Self-similarities and invariant densities for model sets

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ABSTRACT Model sets (also called cut and project sets) are generalizations of lattices. Here we show how the self-similarities of model sets are a natural replacement for the group of translations of a lattice. This leads us to the concept of averaging operators and invariant densities on model sets. We prove that invariant densities exist and that they produce absolutely continuous invariant measures in internal space. We study the invariant densities and their relationships to diffraction, continuous refinement operators, and Hutchinson measures.

1 Model sets and self-similarities

In this paper we introduce the notion of averaging operators on suitable spaces of functions on model sets. An averaging operator encodes information about the entire set of self-similarities with given inflation factor for a given model set. It can be interpreted as a Hilbert-Schmidt operator on the space of continuous functions on the corresponding acceptance window and, remarkably, from this point of view is seen to be an example of the recently studied continuous refinement operators. Using this connection we can determine the spectrum and associated set of eigenfunctions for any inflation factor of any given model set. In particular, the leading eigenvalue 1 gives rise to an *invariant density* for the model set. We derive some properties of the Bragg spectrum of a model set that has been weighted by an invariant density. We also show that an invariant density leads to an absolutely continuous invariant measure on internal space and we relate this measure to a weakly converging sequence of Hutchinson measures. The full mathematical development of this work will appear in [2].

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1.1 Cut and project schemes

We begin with the notion of a cut and project scheme. By definition, this consists of a collection of spaces and mappings:

$$\begin{array}{ccccc} \mathbb{R}^m & \xleftarrow{\pi_1} & \mathbb{R}^m \times \mathbb{R}^n & \xrightarrow{\pi_2} & \mathbb{R}^n \\ & & \cup & & \\ & & \tilde{L} & & \end{array} \tag{1.1}$$

where $\mathbb{R}^m$ and $\mathbb{R}^n$ are two real spaces, π_1 and π_2 are the projection maps onto them, and $\tilde{L} \subset \mathbb{R}^m \times \mathbb{R}^n$ is a lattice. We assume that $\pi_1|_{\tilde{L}}$ is injective and that $\pi_2(\tilde{L})$ is dense in $\mathbb{R}^n$. We call $\mathbb{R}^m$ (resp. $\mathbb{R}^n$) the physical (resp. internal) space. We will assume that $\mathbb{R}^m$ and $\mathbb{R}^n$ are equipped with Euclidean metrics and that $\mathbb{R}^m \times \mathbb{R}^n$ is the orthogonal sum of the two spaces. For x lying in any of these spaces, $|x|$ denotes its length.

A cut and project scheme involves, then, the projection of a lattice into a space of smaller dimension, but a lattice that is transversally located with respect to the projection maps involved.

1.2 Example

A simple, and very useful, example of such a scheme arises from a real quadratic irrationality q. We form the ring $\mathbb{Z}[q] \subset \mathbb{R}$ and let * be the $\mathbb{Z}$-mapping that takes q into its algebraic (quadratic) conjugate. Then the set of points $\widetilde{\mathbb{Z}[q]} := \{(x, x^*) \mid x \in \mathbb{Z}[q]\}$ is a lattice in $\mathbb{R}^2$ and

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R}^2 & \xrightarrow{\pi_2} & \mathbb{R} \\ & & \cup & & \\ & & \widetilde{\mathbb{Z}[q]} & & \end{array} \tag{1.2}$$

where we use the coordinate projections, is a cut a project scheme. An important case of this occurs when $q = \tau := (1+\sqrt{5})/2$.

1.3 Model sets

Let $L := \pi_1(\tilde{L})$ and let

$$(\ )^* : \quad L \longrightarrow \mathbb{R}^n \tag{1.3}$$

be the mapping $\pi_2 \circ (\pi_1|_{\tilde{L}})^{-1}$. This mapping extends naturally to a mapping on the rational span $\mathbb{Q}L$ of L, also denoted by $(\ )^*$. Note that the lattice $\tilde{L}$ can also be written as

$$\tilde{L} \; = \; \{(x, x^*) \mid x \in L\}. \tag{1.4}$$

Now, let $\Omega \subset \mathbb{R}^n$. Define

$$\Lambda \; = \; \Lambda(\Omega) \; := \; \{x \in L \mid x^* \in \Omega\}. \tag{1.5}$$

We call such a set Λ a *model set* (or *cut and project set*) if the following three conditions are fulfilled,

W1 $\Omega \subset \mathbb{R}^n$ is compact.

W2 $\Omega = \overline{\mathrm{int}(\Omega)}$.

W3 The boundary of Ω has Lebesgue measure 0.

The mathematical reasons for studying model sets are that they are very natural generalizations of lattices, they share many properties with them, and they allow symmetries that are otherwise unavailable in lattices of the corresponding dimensions. For example, the following properties are shared by all model sets Λ:

M1 Λ is *uniformly discrete*: that is to say, there is an $r > 0$ so that for all distinct $x, y \in \Lambda$, $|x - y| \geq r$.

