Fields Institute Communications
Volume **00**, 0000

Cut-and-Project Sets in Locally Compact Abelian Groups

Martin Schlottmann
Institut für Theoretische Physik
Universität Tübingen
Auf der Morgenstelle 14
D-72076 Tübingen, Germany
martin.schlottmann@uni-tuebingen.de

Abstract. The cut-and-project formalism in arbitrary locally compact Abelian groups is investigated. It is established that a generalization of the well-known formula for the density of the resulting cut-and-project sets holds. Furthermore, a necessary and sufficient criterion for the existence of a cut-and-project scheme for a given subset of a locally compact Abelian group is provided. It is proved that this scheme is essentially uniquely determined by the local isomorphism class of the set.

1 Introduction

Cut-and-project sets have been introduced by [**8**] as models for harmonious sets and later, in the course of the structure analysis of quasicrystals, rediscovered in a variety of different schemes [**6, 2, 5, 10**]. Whereas the use of arbitrary locally compact Abelian groups as internal spaces was already described by [**8**], physicists, according to their needs, restricted themselves to the consideration of vector spaces only. Because a large part of the basic theory of these sets can be carried out without the linear structure of the real or the internal space, it is worthwhile to establish the essential facts on the formalism in as much generality as possible. The present article is devoted to the exposition of some steps in this direction.

The cut-and-project method as well-known by physicists for the generation of (quasiperiodic) point sets in a finite-dimensional vector space $\mathfrak{V}$ can be described as follows. One supplements the space $\mathfrak{V}$, the "real space", where the point sets live, by a second finite-dimensional vector space $\mathfrak{V}_{\mathrm{int}}$, the "internal space", a lattice Λ in $\mathfrak{V} \oplus \mathfrak{V}_{\mathrm{int}}$ (i.e., a discrete additive subgroup of $\mathfrak{V} \oplus \mathfrak{V}_{\mathrm{int}}$ which is relatively dense in $\mathfrak{V} \oplus \mathfrak{V}_{\mathrm{int}}$, the latter meaning that there is a compact subset K of $\mathfrak{V} \oplus \mathfrak{V}_{\mathrm{int}}$ with $\Lambda + K = \mathfrak{V} \oplus \mathfrak{V}_{\mathrm{int}}$), and a subset W of $\mathfrak{V}_{\mathrm{int}}$, the "window" or "acceptance domain", which is usually assumed to fulfil some conditions such as being bounded and containing interior points. The associated cut-and-project set is defined as

$$P(W) := \{ \pi(q) \mid q \in \Lambda,\ \ \pi_{\mathrm{int}}(q) \in W \}, \tag{1.1}$$

1991 *Mathematics Subject Classification.* Primary: ; Secondary:

where $\pi : \mathfrak{V} \oplus \mathfrak{V}_{\text{int}} \to \mathfrak{V}$ and $\pi_{\text{int}} : \mathfrak{V} \oplus \mathfrak{V}_{\text{int}} \to \mathfrak{V}_{\text{int}}$ are the canonical projections. In other words, $P(W)$ is the projection into the real space of the set $\Lambda \cap (\mathfrak{V} + W)$ of all lattice points which are contained in the strip $\mathfrak{V} + W$.

It was noticed very early that cut-and-project sets have some properties which are desirable from the physical point of view. Among them is the fact that they have a well-defined and uniform density. This means that the number of points in a certain volume is proportional to the volume in the infinite volume limit, and this limit is uniform in the choice where the volume is cut from the pattern. This is important for the interpretation of the infinite set as an approximation to finite physical samples. Furthermore, it turned out that the density is proportional to the volume of the acceptance domain and thereby easy to calculate. The theorem hidden behind this fact was some kind of folklore in the quasicrystal community for a long time. In the present paper, we set out to give a general proof for it, thereby generalizing earlier results [**11**], [**4**].

