.*

[6] We are using the symbol D_4 both for the root system and for the corresponding root lattice. The convex hull of the 24 roots of D_4 is the regular 24-cell [12, p. 292], mentioned in an earlier footnote. RVM would like to take this opportunity to note that in [8] the root system D_4 was inexplicably left out of the classification of subroot systems of the root system of type H_4. All such D_4 subroot systems are conjugate by the Weyl group of H_4.

PROOF: Suppose that $\boldsymbol{a}_1\mathcal{O}\boldsymbol{b}_1 = \boldsymbol{a}_2\mathcal{O}\boldsymbol{b}_2$. Then $\boldsymbol{a}_1^{-1}\boldsymbol{a}_2\mathcal{O}\boldsymbol{b}_2\boldsymbol{b}_1^{-1} = \mathcal{O}$. According to Prop. 2, one of two things may happen:
(i) There is an $r \in K$ so that $r\boldsymbol{a}_1^{-1}\boldsymbol{a}_2 = \boldsymbol{u} \in \mathcal{O}^\times$ (and $r^{-1}\boldsymbol{b}_2\boldsymbol{b}_1^{-1} = \boldsymbol{v} \in \mathcal{O}^\times$). Then, $r\boldsymbol{a}_2 = \boldsymbol{a}_1\boldsymbol{u} \in \mathcal{O}$ gives $r \in \mathcal{o}$ since $\boldsymbol{a}_2$ is primitive. Likewise, $r^{-1}\boldsymbol{a}_1 = \boldsymbol{a}_2\boldsymbol{u}^{-1} \in \mathcal{O}$ gives $r^{-1} \in \mathcal{o}$, so $r \in \mathcal{o}^\times \subset \mathcal{O}^\times$ and we are done in this case.
(ii) We are in the case $\mathcal{O} = \mathbb{J}$ and there is an $r \in K$ so that $2r\boldsymbol{a}_1^{-1}\boldsymbol{a}_2 = (1+\boldsymbol{i})\boldsymbol{u}$, $\boldsymbol{u} \in \mathcal{O}^\times$. This gives $2r\boldsymbol{a}_2 \in \mathcal{O}$ and $r^{-1}\boldsymbol{a}_1 = \boldsymbol{a}_2\boldsymbol{u}^{-1}(1-\boldsymbol{i}) \in \mathcal{O}$ whence $2r, r^{-1} \in \mathbb{Z}$. Thus $r = \pm 1, \pm\frac{1}{2}$. Since $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are both odd and $2 \in (1+\boldsymbol{i})^2\mathcal{O}^\times$, none of these values of r is possible.
The reverse direction is clear. $\square$

We are now in the position to formulate the main result of this section.

Theorem 1 *The number of similarity submodules of $\mathcal{O}$ of a given index is a multiplicative arithmetic function. Its Dirichlet series generating function is given by*

$$F_{\mathcal{O}}(s) \;=\; \frac{(\zeta_{\mathcal{O}}(s))^2}{\zeta_K(4s)} \cdot \begin{cases} \frac{1}{1+4^{-s}}, & \text{if } \mathcal{O} = \mathbb{J} \\ 1, & \text{if } \mathcal{O} = \mathbb{I} \end{cases} . \tag{30}$$

PROOF: As a result of Prop. 2 and Prop. 3, any SSM $\mathcal{M}$ of $\mathcal{O}$ can be uniquely written as

$$\mathcal{M} \;=\; \boldsymbol{a}\mathcal{O}\mathcal{O}\boldsymbol{b}\,, \tag{31}$$

i.e. as a product of a right and a left ideal of $\mathcal{O}$, where $\boldsymbol{a}$ is $\mathcal{O}$-primitive (and also odd if $\mathcal{O} = \mathbb{J}$).