M2 Λ is *relatively dense*: that is to say, there is an $R > 0$ so that for each $x \in \mathbb{R}^m$ the open ball of radius R around x contains a point of Λ.

M3 There is a finite set F so that $\Lambda - \Lambda \subset \Lambda - F$.

M4 $\Lambda - \Lambda$ is a Delone set (i.e. satisfies [**M1, M2**]).

M5 Λ has a well-defined density d, i.e.

$$d = \lim_{s \to \infty} \frac{\#\Lambda_s}{\mathrm{vol}(B_s(0))} = \lim_{s \to \infty} \frac{\#\Lambda_s}{c_m s^m} \tag{1.6}$$

exists, where $B_s(0) := \{x \in \mathbb{R}^m \mid |x| \leq s\}$ and

$$c_m := \frac{\pi^{m/2}}{\Gamma(\frac{m}{2} + 1)} \tag{1.7}$$

is the volume of the unit sphere in $\mathbb{R}^m$.

M6 Λ diffracts. (See section 3.1 for more on this.)

A set with the properties **M1** and **M2** is called a *Delone* set. A lattice is nothing else than a Delone set that is a group. If $F = \{0\}$ then **M3** states that Λ is a group, so **M3** is in fact a generalization of the group law. The limit in (1.6) is easily seen to be independent of the choice of origin for the Euclidean space. What is more, it even exists *uniformly* for sets. This means that for any subset S of Ω with boundary of measure 0, the relative frequency of the points of $(\Lambda_s)^*$ falling into S, as $s \to \infty$, is $\mathrm{vol}(S)/\mathrm{vol}(\Omega)$, and the convergence is uniform with respect to translation of the set S. For more on these properties one may consult [6, 13, 14, 17].

Model sets arise in situations in which one is looking for Delone structures with symmetries that are incompatible with lattices. The most famous example is that of the icosahedral group which cannot appear as the point symmetry of any lattice in 3-space. It is known [15, 5] that if G is any finite group acting irreducibly in $\mathbb{R}^m$ and X is any non-trivial orbit of G, then either G acts crystallographically in $\mathbb{R}^m$, that is to say, there is a G-stable lattice of $\mathbb{R}^m$; or there is a G-stable cut and project set in $\mathbb{R}^m$ that contains the set X. This is the origin of the interest in these sets in the theory of quasicrystals, see [8] for background material.

However, there is a serious price to be paid for moving from lattices to model sets. Lattices, by definition, have an entire lattice of translational symmetries. By comparison, in a model set Λ described by a cut and project scheme (1.1), the set of translational symmetries is the kernel of $()^*$ in (1.3), and in all the standard examples this is in fact $\{0\}$. Fortunately, in many cases of interest, there is nonetheless an abundance of symmetry, as long as one is prepared to consider self-similarities instead of group symmetries.

1.4 Self-similarities

Definition 1 *A* self-similarity *of Λ is an affine linear mapping $t = t_{Q,v}$*

$$t_{Q,v} \;:\quad x \mapsto Qx + v \tag{1.8}$$

on $\mathbb{R}^m$ that maps Λ into itself, where Q is a (linear) similarity and $v \in \mathbb{R}^m$. Thus $Q = qR$, i.e. it is made up of an orthogonal transformation R and an inflation factor *q.*

Let $t_{Q,v}$ be a self-similarity of Λ. If Λ is uniformly discrete, we must have $|q| \geq 1$. This is the reason why we also call such a self-similarity an *affine inflation*. We are interested in the *entire* set of affine inflations with the same similarity factor Q. It is convenient to have $0 \in \Lambda$ and $0 = 0^* \in \mathrm{int}(\Omega)$. Using a translation of Λ by a vector $v_0 \in L$ and the corresponding translation of Ω by $v_0^* \in L^*$, we assume this to be the case. This makes no structural difference to the set of inflations of Λ, but simplifies the algebra: if $0 \in \Lambda$ and $t_{Q,v}$ is an affine inflation, then $v \in L$ and $Q(L) \subset L$. In fact, we are going to also assume that $QL = L$.

Let us then fix once and for all a (linear) similarity transformation Q on $\mathbb{R}^m$ such that $QL = L$. How do we describe the set of all self-similarities with similarity factor Q? In preparation for answering this question, it is useful to note that there are three different ways of looking at the same cut and project set Λ: first as a Delone set in $\mathbb{R}^m$, which we may think of as the *discrete* picture; second as part of the lattice $\tilde{L}$, which we may think of as the *arithmetic* picture; and finally as a dense subset of Ω via the mapping $()^*$, which may be thought of as the *analytic* picture.