Another question arises from the use of cut-and-project sets for the modelling of physical structures. After all, the physical space has three dimensions, and for the description of cut-and-project sets we introduce an entirely new internal space with additional degrees of freedom. Therefore, it would be desirable to determine to which amount the internal space is implied by the structure in the physical space itself. In what follows, we will show that, indeed, the internal space, the lattice, and the window, after some trivial normalization, can be obtained by the cut-and-project set itself by a simple mathematical construction. In other words, we turn the cut-and-project scheme upside down and solve the problem: given a point set P, what is $\mathfrak{V}_{\text{int}}$, Λ, and W. The main problem here is the reconstruction of $\mathfrak{V}_{\text{int}}$ as a topological group. It will be shown that the topology of $\mathfrak{V}_{\text{int}}$ is intimely connected with the question which translation vectors of the physical space map finite portions of the point set into finite portions of the (same) set. The uniqueness of this construction will be a simple corollary.

For both questions, we will choose our setup as general as seems reasonable. That is, after a suggestion of R. V. Moody, we will work in the framework of locally compact Abelian groups.

The present paper is organized roughly as follows. Firstly, we will consider the density of the cut-and-projection sets $P(W)$ which should exist and be proportional to the volume of the window W, provided the window is sufficiently nice, and prove that the expected formula for the density holds (Section 2 and 3). We will then, in Section 4, generalize the framework by allowing arbitrary locally compact Abelian groups as internal space and prove that the density formula remains valid. In Section 5 we set up the general framework allowing arbitrary locally compact Abelian groups for the real space, too, and define a normal form of cut-and-project schemes. A necessary and sufficient criterion for the existence of a cut-and-project scheme for a given point set is given in Section 6, and in Section 7, it is proved that this scheme is essentially determined by the local isomorphism class of the point set.

2 Density of Cut-and-Project Sets

We will begin our investigations with the formula for the density of cut-and-project sets (and its proof) in this and the next section. Our approach uses only elementary facts from the theory of $\mathbb{R}^n$; a different proof of the density formula using Fourier transformation has been given by [**4**]. For a start, let us introduce some notation and conventions.

The closed ball of radius r around the origin in $\mathfrak{V}$ will be denoted by B_r. We will never consider balls in other spaces than the real space, and it will be immaterial which norm in $\mathfrak{V}$ is chosen to define balls. In $\mathfrak{V}$ and $\mathfrak{V}_{\text{int}}$ we choose some translationally invariant (Haar) measures μ and μ_{int}.

The cardinality of a set M will be denoted by $\#M$. If M is a subset of a set on which an addition is defined, we use the abbreviation $t + M := \{\, t + x \mid x \in M \,\}$. Analogously, for two such subsets M_1, M_2: $M_1 + M_2 := \{\, x_1 + x_2 \mid x_1 \in M_1,\ x_2 \in M_2 \,\}$, and so on.

We will freely consider the components $\mathfrak{V}$ and $\mathfrak{V}_{\text{int}}$ of the direct sum $\mathfrak{V} \oplus \mathfrak{V}_{\text{int}} := \mathfrak{V} \oplus \mathfrak{V}_{\text{int}}$ as subsets, subgroups, and subspaces of $\mathfrak{V} \oplus \mathfrak{V}_{\text{int}}$; however, topological terms as the interior W° and the closure $\overline{W}$ of a subset W of $\mathfrak{V}_{\text{int}}$ will always be meant relatively to $\mathfrak{V}_{\text{int}}$ and its topology. A subset W of $\mathfrak{V}_{\text{int}}$ has almost no boundary iff the volume $\mu_{\text{int}}(\partial W)$ of its boundary $\partial W := \overline{W} \backslash W^\circ$ (relatively to $\mathfrak{V}_{\text{int}}$!) vanishes (this is the case iff the characteristic function of W is Riemann integrable).

The density of a discrete point set P in $\mathfrak{V}$ is the average number of points in P per unit volume, i.e.,

$$\lim_{r\to\infty} \frac{\#(P \cap B_r)}{\mu(B_r)}, \tag{2.1}$$

provided this limit exists. We will not be content only with the existence of the limit (2.1), which distinguishes the origin of $\mathfrak{V}$, but will show that the limit

$$\lim_{r\to\infty} \frac{\#(P \cap (x + B_r))}{\mu(B_r)} \tag{2.2}$$

exists *uniformly* in $x \in \mathfrak{V}$ for a large class of cut-and-project sets.