Any right ideal $\boldsymbol{d}\mathcal{O}$ can be written *uniquely* as a product $c(\boldsymbol{d})\mathcal{o}\boldsymbol{a}\mathcal{O}$, where $c(\boldsymbol{d})$ (the content of $\boldsymbol{d}$) is in $\mathcal{o}$ and $\boldsymbol{a} \in \mathcal{O}$ is $\mathcal{O}$-primitive. In addition, we have the formula $[\mathcal{O} : \boldsymbol{d}\mathcal{O}] = [\mathcal{o} : c(\boldsymbol{d})\mathcal{o}]^4 \cdot [\mathcal{O} : \boldsymbol{a}\mathcal{O}]$. Thus the Dirichlet series for the *primitive* right ideals of $\mathcal{O}$ (those $\boldsymbol{a}\mathcal{O}$ with $\boldsymbol{a}$ primitive) is the quotient of two zeta functions,

$$\zeta_{\mathcal{O}}(s)/\zeta_K(4s)\,. \tag{32}$$

In the case $\mathcal{O} = \mathbb{I}$, the factorization (31) leads at once to $F_{\mathbb{I}}(s) = (\zeta_{\mathbb{I}}(s))^2/\zeta_{\mathbb{Q}(\tau)}(4s)$.

In the case $\mathcal{O} = \mathbb{J}$, writing (32) explicitly as an Euler product, we find that the contribution for the prime 2 is $(1+4^{-s})$. This corresponds to the fact that the primitive right ideals are either of the form $\boldsymbol{a}\mathbb{J}$ with $\boldsymbol{a}$ odd or of the form $\boldsymbol{a}(1+\boldsymbol{i})\mathbb{J}$ with $\boldsymbol{a}$ odd. The latter ones are to be removed from the counting (because $(1+\boldsymbol{i})\mathbb{J} = \mathbb{J}(1+\boldsymbol{i})$ otherwise leads to doubly counting them), and the corresponding Dirichlet series is obtained by removing the factor $(1+4^{-s})$. This gives the result claimed. $\square$

Results: lattices

Let us now consider the Hurwitzian case $\mathcal{O} = \mathbb{J}$ in detail, and also the lattice $\mathcal{L} = \mathbb{Z}^4$. If $\boldsymbol{x} = (1 + \boldsymbol{i})$, we have $\boldsymbol{x}\mathcal{O} = \mathcal{O}\boldsymbol{x}$ and also $\boldsymbol{x}\mathcal{L} = \mathcal{L}\boldsymbol{x}$. These lattices are related by

$$\boldsymbol{x}\mathcal{O} \overset{2}{\subset} \mathcal{L} \overset{2}{\subset} \mathcal{O} \tag{33}$$

where the integer on top of the inclusion symbol is the corresponding index.

Lemma 2 *If $\boldsymbol{a}\mathcal{L}\boldsymbol{b} \subset \mathcal{L}$, then there exist $\boldsymbol{a}_1, \boldsymbol{b}_1 \in \mathcal{O}$, with $\boldsymbol{a}_1$ odd and $\mathcal{O}$-primitive, such that $\boldsymbol{a}\mathcal{L}\boldsymbol{b} = \boldsymbol{a}_1\mathcal{L}\boldsymbol{b}_1$.*

Proof: Let $\boldsymbol{a}\mathcal{L}\boldsymbol{b} \subset \mathcal{L}$. We can assume, without loss of generality, that $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{H}(\mathbb{Q})$ and, by using a suitable scaling and the fact that $\boldsymbol{x}\mathcal{L} = \mathcal{L}\boldsymbol{x}$, we may even assume that $\boldsymbol{a} \in \mathcal{O}$ and that $\boldsymbol{a}$ is $\mathcal{O}$-primitive and odd. Since $\boldsymbol{a}\mathcal{O}\boldsymbol{x}\boldsymbol{b} \subset \boldsymbol{a}\mathcal{L}\boldsymbol{b} \subset \mathcal{O}$, the conditions on $\boldsymbol{a}$ already show that $\boldsymbol{x}\boldsymbol{b} \in \mathcal{O}$ (see Remark 1). Write $\boldsymbol{b} = \boldsymbol{x}^{-1}\boldsymbol{c}$ with $\boldsymbol{c} \in \mathcal{O}$.