As an illustration of these ideas, note that Q naturally gives rise to an automorphism $\tilde{Q}$ of the lattice $\tilde{L}$, i.e. an element of $GL_{\mathbb{Z}}(\tilde{L})$, and a linear mapping Q^* of $\mathbb{R}^n$ that maps Ω into itself. From the arithmetic nature of $\tilde{Q}$ we deduce that the eigenvalues of Q and Q^* are algebraic integers and from the compactness of Ω that Q^* is contractive. Furthermore, one can deduce [2] that Q^* is diagonalizable from the corresponding property of Q. Strictly speaking we can only deduce that the eigenvalues of Q^* do not exceed 1 in absolute value, but we will always assume that in fact they are less than one in absolute value. In the sequel we will normally denote the contraction Q^* by A to match various sources we will refer to frequently.

Define

$$\Omega_Q \ := \ \{u \in \mathbb{R}^n \mid A\Omega + u \subset \Omega\}\,, \tag{1.9}$$

we say that Q is *compatible* with Λ if $\mathrm{int}(\Omega_Q) \neq \emptyset$. In this article, we shall always assume that not only Ω, but also Ω_Q is Riemann measurable, i.e. $\partial\Omega_Q$ has zero Lebesgue measure. Interpreting (1.8) on the window side we obtain:

Proposition 1 *Let $\Lambda = \Lambda(\Omega)$ be a model set based on a window Ω that satisfies the window conditions* **W1** – **W3**. *Let Q be a similarity compatible with Λ. Then the set $\mathcal{T}_Q$ of affine inflations with the same similarity Q is the set of mappings $t_{Q,v}$ where v runs through the set*

$$T_Q \ = \ \{v \in L \mid v^* \in \Omega_Q\}\,. \tag{1.10}$$

In particular, T_Q is also a model set.

Let us pause to consider the special situation where Ω is convex. In this case, Ω_Q is also convex and hence satisfies the conditions that we need. If in addition $Q^* = \varepsilon \cdot \mathbb{1},\ 0 < \varepsilon < 1$, which actually is often the case in examples of physical relevance, one obtains

$$\Omega_Q \ = \ (1-\varepsilon)\Omega\,. \tag{1.11}$$

If $-1 < \varepsilon < 0$, but $\Omega = -\Omega$, (1.11) is still true if ε is replaced by $|\varepsilon|$. This happens in our Examples.

1.5 Example

In Example 1.2 above, take $q = \tau$. A simple model set is defined by

$$\Lambda \ = \ \{x \in \mathbb{Z}[\tau] \mid x' \in [-1,1]\}. \tag{1.12}$$

Let us look at the inflation factor $q = \tau$. Multiplication by q determines the contraction $A = \tau' \cdot \mathbb{1}$, $\tau' = -1/\tau$, on the internal side and

$$\Omega_\tau \ = \ \{u \in \mathbb{R} \mid -\frac{1}{\tau} \cdot [-1,1] + u \subset [-1,1]\} \ = \ [-1 + \frac{1}{\tau}\,, 1 - \frac{1}{\tau}]\,. \tag{1.13}$$

Thus, for all $v \in \mathbb{Z}[\tau]$ with $v' \in [-1/\tau^2, 1/\tau^2]$,

$$t_{\tau,v} \ : \quad x \mapsto \tau x + v \tag{1.14}$$

is an inflation of Λ, and

$$t^*_{\tau,v} \; : \quad y \mapsto \frac{-y}{\tau} + v' \tag{1.15}$$

is the corresponding contraction in internal space.

2 Averaging operators and invariant densities

One of the most famliar techniques in the theory of group representations is the use of group averages in order to produce invariants. Thus for a finite group G one typically invokes the process

$$F \;\mapsto\; \frac{1}{\#G} \sum_{g\in G} g\cdot F \tag{1.16}$$

which averages the function F over the group, where $g\cdot F(x) := F(g^{-1}x)$. We intend to do exactly the same thing replacing G by the set of all self-similarities $\mathcal{T}_Q$ of a model set Λ. Since $\mathcal{T}_Q$ will be infinite we have to be a little careful in averaging.

For any subset $T \subset \mathbb{R}^m$ and for any $s \geq 0$, we thus define

$$T_s \;:=\; \{x \in T \mid |x| \leq s\} \tag{1.17}$$

where $|x| = (x\cdot x)^{1/2}$ is the standard Euclidean norm on $\mathbb{R}^m$.