Let W be a subset of $\mathfrak{V}_{\text{int}}$. Then, obviously,

$$\#(P(W) \cap (x + B_r)) = \#(\Lambda \cap (x + B_r + W)), \tag{2.3}$$

if π is one-to-one when restricted to Λ, i.e., if $\Lambda \cap \mathfrak{V}_{\text{int}} = 0$. Intuitively, the number of lattice points inside $x + B_r + W$ should be proportionally to the volume of $x + B_r + W$, i.e., proportional to $\mu(B_r)\mu_{\text{int}}(W)$, at least in the limit of infinite volume. However, it is not clear, and, in general, not true, that this infinite volume limit may be carried out by keeping the window W fixed and increasing only r. But, if the lattice looks dense if one observes only its components in the internal space, then one may hope that this special form of infinite volume limit makes sense and, furthermore, does not depend on any displacement x.

These considerations will be made precise in the next section, and what we will show in fact is

Proposition 2.1 *(cf. Lemma 2.7, p. 38 in* [**11**]*) Let $\mathfrak{V} \cong \mathbb{R}^n$, $\mathfrak{V}_{int} \cong \mathbb{R}^m$. Let Λ be a lattice in $\mathfrak{V} \oplus \mathfrak{V}_{int}$, $|\Lambda|$ be the volume of a measurable fundamental domain of Λ in $\mathfrak{V} \oplus \mathfrak{V}_{int}$ with respect to the product $\mu \otimes \mu_{int}$ of the invariant measures μ, μ_{int} on $\mathfrak{V}$, $\mathfrak{V}_{int}$, respectively. Assume that the projection image $\pi_{int}(\Lambda)$ of the lattice is dense in the internal space $\mathfrak{V}_{int}$. If W is a bounded subset of $\mathfrak{V}_{int}$ with almost no boundary, then*

$$\lim_{r\to\infty} \frac{\#(\Lambda \cap (t + B_r + W))}{\mu(B_r)} = \frac{\mu_{int}(W)}{|\Lambda|} \tag{2.4}$$

uniformly in $t \in \mathfrak{V} \oplus \mathfrak{V}_{int}$.

According to (2.3), this obviously settles the question of the existence and the value of the limit in (2.2) in the case that $\Lambda \cap \mathfrak{V}_{\rm int}$ is trivial and $\overline{\pi_{\rm int}(\Lambda)} = \mathfrak{V}_{\rm int}$. Furthermore, as we have also allowed for translations of the window W in (2.4), Proposition 2.1 gives simultaneously a characterization of the uniformity of the density of the projection of the lattice in the internal space.

Proposition 2.1 will be generalized later on to the case where $\mathfrak{V}_{\rm int}$ is an arbitrary locally compact Abelian group (keeping the other conditions fixed). In this context, we will also learn how the case that $\Lambda \cap \mathfrak{V}_{\rm int}$ is non-trivial or the projection of Λ is not dense in $\mathfrak{V}_{\rm int}$ can be treated.

3 Proof of Proposition 2.1

Without loss of generality, we may assume that the volume of measurable fundamental domains of Λ in $\mathfrak{V} \oplus \mathfrak{V}_{\rm int}$ equals 1. Let us introduce the notation

$$\#_{r,t} W := \frac{\#(\Lambda \cap (t + B_r + W))}{\mu(B_r)}, \tag{3.1}$$

where $r \geq 0$ and $t \in \mathfrak{V} \oplus \mathfrak{V}_{\rm int}$. We will call a subset W of $\mathfrak{V}_{\rm int}$ a *udp* set (for: having the uniform density property), iff it is bounded and for each strictly positive ε there is some radius $r_0 \geq 0$ such that for all vectors $t \in \mathfrak{V} \oplus \mathfrak{V}_{\rm int}$ and all radii $r \geq r_0$ the estimate

$$|\#_{r,t} W - \mu_{\rm int}(W)| < \varepsilon \tag{3.2}$$

holds.

The density formula amounts to the assertion that every bounded subset of $\mathfrak{V}_{\rm int}$ with almost no boundary is in fact a *udp* set. We start the proof with several observations which, with one exception, require only short argumentation.

1) If W is *udp* and $c \in \mathfrak{V}_{\rm int}$, then $c + W$ is *udp*, as follows from the very definition.