Consider $[\mathcal{L} : \boldsymbol{a}\mathcal{L}\boldsymbol{b}] = |\boldsymbol{a}|^4\,|\boldsymbol{b}|^4 = \frac{|\boldsymbol{a}|^4\,|\boldsymbol{c}|^4}{4} \in \mathbb{Z}$. Since $\boldsymbol{a}$ is odd, it follows that $4 \mid |\boldsymbol{c}|^2$ and hence $\boldsymbol{x} \mid \boldsymbol{c}$, because any even element in $\mathbb{J}$ is of the form $(1 + \boldsymbol{i})^r$ times an odd element [20]. Consequently, $\boldsymbol{b} \in \mathcal{O}$ and we conclude that $\boldsymbol{a}\mathcal{L}\boldsymbol{b} \subset \mathcal{L}$ implies that we can rearrange the quaternions in the way claimed. □

This provides a link between the SSL problems for $\mathcal{L}$ and for $\mathcal{O}$. The difference between the counting arises as follows. The symmetry group of $\mathcal{O}$ is isomorphic with the Weyl group of F_4 (see above), while that of $\mathcal{L}$, which is the Weyl group of B_4, is a subgroup of index 3. As we shall show, this only influences the number of SSLs of even index when going from $\mathcal{O}$ to $\mathcal{L}$.

Theorem 2 *The possible indices of similarity sublattices of hypercubic lattices in 4D are precisely the squares of rational integers. The number of SSLs of given index is a multiplicative arithmetic function. For the case of $\mathbb{J} = D_4^*$, the corresponding Dirichlet series generating function $F_{\mathbb{J}}$ reads*

$$F_{\mathbb{J}}(s) \;=\; \frac{(\zeta_{\mathbb{J}}(s))^2}{(1+4^{-s})\,\zeta(4s)} \;=\; \frac{(1-2^{1-2s})^2}{1+4^{-s}} \cdot \frac{(\zeta(2s)\zeta(2s-1))^2}{\zeta(4s)}\,. \tag{34}$$

The same series also applies to the lattice D_4, while for the primitive hypercubic lattice, $\mathbb{Z}^4$, it reads

$$F_{\mathbb{Z}^4}(s) \;=\; (1 + \frac{2}{4^s}) \cdot F_{\mathbb{J}}(s)\,. \tag{35}$$

Proof: The statement about $F_{\mathbb{J}}(s)$ follows directly from Theorem 1 and from Eq. (8). It is rather easy to see from the Euler product representation that precisely all squares of integers occur as indices.

In order to extend this to $\mathcal{L}$, we have to understand how the different symmetries lead to different countings. Assume $\boldsymbol{a}\mathcal{L}\boldsymbol{b} \subset \mathcal{L}$. Due to Lemma 2, we may assume that $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{O}$ with $\boldsymbol{a}$ odd and $\mathcal{O}$-primitive, i.e. we assume canonical form.

By Prop. 3, $\boldsymbol{a}\mathcal{O}\boldsymbol{b} = \boldsymbol{a}_1\mathcal{O}\boldsymbol{b}_1$ if and only if there are units $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{O}^\times$ with $\boldsymbol{a}_1 = \boldsymbol{a}\boldsymbol{u}$ and $\boldsymbol{b}_1 = \boldsymbol{v}\boldsymbol{b}$. However, the unit group of $\mathcal{L}$ is only the quaternion group, $Q = \{\pm 1, \pm \boldsymbol{i}, \pm \boldsymbol{j}, \pm \boldsymbol{k}\}$, and Q is a normal subgroup of $\mathcal{O}^\times$ with $\mathcal{O}^\times / Q \simeq \mathbb{Z}/3\mathbb{Z}$. We may take $\boldsymbol{t} = (1,1,1,1)/2$ as a suitable representative of a generator of this cyclic group in $\mathcal{O}^\times$ [16, § 26]. Note that, for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{O}^\times$, we have $\boldsymbol{a}\boldsymbol{u}\mathcal{L}\boldsymbol{v}\boldsymbol{b} = \boldsymbol{a}\mathcal{L}\boldsymbol{b}$ if and only if $\boldsymbol{u}\boldsymbol{v} \in Q$. So the single SSL $\boldsymbol{a}\mathcal{O}\boldsymbol{b}$ of $\mathcal{O}$ *may* give rise to three different SSLs of $\mathcal{L}$, namely to $\boldsymbol{a}\mathcal{L}\boldsymbol{b}$, $\boldsymbol{a}\mathcal{L}\boldsymbol{t}\boldsymbol{b}$, $\boldsymbol{a}\mathcal{L}\boldsymbol{t}^2\boldsymbol{b}$. Whether or not this happens depends on whether or not the latter two are actually in $\mathcal{L}$.