2.1 Averaging operators

Definition 2 *Let $\Lambda = \Lambda(\Omega)$ be a model set based on a window Ω that satisfies the window conditions* **W1** – **W3**. *Let Q be a similarity (with inflation factor q) that is compatible with Λ, let $\mathcal{T} = \mathcal{T}_Q$ be the set of all self-similarities of Λ with similarity Q, and let $T = T_Q$ be the corresponding set of translations. Let $p : L \to \mathbb{R}$ be any function on L that vanishes off Λ. Then the average of p over $\mathcal{T}$ is defined to be*

$$(\mathcal{A}p)(x) \;=\; \lim_{s\to\infty} \frac{|\det(Q)|}{\#T_s} \sum_{v\in T_s} p(t_v^{-1}x) \tag{1.18}$$

provided this limit exists. We say that p is a Q-invariant density on Λ if

ID1 *p is non-negative on Λ,*

ID2 *$\mathcal{A}p = p$, and*

ID3 *p is normalized, i.e.*

$$\lim_{s\to\infty} \frac{1}{\#\Lambda_s} \sum_{x\in\Lambda_s} p(x) \;=\; 1\,. \tag{1.19}$$

Note that for $x \in \Lambda$ and $v \in T$, $t_v^{-1}x \in L$ but $t_v^{-1}x$ does not in general lie in Λ. Thus in (1.18) one can expect that many of the summands on the right hand side will be 0 because p vanishes off Λ. Note also that (1.19) is a normalization per point of Λ. This can be changed to a normalization per unit volume if necessary because the density d of points of Λ exists.

Let $\mathcal{C}(\Omega)$ be the space of all continuous complex-valued functions on $\mathbb{R}^n$ with support in Ω. Via the mapping $()^*$ of (1.3) we obtain a space $\mathcal{C}(\Lambda)$ of functions on L, vanishing off Λ: for $f \in \mathcal{C}(\Omega)$, we define $p = p_f \in \mathcal{C}(\Lambda)$ by

$$p(x) \; = \; \mathrm{vol}(\Omega) \cdot f(x^*)\,, \tag{1.20}$$

for $x \in \Lambda$, and $p(x) = 0$ otherwise. Here, the normalization constant $\mathrm{vol}(\Omega)$ is thrown in to make things more convenient later on.

Bearing in mind (1.20), we rewrite (1.18) as a new averaging operator on $\mathcal{C}(\Omega)$:

$$(\mathcal{A}f)(x^*) \; = \; \lim_{s\to\infty} \frac{|\det(Q)|}{\#T_s} \sum_{v\in T_s} f((t_v^*)^{-1}x^*) \tag{1.21}$$

for all $x^* \in \Lambda^*$.

Now, it is well-known [16, 17, 6] that the points $\pi_2(\tilde{L})$ are uniformly distributed in $\mathbb{R}^n$, in the sense described in Section 1.3. In particular, the points of T^* are uniformly distributed in Ω_Q, hence $\lim_{s\to\infty} \overline{(T_s)^*} = \Omega_Q$. Using Weyl's theorem [18, 12], the continuity of f, and the fact that Ω_Q is Riemann integrable, we obtain

$$(\mathcal{A}f)(x^*) \; = \; \frac{|\det(Q)|}{\mathrm{vol}(\Omega_Q)} \int_{\Omega_Q} f(A^{-1}(x^* - u))du\,. \tag{1.22}$$

Using the fact that $\det(Q) = \det(A)^{-1}$ (remember that $A = Q^*$) and introducing the normalized indicator (or characteristic) functions

$$X_S \; := \frac{\mathbf{1}_S}{\mathrm{vol(S)}} \tag{1.23}$$

defined for all measurable subsets S of $\mathbb{R}^n$, we can rewrite (1.22) in a number of equivalent ways:

$$\begin{aligned}
(\mathcal{A}f)(x) \;&=\; \int_{\mathbb{R}^n} X_{Ay+\Omega_Q}(x) f(y)dy \\
&=\; \frac{1}{|\det(A)|} \int_{\mathbb{R}^n} X_{A^{-1}\Omega_Q}(A^{-1}x - y) f(y)dy \\
&=\; \frac{1}{|\det(A)|} \int_{\mathbb{R}^n} X_{\Omega_Q}(x - y) f(A^{-1}y)dy\,.
\end{aligned} \tag{1.24}$$

This shows that the averaging operator $\mathcal{A}$ is a *continuous refinement operator* in the sense of [9] in the one-dimensional case and [10] in the multi-dimensional case. In Fourier space, by application of the convolution theorem, (1.24) reads as

$$\widehat{(\mathcal{A}f)}(k) = \widehat{X_{\Omega_Q}}(k)\hat{f}(A^t k) \tag{1.25}$$

where $\hat{f}(k) := \int_{\mathbb{R}^n} e^{-2\pi i k\cdot x} f(x)dx$ and $f(x) = \int_{\mathbb{R}^n} e^{2\pi i k\cdot x}\hat{f}(k)dk$.