2) A simple application of the triangle inequality to (3.2) shows that for two *udp* sets W_1, W_2:
a) If W_1 and W_2 are disjoint, then their union is *udp*,
b) If $W_1 \subseteq W_2$ then the complement $W_2 \backslash W_1$ of W_1 in W_2 is *udp*.

3) If there are sequences $(L_i)_{i\in\mathbb{N}}$ and $(M_i)_{i\in\mathbb{N}}$ of *udp* sets with the property that $\mu_{\rm int}(M_i \backslash L_i) \to 0$ as $i \to \infty$ and $L_i \subseteq W \subseteq M_i$ ($i \in \mathbb{N}$), then W is *udp*. This holds because we then have the estimates

$$\begin{aligned} \#_{r,t} W - \mu_{\rm int}(W) &\leq \#_{r,t} M_i - \mu_{\rm int}(M_i) + \mu_{\rm int}(M_i \backslash L_i), &(3.3)\\ \mu_{\rm int}(W) - \#_{r,t} W &\leq \mu_{\rm int}(L_i) - \#_{r,t} L_i + \mu_{\rm int}(M_i \backslash L_i), &(3.4)\end{aligned}$$

from which the inequality

$$\begin{aligned} |\#_{r,t} W - \mu_{\rm int}(W)| \leq\ & |\#_{r,t} L_i - \mu_{\rm int}(L_i)| + |\#_{r,t} M_i - \mu_{\rm int}(M_i)| \\ & + \mu_{\rm int}(M_i \backslash L_i) \end{aligned} \tag{3.5}$$

is easily derived.

4) For every open set $U \subseteq \mathfrak{V}_{\rm int}$ and every compact $K \subseteq U$, there is a *udp* set D with almost no boundary and $K \subseteq D \subseteq U$.
This statement will be proved below.

Before we begin to show 4), let us quickly get the density formula out of 1)–4). Let W be a bounded subset of $\mathfrak{V}_{\rm int}$ and $\varepsilon > 0$. First of all, we find a *udp* set $C \supseteq \overline{W}$ with almost no boundary; we may even assume that $\overline{W} \subseteq C^\circ$. There are compact sets $K_1 \subseteq W^\circ$ and $K_2 \subseteq (C^\circ \backslash \overline{W})$ such that $\mu_{\text{int}}(W^\circ \backslash K_1) < \varepsilon$ and $\mu_{\text{int}}((C^\circ \backslash \overline{W}) \backslash K_2) < \varepsilon$. With the help of 4), we find *udp* sets L and D with $K_1 \subseteq L \subseteq W^\circ$ and $K_2 \subseteq D \subseteq (C^\circ \backslash \overline{W})$. According to 2), $M := C \backslash D$ is *udp*, and one calculates $\mu_{\text{int}}(W^\circ \backslash L) < \varepsilon$ and

$$\begin{aligned} \mu_{\text{int}}(M \backslash \overline{W}) &= \mu_{\text{int}}((C \backslash \overline{W}) \backslash D) \\ &= \mu_{\text{int}}((C^\circ \backslash \overline{W}) \backslash D) < \varepsilon \end{aligned} \tag{3.6}$$

(because $\mu_{\text{int}}(\partial C)$ vanishes) and, therefore,

$$\mu_{\text{int}}(M \backslash L) < \mu_{\text{int}}(\overline{W} \backslash W^\circ) + 2\varepsilon = \mu_{\text{int}}(\partial W) + 2\varepsilon. \tag{3.7}$$

If, now, $\mu_{\text{int}}(\partial W)$ vanishes, then this estimate in connection with 3) gives the desired result.

We finally have to show statement 4). Let $q_1, \ldots, q_m$ be elements of the lattice Λ such that $\pi_{\text{int}}(q_1), \ldots, \pi_{\text{int}}(q_m)$ form a basis of $\mathfrak{V}_{\text{int}}$, let

$$W := [\pi_{\text{int}}(q_1), \ldots, \pi_{\text{int}}(q_m)[\,, \tag{3.8}$$

where we introduce the notation

$$[p_1, \ldots, p_k[:= \{ \sum_{i=1}^{k} \lambda_i p_i \mid 0 \le \lambda_i < 1 \ \ (i = 1, \ldots, k) \}. \tag{3.9}$$