Now, if $\boldsymbol{a}\mathcal{L}\boldsymbol{b}$ and $\boldsymbol{a}\mathcal{L}\boldsymbol{t}\boldsymbol{b}$ are both in $\mathcal{L}$, then so is $\boldsymbol{a}(\mathcal{L} + \mathcal{L}\boldsymbol{t})\boldsymbol{b} = \boldsymbol{a}\mathcal{O}\boldsymbol{b}$, whence also $\boldsymbol{a}\mathcal{L}\boldsymbol{t}^2\boldsymbol{b} \subset \mathcal{L}$. Similarly, $\boldsymbol{a}\mathcal{L}\boldsymbol{t}^2\boldsymbol{b} \subset \mathcal{L}$ implies $\boldsymbol{a}\mathcal{L}\boldsymbol{t}\boldsymbol{b} \subset \mathcal{L}$, too. Thus $\boldsymbol{a}\mathcal{O}\boldsymbol{b} \subset \mathcal{O}$ gives rise to 3 different SSLs of $\mathcal{L}$ if and only if $\boldsymbol{a}\mathcal{O}\boldsymbol{b} \subset \mathcal{L}$.

However, $\boldsymbol{a}\mathcal{O}\boldsymbol{b} \subset \mathcal{L}$ implies that $[\mathcal{O} : \boldsymbol{a}\mathcal{O}\boldsymbol{b}]$ is even because $[\mathcal{O} : \mathcal{L}] = 2$. Conversely, $[\mathcal{O} : \boldsymbol{a}\mathcal{O}\boldsymbol{b}]$ even implies that $|\boldsymbol{a}|^4\,|\boldsymbol{b}|^4$ is divisible by 4, so $\boldsymbol{a}$ or $\boldsymbol{b}$ must be even and hence $\boldsymbol{a}\mathcal{O}\boldsymbol{b} \subset \boldsymbol{x}\mathcal{O} \subset \mathcal{L}$. In short, the 3 SSL situation occurs if and only if $[\mathcal{O} : \boldsymbol{a}\mathcal{O}\boldsymbol{b}]$ is even. As a consequence, the counting function for $\mathcal{L}$ is still multiplicative, and the modification in the Euler product expansion occurs only in the factor that belongs to the prime 2. It is easy to check that the result is that given in the Theorem. □

If we take into account that the possible indices are always squares, it is reasonable to define the appropriate coefficients as follows,

$$F_{\mathcal{L}}(s) \;=\; \sum_{m=1}^{\infty} \frac{f_{\mathcal{L}}(m)}{m^{2s}}\,. \tag{36}$$

So, the coefficients actually are

$$f_{\mathcal{L}}(m) \;=\; \left|\{\mathcal{L}' \text{ is SSL of } \mathcal{L} \mid [\mathcal{L} : \mathcal{L}'] = m^2\}\right|. \tag{37}$$

To simplify explicit formulas here and later on, we introduce the function

$$g(n,r) \;:=\; (r+1)n^r + 2\frac{1-(r+1)n^r + r\,n^{r+1}}{(n-1)^2} \tag{38}$$

for integers $r \geq 0$ and $n > 1$. Note that $g(n,0) = 1$. An explicit expansion of the Euler factors now gives the following result.

Corollary 1 *The arithmetic function $f_{\mathbb{J}}(m)$ is multiplicative. It is given by*

$$f_{\mathbb{J}}(p^r) \;=\; \begin{cases} 1, & \textit{if } p = 2 \\ g(p,r), & \textit{if } p \textit{ is an odd prime} \end{cases} \tag{39}$$

where $r \geq 0$ in all cases. Similarly, $f_{\mathbb{Z}^4}(m)$ is a multiplicative arithmetic function. It is related to $f_{\mathbb{J}}(m)$ via

$$f_{\mathbb{Z}^4}(m) \;=\; \begin{cases} f_{\mathbb{J}}(m), & m \text{ odd} \\ 3 \cdot f_{\mathbb{J}}(m), & m \text{ even.} \end{cases} \tag{40}$$

The first few terms of $F_{\mathbb{J}}(s)$ read explicitly

$$F_{\mathbb{J}}(s) \;=\; 1 + \tfrac{1}{4^s} + \tfrac{8}{9^s} + \tfrac{1}{16^s} + \tfrac{12}{25^s} + \tfrac{8}{36^s} + \tfrac{16}{49^s} + \tfrac{1}{64^s} + \tfrac{41}{81^s} + \tfrac{12}{100^s} + \tfrac{24}{121^s} + \tfrac{8}{144^s} + \cdots \tag{41}$$

while those for $F_{\mathbb{Z}^4}(s)$ follow easily from (40). They are now listed as [29, sequence A 045771] and [29, sequence A 035292], respectively. Note that Eq. (39) implies that the SSMs of $\mathbb{J}$ of index 4^r are unique – they are, in fact, just the 2-sided ideals $(1+\boldsymbol{i})^r\mathbb{J}$.