2.2 Invariant densities

At this point, we can determine a function $f = f_p$ corresponding to an *invariant density* p. We are looking for a 1-eigenfunction of the operator $\mathcal{A}$. The normalization condition **ID3**, seen on the window side, becomes, using Weyl again, $\int_{\mathbb{R}^n} f(u)du = 1$. Iteration of (1.25) leads to

$$\hat{f}(k) \;=\; \hat{X}_{\Omega_Q}(k)\cdot\hat{X}_{\Omega_Q}(A^t k)\cdot\ldots\cdot\hat{X}_{\Omega_Q}((A^t)^N k)\cdot\hat{f}((A^t)^{N+1}k)\,. \quad (1.26)$$

Since A is a contraction, so is A^t. Consequently, $(A^t)^{N+1}k \to 0$ as $N\to\infty$. Since $\hat{f}$ and $\hat{X}_{\Omega_Q}$ are C^∞ and $\hat{f}(0) = 1$, we can take the limit and obtain

$$\hat{f}(k) \;=\; \prod_{N=0}^{\infty}\hat{X}_{\Omega_Q}((A^t)^N k) \;=\; \prod_{N=0}^{\infty}\frac{\hat{\mathbf{1}}_{\Omega_Q}((A^t)^N k)}{\mathrm{vol}(\Omega_Q)}\,. \quad (1.27)$$

This is an infinite product with compact convergence that solves our problem: $\hat{f}$ is an infinite product of C^∞-functions, and is itself C^∞. The function f is now the inverse Fourier transform of $\hat{f}$. By construction, it is the Radon-Nikodym derivative, and hence the density, of an absolutely continuous invariant (1.43) measure (with respect to Lebesgue measure) in internal space. Again applying the convolution theorem we arrive at:

Proposition 2 *Let $\Lambda = \Lambda(\Omega)$ be a model set based on a window Ω that satisfies the window conditions* **W1** – **W3**. *Let Q be a similarity compatible with Λ and let $A := Q^*$. Then there is a unique Q-invariant density p for Λ lying in $\mathcal{C}(\Lambda)$. This is given through $p = p_f$, see* (1.20)*, where f is given by the infinite convolution product*

$$f \;=\; \mathop{\ast}_{N=0}^{\infty}\frac{\mathbf{1}_{A^N\Omega_Q}}{\mathrm{vol}(A^N\Omega_Q)}\,. \quad (1.28)$$

Note that this convolution of characteristic functions defines a C^∞ function with compact support contained in Ω.

If $Q, Q^2, \ldots,$ are all compatible with Λ then it is instructive to look at the corresponding invariant densities $f = f_{(1)}, f_{(2)}, \ldots$ The sets $\{\Omega_{Q^n}\}$ are increasing and $\overline{\bigcup_n \Omega_{Q^n}} = \Omega$. The functions of the sequence $\{\hat{f}_{(k)}\}$ become increasingly concentrated around 0 and it is natural to expect $\lim_{s\to\infty} f_{(s)} = \mathbf{1}_\Omega/\mathrm{vol}(\Omega)$. This is illustrated in the case of our Example by the sequence of graphs of Fig.1.

2.3 Example

We continue with Example 1.5 from above and determine the invariant density (or rather its Fourier transform) on $[-1,1]$ corresponding to the inflation factor τ. If f is this density, then equation (1.25) becomes

$$\hat{f}(k) \;=\; \frac{\tau^2}{2}\hat{\mathbf{1}}_{[-1/\tau^2,1/\tau^2]}(k)\hat{f}(-k/\tau)\,. \tag{1.29}$$

Routine calculation gives:

$$\frac{1}{2a}\hat{\mathbf{1}}_{[-a,a]}(k) \;=\; \frac{1}{2a}\int_{-a}^{a} e^{-2\pi i k x}dx \;=\; \frac{\sin(2\pi a k)}{2\pi a k}\,. \tag{1.30}$$

So, we obtain

$$\hat{f}(k) \;=\; \frac{\sin\left(\frac{2\pi k}{\tau^2}\right)}{\left(\frac{2\pi k}{\tau^2}\right)}\hat{f}(-k/\tau) \;=\; \ldots \;=\; \prod_{N=2}^{\infty}\frac{\sin\left(\frac{2\pi k}{\tau^N}\right)}{\left(\frac{2\pi k}{\tau^N}\right)}\,. \tag{1.31}$$

The calculations for other inflation factors are similar; Fig. 1 shows some of the resulting invariant densities.

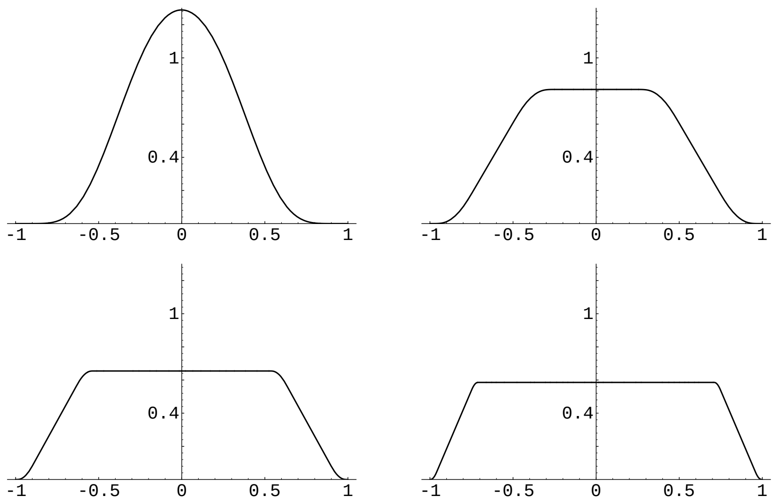

FIGURE 1. Invariant densities for the model set of Example 1.5, for inflation factors τ, τ^2, τ^3 and τ^4.