(For linearly independent $p_1, \ldots, p_k$, the set $[p_1, \ldots, p_k[$ is the half-open parallelotope spanned by $p_1, \ldots, p_k$.) If we succeed in showing that every W as in (3.8) is *udp*, then 4) results as follows. If $K \subseteq U \subseteq \mathfrak{V}_{\text{int}}$, K compact and U open, then we may choose W of the form (3.8) such that $K + W - W \subseteq U$, because $\pi_{\text{int}}(\Lambda)$ is dense in $\mathfrak{V}_{\text{int}}$. The elements of the set

$$\{ W + \sum_{i=1}^{m} n_i \pi_{\text{int}}(q_i) \mid n_i \in \mathbb{Z}, \ (1 \le i \le m) \} \tag{3.10}$$

are pairwise disjoint and cover $\mathfrak{V}_{\text{int}}$. Then, K is covered by those finitely many elements of (3.10) which intersect K and which are, therefore, contained in U. The union D of these sets is *udp* according to 1) and 2), $K \subseteq D \subseteq U$, and, clearly, D has almost no boundary.

Let, now, W be as in (3.8). It is always possible to complete $q_1, \ldots, q_m$ to a basis $q_1, \ldots, q_{m+n}$ of $\mathfrak{V} \oplus \mathfrak{V}_{\text{int}}$ consisting of lattice points. The integral linear combinations of the vectors $q_1, \ldots, q_{m+n}$ constitute a certain sublattice $\tilde{\Lambda}$ of Λ of finite index which is precisely the number $\#(\Lambda \cap E)$ for each fundamental domain E of $\tilde{\Lambda}$, and this number, in turn, equals the volume of E because we have normalized the volume of (measurable) fundamental domains of Λ to be unity. Examples of such fundamental domains of $\tilde{\Lambda}$ are all sets of the form $t + E_s$ where $t \in \mathfrak{V} \oplus \mathfrak{V}_{\text{int}}$ and $E_s := [q_1, \ldots, q_{m+n}[\,$.

Our aim now will be to deform such sets where the number of lattice points which they contain is easily handled into sets which are properly aligned with our projection strip. This will be achieved with the help of two transformations. Let S_1 be the linear transformation defined by

$$S_1(\pi_{\text{int}}(q_i)) := q_i \ \ (i = 1, \ldots, m), \qquad S_1(x) = x \ \ (x \in \mathfrak{V}). \tag{3.11}$$

This transformation obviously fulfils the equation $(S_1 - 1)^2 = 0$, therefore the determinant of S_1 is 1 ($S_1 - 1$ is upper triangular with vanishing diagonal elements in some convenient basis). The subspace $\mathfrak{V}$ is not moved at all by S_1, therefore,

$\mathfrak{V} \oplus \mathfrak{V}_{\text{int}} = \mathfrak{V} \oplus \mathfrak{V}'$ if we set $\mathfrak{V}' := S_1\mathfrak{V}_{\text{int}}$. We denote the canonical projection $\mathfrak{V} \oplus \mathfrak{V}' \to \mathfrak{V}$ by π'. Then, because $q_1, \ldots, q_m$ constitute a basis of $\mathfrak{V}'$, we see that $\pi'(q_{m+1}), \ldots, \pi'(q_{m+n})$ are a basis of $\mathfrak{V}$; let $E := [\pi'(q_{m+1}), \ldots, \pi'(q_{m+n})[$.

We define a second linear transformation of $\mathfrak{V} \oplus \mathfrak{V}_{\text{int}}$ by

$$S_2(q_i) := \pi'(q_i) \quad (i = m+1, \ldots, m+n), \qquad S_2(x) = x \quad (x \in \mathfrak{V}'). \tag{3.12}$$