Let us now, in line with the previous examples, briefly consider the asymptotic behaviour of the coefficients. Since $\zeta(s) \neq 0$ in $\{\mathrm{Re}(s) \geq 1\}$, it is clear that $F_{\mathcal{L}}(s)$ is meromorphic in the same half-plane, with only one pole which is of second order and located at $s = 1$. This is true both of $\mathcal{L} = \mathbb{J}$ and $\mathcal{L} = \mathbb{Z}^4$. Using the Dirichlet series (36) and applying the results from the Appendix to $F_{\mathcal{L}}(s/2)$, we get the following

Corollary 2 *The coefficients $f_{\mathcal{L}}(m)$ grow faster than linear on average for large m, and we have the asymptotic behaviour*

$$\sum_{m \leq x} f_{\mathcal{L}}(m) \;\simeq\; C_{\mathcal{L}} \cdot x^2 \log(x) \qquad \textit{(as } x \to \infty\textit{)} \tag{42}$$

where the constant is given by

$$C_{\mathcal{L}} \;=\; \frac{1}{2}\,\mathrm{res}_{s=1}\left((s-1)F_{\mathcal{L}}(s)\right) \;=\; \frac{1}{4} \cdot \begin{cases} 1, & \text{if } \mathcal{L} = \mathbb{J} \\ \frac{3}{2}, & \text{if } \mathcal{L} = \mathbb{Z}^4 \end{cases} . \tag{43}$$

Note that this really is an asymptotic result, and that, numerically, the estimates of $C_{\mathcal{L}}$ converge rather slowly (from above) to the values given in the Corollary.

Results: modules

Let us first state the result for the icosian ring $\mathbb{I}$ itself.

Theorem 3 *The possible indices of similarity submodules of the icosian ring $\mathbb{I}$ are the squares of rational integers that can be represented by the quadratic form $x^2 + xy - y^2$. The number of SSMs of given index is a multiplicative arithmetic function, its Dirichlet series generating function reads*

$$F_{\mathbb{I}}(s) \;=\; \frac{(\zeta_{\mathbb{I}}(s))^2}{\zeta_K(4s)} \;=\; \frac{(\zeta_K(2s)\zeta_K(2s-1))^2}{\zeta_K(4s)} \tag{44}$$

where $K = \mathbb{Q}(\tau)$.

The proof follows immediately from Theorem 1 in combination with Eq. (12). Note that the possible indices are just the squares of the possible norms of ideals in $\mathbb{Q}(\tau)$ and hence of the form given. Taking this into account, we write

$$F_{\mathbb{I}}(s) \;=\; \sum_{m=1}^{\infty} \frac{f_{\mathbb{I}}(m)}{m^{2s}} \tag{45}$$

in analogy to above, and obtain, by an explicit expansion of the Euler factors, the following result (compare [29, sequence A 035284]).

Corollary 3 *The arithmetic function $f_{\mathbb{I}}(m)$ is multiplicative and given by*

$$f_{\mathbb{I}}(p^r) \;=\; \begin{cases} g(5,r), & \text{if } p=5 \\ 0, & \text{if } p \equiv \pm 2 \bmod 5 \text{ and } r \text{ is odd} \\ g(p^2,\ell), & \text{if } p \equiv \pm 2 \bmod 5 \text{ and } r = 2\ell \\ \sum_{s=0}^{r} g(p,s)g(p,r-s), & \text{if } p \equiv \pm 1 \bmod 5 \end{cases} \tag{46}$$

where always $r \geq 0$ and g is the function defined in Eq. (38).