2.4 Eigenvalues and eigenfunctions

According to [10], if the eigenvalues of A^t are $\{\alpha_1,\ldots,\alpha_n\}$, $0<|\alpha_i|<1$, then the spectrum of $\mathcal{A}$ is

$$\mathrm{spec}(\mathcal{A}) = \{\alpha^{\boldsymbol{a}} \mid \boldsymbol{a}\in\mathbb{Z}^n_{\geq 0}\} \tag{1.32}$$

where $\alpha^{\boldsymbol{a}} := \alpha_1^{a_1} \cdot \ldots \cdot \alpha_n^{a_n}$. Furthermore, the multiplicities are those suggested by the notation:

$$\operatorname{mult}(\lambda) = \#\{\boldsymbol{a} \mid \alpha^{\boldsymbol{a}} = \lambda\}\,. \tag{1.33}$$

In particular, if $\alpha_1 = \ldots = \alpha_n = \alpha$, then $\lambda = \alpha^{|\boldsymbol{a}|}$, $|\boldsymbol{a}| = \alpha_1 + \ldots + \alpha_n$, and

$$\operatorname{mult}(\lambda) = \begin{pmatrix} |\boldsymbol{a}| + n - 1 \\ |\boldsymbol{a}| \end{pmatrix},$$

a formula that is well-known from the n-dimensional harmonic oscillator.

REMARK: In [9, 10] A is assumed to be diagonalizable and all the eigenvalues are assumed real. The diagonalizability of Q guarantees that of $Q^* = A$ [2]. By allowing complex-valued functions to enter the picture, it is not necessary to assume that the eigenvalues of A are real. Nor do we wish to impose such an assumption since we do not have this type of control over the spectrum of A.

It is quite easy to find eigenfunctions representing these eigenvalues in terms of the invariant density f. To do so, we choose an eigenbasis $\{v_1, \ldots, v_n\}$ for A^t in $\mathbb{C}^n$, using the fact that A^t is diagonalizable. Any $k \in \mathbb{C}^n$ can be written as $k = \kappa_1 v_1 + \ldots + \kappa_n v_n$ and the $\kappa_j = \kappa_j(k)$ are the corresponding coordinate functions (which we allow to be $\mathbb{C}$-valued). Now, fix $\boldsymbol{b} \in \mathbb{Z}^n_{\geq 0}$, define $\kappa^{\boldsymbol{b}} = \kappa_1^{b_1} \cdot \ldots \cdot \kappa_n^{b_n}$, and let $\hat{u}(k) := (\kappa(k))^{\boldsymbol{b}} \hat{f}(k)$. Then we obtain from (1.25)

$$\begin{aligned} \widehat{\mathcal{A}u}(k) &= \widehat{X_{\Omega_Q}}(k)\hat{u}(A^t k) \;=\; \widehat{X_{\Omega_Q}}(k)(\kappa(A^t k))^{\boldsymbol{b}} \hat{f}(A^t k) \\ &= \alpha^{\boldsymbol{b}}\,(\kappa(k))^{\boldsymbol{b}} \hat{f}(k) \;=\; \alpha^{\boldsymbol{b}}\,\hat{u}(k). \end{aligned} \tag{1.34}$$

Returning from the Fourier domain, this enables us to write down the eigenfunctions of $\mathcal{A}$. If $\{v_1^\dagger, \ldots, v_n^\dagger\}$ denotes the dual basis (i.e. $v_i^\dagger \cdot v_j = \delta_{ij}$), we define the directional derivative $\mathrm{D}_j := v_j^\dagger \cdot \nabla$ and obtain

Proposition 3 *The partial derivatives*

$$\mathrm{D}^{\boldsymbol{b}} f,\ \boldsymbol{b} \in \mathbb{Z}^n_{\geq 0} \tag{1.35}$$

are eigenfunctions of the refinement operator $\mathcal{A}$, with eigenvalue $\alpha^{\boldsymbol{b}}$.

Some of these derivatives, calculated for our guiding Example, are shown in Figure 2.