Again, $(S_2 - 1)^2 = 0$, and one calculates $S_2^{-1} = 1 - (S_2 - 1)$. The point now is that $S_2(E_s) = E + W'$ (with $W' := S_1(W) = [q_1, \ldots, q_m[$) is a fundamental domain of $\tilde{\Lambda}$ which is equivalent to the statement that for each point x of $\mathfrak{V} \oplus \mathfrak{V}_{\text{int}}$ there is exactly one $q \in \tilde{\Lambda}$ and exactly one $f \in E_s$ such that

$$S_2^{-1}x = S_2^{-1}q + f. \tag{3.13}$$

For this, let $\mathfrak{V}''$ be the subspace of $\mathfrak{V} \oplus \mathfrak{V}_{\text{int}}$ generated by $q_{m+1}, \ldots, q_{m+n}$. Then, if we set $\Lambda' := \tilde{\Lambda} \cap \mathfrak{V}'$ and $\Lambda'' := \tilde{\Lambda} \cap \mathfrak{V}''$, we have $\tilde{\Lambda} = \Lambda' \oplus \Lambda''$. If we write $x = x' + x''$ for the unique decomposition of $x \in \mathfrak{V} \oplus \mathfrak{V}_{\text{int}} = \mathfrak{V}' \oplus \mathfrak{V}''$ into its components in $\mathfrak{V}'$ and $\mathfrak{V}''$, we see that (3.13) is the same as

$$x' + (S_2^{-1} - 1)x'' + x'' = q' + (S_2^{-1} - 1)q'' + q'' + f' + f'', \tag{3.14}$$

because S_2^{-1} does not affect $\mathfrak{V}'$. Furthermore, because the image of $S_2^{-1} - 1 = 1 - S_2$ is contained in $\mathfrak{V}'$, we arrive at the following decomposition of (3.14) into components:

$$x'' = q'' + f'' \tag{3.15}$$

$$x' + (S_2^{-1} - 1)x'' = q' + (S_2^{-1} - 1)q'' + f'. \tag{3.16}$$

Obviously, $f \in E_s$ is equivalent to the condition that $f' \in W'$ and that $f'' \in W'' := [q_{m+1}, \ldots, q_{m+n}[$; here, W' and W'' are fundamental domains of Λ', Λ'' in $\mathfrak{V}'$, $\mathfrak{V}''$, respectively. Therefore, there is a unique solution of (3.15) with $q'' \in \Lambda''$ and $f'' \in W''$, and, given this solution, we have a unique solution of (3.16) with $q' \in \Lambda'$ and $f' \in W'$.

So, $S_2(E_s)$ is indeed a fundamental domain of $\tilde{\Lambda}$, therefore,

$$\begin{aligned} \#(\Lambda \cap (t + E + W')) &= \mu \otimes \mu_{\text{int}}(E + W') = \mu \otimes \mu_{\text{int}}(S_1(E + W)) \\ &= \mu \otimes \mu_{\text{int}}(E + W) \\ &= \mu(E)\mu_{\text{int}}(W), \end{aligned} \tag{3.17}$$

for each $t \in \mathfrak{V} \oplus \mathfrak{V}_{\text{int}}$.

From (3.17) one concludes that

$$\#(\Lambda \cap (t + F + W')) = \mu(F)\mu_{\text{int}}(W), \tag{3.18}$$

for each $t \in \mathfrak{V} \oplus \mathfrak{V}_{\text{int}}$ and for every subset F of $\mathfrak{V}$ which can be written as a disjoint union of finitely many sets of the form $s + E$ ($s \in \mathfrak{V}$).

Because W is bounded, one finds some non-negative number δ such that $|x - S_1x| \leq \delta$, if $x \in W$. Then, for $r \geq \delta$ and $t \in \mathfrak{V} \oplus \mathfrak{V}_{\text{int}}$, we have

$$t + B_{r-\delta} + W' \subseteq t + B_r + W \subseteq t + B_{r+\delta} + W', \tag{3.19}$$

remembering that $W' = S_1W$.

Let ε be some number which exceeds the diameter of E. Then, for $r > \delta + \varepsilon$, we may cover $B_{r-\delta-\varepsilon}$ by a finite disjoint union $F_l(r)$ of sets of the form $s + E$

contained in $B_{r-\delta}$, and, in the same fashion, cover $B_{r+\delta}$ by a finite disjoint union $F_u(r)$ of sets of the form $s+E$ contained in $B_{r+\delta+\varepsilon}$. It follows that

$$\frac{\mu(F_u(r)\backslash F_l(r))}{\mu(B_r)} \leq \frac{\mu(B_{r+\delta+\varepsilon}\backslash B_{r-\delta-\varepsilon})}{\mu(B_r)} \to 0 \quad (r\to\infty). \tag{3.20}$$