The first few terms of $F_{\mathbb{I}}(s)$ read explicitly

$$F_{\mathbb{I}}(s) \;=\; 1 + \tfrac{10}{16^s} + \tfrac{12}{25^s} + \tfrac{20}{81^s} + \tfrac{48}{121^s} + \tfrac{66}{256^s} + \tfrac{80}{361^s} + \tfrac{120}{400^s} + \tfrac{97}{625^s} + \tfrac{120}{841^s} + \tfrac{128}{961^s} + \cdots \tag{47}$$

Let us briefly look at the 10 SSMs of $\mathbb{I}$ of index 16. From Eq. (13) it is obvious that they are just the 5 left ideals $\mathbb{I}\boldsymbol{a}$ and the 5 right ideals $\boldsymbol{a}\mathbb{I}$, with suitable generators $\boldsymbol{a}$ with $\mathrm{N}(|\boldsymbol{a}|^2) = 16$. Note that none of them is 2-sided.

Finally, we can again determine the asymptotic behaviour along the lines used before. $F_{\mathbb{I}}(s/2)$ is holomorphic in the half-plane $\{\mathrm{Re}(s) \geq 2\}$, with a single second-order pole at $s = 2$. With the results from the Appendix, we then obtain

Corollary 4 *For $x \to \infty$, the asymptotic behaviour of the coefficients $f_{\mathbb{I}}(m)$ is*

$$\sum_{m \leq x} f_{\mathbb{I}}(m) \;\sim\; \frac{6(\log(\tau))^2}{5\sqrt{5}}\, x^2 \log(x) \;\simeq\; 0.124271\, x^2 \log(x)\,. \tag{48}$$

At this point, it would be interesting to relate these findings to the corresponding ones for the $\mathbb{Z}[\tau]$-modules $\boldsymbol{\mathcal{L}} = \mathbb{Z}[\tau]^4$ and $\boldsymbol{\mathcal{M}} = \mathbb{J}[\tau] = \mathbb{I} \cap \mathbb{I}'$. Since this requires a lot more effort than in the previous case ($\mathbb{Z}^4$ versus $\mathbb{J}$), we postpone it, and rather state the result for the cubian[7] maximal order $\mathbb{K}$ in $\mathbb{H}(\mathbb{Q}(\sqrt{2}))$.

[7] The term “octonian” would be more natural a choice, but it has already been taken!

Theorem 4 *The possible indices of similarity submodules of the cubian ring $\mathbb{K}$ are the squares of rational integers that can be represented by the quadratic form x^2-2y^2. The number of SSMs of given index is a multiplicative arithmetic function, its Dirichlet series generating function reads*

$$
\begin{aligned}
F_{\mathbb{K}}(s) &= \frac{(\zeta_{\mathbb{K}}(s))^2}{\zeta_K(4s)} = \frac{(\zeta_K(2s)\zeta_K(2s-1))^2}{\zeta_K(4s)} \qquad (49)\\
&= 1+\tfrac{6}{4^s}+\tfrac{22}{16^s}+\tfrac{32}{49^s}+\tfrac{66}{64^s}+\tfrac{20}{81^s}+\tfrac{192}{196^s}+\tfrac{178}{256^s}+\tfrac{72}{289^s}+\tfrac{120}{324^s}+\tfrac{96}{529^s}+\tfrac{52}{625^s}+\cdots
\end{aligned}
$$

where $K=\mathbb{Q}(\sqrt{2})$.

The proof follows directly from the proof of Theorem 1, since literally every step taken for the icosian ring translates into one here, with $K=\mathbb{Q}(\sqrt{2})$, $\mathcal{o}=\mathbb{Z}[\sqrt{2}]$, $\mathcal{O}=\mathbb{K}$, and $\mathcal{L}=\mathbb{Z}[\sqrt{2}]^4$. Note that now, since 2 is not a prime in $\mathbb{Z}[\sqrt{2}]$ (it actually ramifies there), $\mathfrak{P}_1=\sqrt{2}\,\mathbb{K}=(2+\sqrt{2})\mathbb{K}$ is the prime ideal of $\mathbb{K}$ sitting on top of the prime ideal $(2+\sqrt{2})\mathbb{Z}[\sqrt{2}]$, and the arguments in the proofs have to be adjusted accordingly.