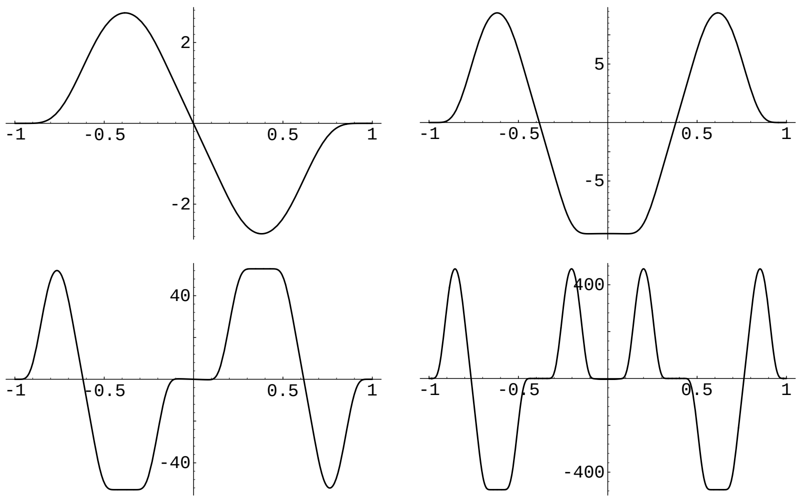

FIGURE 2. The first 4 derivatives of the invariant density of Example 1.5 for inflation factor τ.

3 Further remarks

3.1 Diffraction and a product formula

The product formula (1.27) has an interpretation on the physical side of our picture. To see this, we define functions g_s and h_s by

$$\begin{aligned} g_s(k) &= \frac{1}{\#T_s} \sum_{v \in T_s} e^{-2\pi i k \cdot v} \\ h_s(k) &= \frac{1}{\#\Lambda_s} \sum_{w \in \Lambda_s} p(w) e^{-2\pi i k \cdot w} \end{aligned} \tag{1.36}$$

and, furthermore, functions g and h by

$$\begin{aligned} g(k) &= \lim_{s \to \infty} g_s(k) \\ h(k) &= \lim_{s \to \infty} h_s(k) \,. \end{aligned} \tag{1.37}$$

The existence of g as a well-defined function on $\mathbb{R}^m$ is a known consequence of the fact that T is a model set. We now have [2]:

Proposition 4 $\quad h(k) \;=\; \prod_{N=0}^{\infty} g((Q^t)^N k).$

The significance of the functions g and h appears in the context of diffraction. Suppose that $w : L \to \mathbb{R}_{\geq 0}$ is some bounded non-negative function that vanishes off Λ. Define the tempered distribution

$$\mu_w \;=\; \sum_{x \in \Lambda} w(x) \delta_x \tag{1.38}$$

where δ_x is the Dirac measure at x. The limit, as $s \to \infty$, of the averaged auto-correlation of this measure, which exists for model sets, is the *auto-correlation measure* of Λ (also called its Patterson function, though it is a distribution)

$$\gamma_w \;=\; \lim_{s\to\infty} \frac{1}{\#\Lambda_s} \sum_{x,y\in\Lambda_s} w(x)w(y)\delta_{x-y} \; . \tag{1.39}$$

Its Fourier transform is a positive measure $\hat{\gamma}_w$ which is the *diffraction pattern* of Λ. The point part of this measure is the *Bragg spectrum* of Λ. In the case that w is the indicator function (i.e constant value 1) on a model set Λ then $\hat{\gamma}_w$ is a pure point measure and we say that Λ has a *pure point spectrum.* In any case, the Bragg spectrum can be calculated from the simpler *function*

$$g(\cdot\,;w) \;:\quad g(k;w) \;=\; \lim_{s\to\infty} \frac{1}{\#\Lambda_s} \sum_{x\in\Lambda_s} w(x) e^{-2\pi i x\cdot k} \tag{1.40}$$

provided that this limit exits everywhere. In fact [6]

$$|g(k;w)|^2 \;=\; \hat{\gamma}_w(\{k\}), \text{ for all } k \in \mathbb{R}^m \; . \tag{1.41}$$

Our functions g and h correspond to the cases when $w = \mathbf{1}_\Lambda$ and w is the invariant density p, respectively. In particular, h allows us an exact description of the intensities of the Bragg spectrum of (Λ, p). Notice from the product formula that its support necessarily lies inside the support of the Bragg spectrum of Λ itself. It goes without saying that both g and h are highly discontinuous functions which, for a model set, are non-zero only on a dense point set of zero Lebesgue measure. This set is contained in the so-called *Fourier module* of the model set, i.e. the in the set $\pi_1(\tilde{L}^o)$, where $\tilde{L}^o = \{y \in \mathbb{R}^{m+n} \mid x \cdot y \in \mathbb{Z} \text{ for all } x \in \tilde{L}\}$ is the dual lattice of $\tilde{L}$.