One can again work out the coefficients $f_{\mathbb{K}}(m)$ explicitly (see [29, sequence A 035285])

$$
f_{\mathbb{K}}(p^r) = \begin{cases} g(2,r), & \text{if } p=2\\ 0, & \text{if } p\equiv\pm3\ (8) \text{ and } r \text{ is odd}\\ g(p^2,\ell), & \text{if } p\equiv\pm3\ (8) \text{ and } r=2\ell\\ \sum_{s=0}^{r} g(p,s)g(p,r-s), & \text{if } p\equiv\pm1\ (8)\end{cases} \qquad (50)
$$

and the asymptotic behaviour, using the Appendix, is

$$
\sum_{m\le x} f_{\mathbb{K}}(m) \;\sim\; \frac{15(\log(1+\sqrt{2}))^2}{22\sqrt{2}}\, x^2\log(x) \;\simeq\; 0.374519\, x^2\log(x)\,. \qquad (51)
$$

Concluding remarks

As we have demonstrated above, the similarity submodules of certain 4D $\mathbb{Z}$-modules can be classified by means of algebraic methods based on quaternionic algebras and their maximal orders. Together with the results of [5, 6], this essentially covers the cases related to root systems in dimensions $d\le 4$.

Although we did not emphasize it, one can also determine the actual semigroups of self-similarities of these modules explicitly, notably through the canonical representation of SSMs (Prop. 2) and their uniqueness up to symmetries (Prop. 3). We have described this in more detail for other cases [5], and the interested reader will find no difficulty to extend that approach to this situation.

One application is concerned with the symmetries of coloured versions of the lattices and modules under consideration. Assume that L has a non-trivial (irreducible) point symmetry, and a sublattice L' which is the image of a self-similarity of L of index $m = [L : L'] > 1$. If we assign m different colours to the cosets of L', certain subgroups of the point group of L' (which is conjugate to that of L) will give rise to a *colour symmetry* in the sense that their elements induce a unique, global permutation of the colours, compare [33, 25] and references therein.

This is also closely related to the classification of coincidence site submodules, i.e. of submodules that can be written as the intersection of the original module with a rotated copy of itself, see [2] for background and some recent results. Here are several open questions, particularly in spaces of even dimension, which the above results should help to solve for dimension four.

Finally, one would like to know to what extent a generalization of our results is possible. The root lattices seem to form a sufficiently well-behaved class of objects to try, and some partial answers on the existence of similarity sublattices and their possible indices are given in [9]. We are, however, not aware of general results along the lines discussed here, i.e. including the determination of the number of SSLs of a given index, nor even of a method to overcome the dependence on special features such as the arithmetic of quaternions.

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Appendix

In what follows, we briefly summarize the results from analytic number theory that we need to determine certain asymptotic properties of the coefficients of Dirichlet series generating functions. For the general background, we refer to [1] and [36].

Consider a Dirichlet series of the form $F(s) = \sum_{m=1}^{\infty} a(m) m^{-s}$. We are mainly interested in the quantity $A(x) = \sum_{m \le x} a(m)$ and its behaviour for large x. Let us give one classical result (based upon Tauberian theorems) for the case that $a(m)$ is real and non-negative.

Theorem 5 *Let $F(s)$ be a Dirichlet series with non-negative coefficients which converges for $\mathrm{Re}(s) > \alpha > 0$. Suppose that $F(s)$ is holomorphic at all points of the line $\{\mathrm{Re}(s) = \alpha\}$ except at $s = \alpha$. Here, when approaching α from the half-plane right of it, we assume $F(s)$ to have a singularity of the form $F(s) = g(s) + h(s)/(s-\alpha)^{n+1}$ where n is a non-negative integer, and both $g(s)$ and $h(s)$ are holomorphic at $s = \alpha$. Then we have, as $x \to \infty$,*

$$A(x) \;:=\; \sum_{m \le x} a(m) \;\sim\; \frac{h(\alpha)}{\alpha \cdot n!}\, x^{\alpha} \left(\log(x)\right)^{n}. \tag{52}$$

The proof follows easily from Delange's theorem, e.g. by taking $q = 0$ and $\omega = n$ in Tenenbaum's formulation of it, see [34, ch. II.7, Thm. 15] and references given there.

Note that Delange's theorem is a lot more general in that is still gives results for other local behaviour of $F(s)$ in the neighbourhood of $s = \alpha$, in particular for n not an integer and even for combinations with logarithmic singularities. Let us also point out that there are various extensions to Dirichlet series with complex coefficients, e.g. Thm. 1 on p. 311 of [24], and even stronger results (with good error estimates) for multiplicative arithmetic functions $a(m)$ with values in the unit disc, see [34, ch. I, § 3.8 and ch. III, § 4.3] for details.

The critical assumption in Theorem 5 is the behaviour of $F(s)$ along the entire line $\{\mathrm{Re}(s) = \alpha\}$. In all cases that appear in this article, this can be checked explicitly. To do so, we have to know a few properties of the Riemann zeta function, $\zeta(s)$, and of the Dedekind zeta functions of $\mathbb{Q}(\tau)$ and $\mathbb{Q}(\sqrt{2})$. It is well known that $\zeta(s)$ is a meromorphic function in the complex plane, and that is has a sole simple pole at $s = 1$ with residue 1 [1, Thm. 12.5(a)]. It has no zeros in the half-plane $\{\mathrm{Re}(s) \ge 1\}$ [34, ch. II.3, Thm. 9]. The values of $\zeta(s)$ at positive even integers are known [1, Thm. 12.17] and we have

$$\zeta(2) \;=\; \frac{\pi^2}{6} \quad , \quad \zeta(4) \;=\; \frac{\pi^4}{90}. \tag{53}$$

This is all we need to know for this case.

The Dedekind zeta function of $K = \mathbb{Q}(\tau)$ has some similarly nice properties. It follows from [35, Thm. 4.3] or from [36, §11, Eq. (10)] that it can be written as

$$\zeta_{\mathbb{Q}(\tau)}(s) \;=\; \zeta(s) \cdot L(s,\chi) \tag{54}$$

where $L(s,\chi)$ is the L-series of the primitive Dirichlet character [1, Ch. 6.8] χ defined by

$$\chi(n) \;=\; \begin{cases} 0, & n \equiv 0 \ (5) \\ 1, & n \equiv \pm 1 \ (5) \\ -1, & n \equiv \pm 2 \ (5) \end{cases} \;. \tag{55}$$

Since χ is not the principal character, $L(s,\chi) = \sum_{m=1}^{\infty} \chi(m)\, m^{-s}$ is an entire function [1, Thm. 12.5]. Consequently, $\zeta_{\mathbb{Q}(\tau)}(s)$ is meromorphic, and its only pole is simple and located at $s = 1$. The residue is $L(1,\chi)$ and from [35, Thm. 4.9] we get

$$\mathrm{res}_{s=1}\, \zeta_{\mathbb{Q}(\tau)}(s) \;=\; L(1,\chi) \;=\; \frac{2\log(\tau)}{\sqrt{5}} \;\simeq\; 0.430409\,. \tag{56}$$

Since $L(s,\chi) \neq 0$ for $\mathrm{Re}(s) > 1$, see [35, p. 31], $\zeta_{\mathbb{Q}(\tau)}(s)$ cannot vanish there either. Also, one can again calculate the values of $\zeta_{\mathbb{Q}(\tau)}(s)$ at positive even integers. This is done by means of the functional equation of $L(s,\chi)$ [35, p. 30] and the knowledge of the values of L-functions at negative integers in terms of generalized Bernoulli numbers [35, Thm. 4.2], see [35, Prop. 4.1] for a formula for them. Working this out explictly for $s = 2$ and $s = 4$ gives

$$\zeta_{\mathbb{Q}(\tau)}(2) \;=\; \frac{2\pi^4}{75\sqrt{5}} \quad , \quad \zeta_{\mathbb{Q}(\tau)}(4) \;=\; \frac{4\pi^8}{16875\sqrt{5}}\,. \tag{57}$$

Let us add that these results, and those to follow, can also be found, in rather explicit form, in §9 and §11 of [36].

In the same way, one can determine the zeta function of $\mathbb{Q}(\sqrt{2})$ and its properties. One has $\zeta_{\mathbb{Q}(\sqrt{2})}(s) = \zeta(s) L(s,\chi)$, now with the primitive Dirichlet character $\chi(n) = 0, 1, -1$ for n even, $n \equiv \pm 1$ (8), $n \equiv \pm 3$ (8), respectively. This zeta function has again only one simple pole, at $s = 1$, with residue $L(1,\chi) = \log(1+\sqrt{2})/\sqrt{2} \simeq 0.623225$. Finally, we have

$$\zeta_{\mathbb{Q}(\sqrt{2})}(2) \;=\; \frac{\pi^4}{48\sqrt{2}} \quad , \quad \zeta_{\mathbb{Q}(\sqrt{2})}(4) \;=\; \frac{11\pi^8}{69120\sqrt{2}}\,. \tag{58}$$

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