3.2 Hutchinson measures

Let f be the invariant density of $\mathcal{C}(\Omega)$ corresponding to the compatible similarity Q on $\Lambda = \Lambda(\Omega)$. There is a corresponding measure $\mu = \mu_f$, with support contained in Ω, defined by

$$\mu_f(Y) \;=\; \int_{\mathbb{R}^n} \mathbf{1}_Y(x)\mu_f(dx) \;=\; \int_{\mathbb{R}^n} \mathbf{1}_Y(x) f(x) dx \, . \tag{1.42}$$

We have $\mu_f(\Omega) = 1$. The measure μ_f is invariant in the sense that, if we define $t_v^* \cdot \mu_f$ by $t_v^* \cdot \mu_f(Y) = \mu_f((t_v^*)^{-1}(Y))$, then

$$\mu_f \;=\; \lim_{s\to\infty} \frac{1}{\#T_s} \sum_{v\in T_s} t_v^* \cdot \mu_f \; . \tag{1.43}$$

Now fix some $s > 0$ and consider the finite set of contractions t_v^* on Ω, where $v \in T_s$. According to [7] there is a unique non-negative Borel measure μ_s on Ω for which $\mu_s(\Omega) = 1$ and which is *invariant* in the sense that

$$\mu_s \ = \ \frac{1}{\# T_s} \sum_{v \in T_s} t_v^* \cdot \mu_s \ . \tag{1.44}$$

Furthermore, this measure is the unique fixed point of the process on the set of regular Borel measures on Ω with mass 1 that averages a measure by the right hand side of (1.44). We call this a *Hutchinson measure.* Using the Levy continuity theorem [1] one can show

Proposition 5 *The sequence of Hutchinson measures fulfils:*

(i) $\hat{\mu}_s(k) = \prod_{N=0}^{\infty} g_s((A^t)^N k)$, *for any* $s > 0$*;*

(ii) $\{\hat{\mu}_s\} \to \hat{\mu}_f$, *where the convergence is uniform on compact sets;*

(iii) $\{\mu_s\} \to \mu_f$, *in the sense of weak convergence, i.e.* $\{\mu_s(\varphi)\} \to \mu_f(\varphi)$ *for all* $\varphi \in \mathcal{C}(\Omega)$.

In particular, for any function $\varphi \in \mathcal{C}(\Omega)$, $\mu_s(\varphi)$, for large s, is a good estimate of $\mu_f(\varphi)$. Also, starting from *any* probability measure on Ω, the Hutchinson iteration of (1.44) will converge to μ_s and this procedure, for large s, will also give a good approximation to μ_f.

3.3 The topology of L and Λ

The space of functions $\mathcal{C}(\Lambda)$ on which the averaging operator $\mathcal{A}$ acts seems strange. It is defined in terms of the topology of Ω in $\mathbb{R}^n$ and this is very different from the discrete topology that we see on Λ induced by the topology of its ambient space $\mathbb{R}^m$. The appropriate topology for Λ (and L) is defined intrinsically as follows [17]. For each compact set K of $\mathbb{R}^m$ define

$$N_K \ = \ N_K(\Lambda) \ := \ \{v \in L \mid v + (\Lambda \cap K) = \Lambda \cap (v + K)\}. \tag{1.45}$$

Thus N_K is the set of vectors v for which translation by v is a bijection of the K-patch of Λ and onto the $(v+K)$-patch of Λ. Note that the mapping $K \mapsto N_K$ is inclusion reversing.

Proposition 6 [16, 17] *Suppose that* $L^* \cap \partial\Omega = \emptyset$*. Then the collection of sets* $\{N_K \mid K \subset \mathbb{R}^m,\ K \text{ compact}\}$ *is a basis of neighbourhoods of* 0 *for a topological group structure on* L*. Furthermore,* $\mathbb{R}^n$ *with its standard topology is the completion of* L *under the mapping* $()^* : L \to \mathbb{R}^n$*. With this topology on* L*, the space* $\mathcal{C}(\Lambda)$ *is precisely the space of continuous functions on* L *whose support lies in* Λ.

The intuition behind continuity of a function ϕ (defined on L) with respect to this topology is this: if translation by v is a bijection of two "large" patches of Λ, then $\phi(x+v) - \phi(x)$ is (uniformly) small.

4 Outlook

The existence of *positive* invariant densities naturally suggests probabilistic interpretations. In [3] we study the spectral properties of certain stochastic sets whose sites are selected from that of a model set on a probabilistic basis according to the density p. Effectively, this gives a distribution of points which, after $*$-mapping them into internal space, looks like our window shaped by the invariant density f. This may provide an alternative explanation of the recently made observations of such profiles in real data [11]. Furthermore, as such sets do have finite entropy density, they might be useful for further models of entropic stabilization of quasicrystals.

In this article, we have focused on one-component model sets. It is important for multi-component or multi-coloured systems to be able to realize the similarity-averaging process and the existence of invariant densities in a matrix generalization of what we have done here. This means that we have a finite family of model sets based on cosets of a common $\mathbb{Z}$-module and matrices of similarity maps between them. In fact this set-up results in matrix continuous refinement operators and ultimately again in the existence of invariant densities. What is particularly interesting in this case is the appearance of a Markov matrix of weights for the contributions from the various windows relative to each other. An exposition of this will appear in [4].

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