We may now calculate with (3.19)

$$\begin{aligned} \frac{\#(\Lambda\cap(t+F_l(r)+W'))}{\mu(B_r)} &\leq \#_{r,t}W \\ &\leq \frac{\#(\Lambda\cap(t+F_u(r)+W'))}{\mu(B_r)}, \end{aligned} \tag{3.21}$$

and, therefore, according to (3.18)

$$\begin{aligned} \mu_{\rm int}(W)(\mu(F_l(r))/\mu(B_r)-1) &\leq \#_{r,t}W-\mu_{\rm int}(W) \\ &\leq \mu_{\rm int}(W)(\mu(F_u(r))/\mu(B_r)-1) \end{aligned} \tag{3.22}$$

which amounts to

$$|\#_{r,t}W-\mu_{\rm int}(W)| \leq \mu_{\rm int}(W)\frac{\mu(F_u(r)\backslash F_l(r))}{\mu(B_r)} \tag{3.23}$$

and this tends to zero uniformly in $t\in\mathfrak{V}\oplus\mathfrak{V}_{\rm int}$ as r goes to infinity. Hence, W is a *udp* set. $\square$

4 The General Density Formula

Already the density formula as proved above shows the shortcomings of the restriction that the internal space should be a vector space, as we cannot very elegantly handle the case where the lattice intersects the internal space non-trivially or projects not densely in the internal space. Therefore, we now increase the generality of our schemes by considering arbitrary locally compact Abelian groups as internal spaces.

In the sequel we will need one fact or other on topological Abelian groups. We will indicate the proofs of them where it is possible in one line, but we will have to rely on theorems where this is not possible. In the Appendix, we present a short summary of the basic facts we need on topological groups. The reader interested in some background literature may be referred to standard textbooks (e.g., [**3**], where he will find almost everything we need here), or, for some introductory expositions, the article [**9**] in the present volume.

This section is devoted to the proof of

Theorem 1 *(Density Formula) Let $\mathfrak{V}\cong\mathbb{R}^n$ and $\mathfrak{G}_{int}$ be a locally compact Abelian group. Let Λ be a lattice in $\mathfrak{V}\oplus\mathfrak{G}_{int}$, $|\Lambda|$ be the volume of a measurable fundamental domain of Λ in $\mathfrak{V}\oplus\mathfrak{G}_{int}$ with respect to the product $\mu\otimes\mu_{int}$ of the Haar measures μ, μ_{int} on $\mathfrak{V}$, $\mathfrak{G}_{int}$, respectively. Assume that $\pi_{int}(\Lambda)$ is dense in $\mathfrak{G}_{int}$. If W is a totally bounded subset of $\mathfrak{G}_{int}$ with almost no boundary, then*

$$\lim_{r\to\infty}\frac{\#(\Lambda\cap(t+B_r+W))}{\mu(B_r)} = \frac{\mu_{int}(W)}{|\Lambda|} \tag{4.1}$$

uniformly in $t\in\mathfrak{V}\oplus\mathfrak{G}_{int}$.

(A lattice in a locally compact Abelian group is a discrete relatively dense subgroup.)

A quick examination of the proof of the special density formula in Section 3 reveals that the only serious thing which remains to be proved is the generalization of the observation 4) of Section 3, i.e.,

Lemma 4.1 *Under the assumptions of Theorem 1: For every open $U \subseteq \mathfrak{G}_{int}$ and every compact $K \subseteq U$, there is a* udp *set W with almost no boundary and $K \subseteq W \subseteq U$.*

In this general context, a *udp* set is a totally bounded subset W of $\mathfrak{G}_{\text{int}}$ such that (4.1) holds uniformly in $t \in \mathfrak{G}_{\text{int}}$.

Proof of Lemma 4.1 (and, therefore, Theorem 1): Let us temporarily call *udp* group every locally compact Abelian group $\mathfrak{G}_{\text{int}}$ such that Lemma 4.1 is true for every choice of Λ with $\overline{\pi_{\text{int}}(\Lambda)} = \mathfrak{G}_{\text{int}}$. A result of Section 3 is that for *udp* groups Theorem 1 holds. Furthermore, from Section 3, we already know that every $\mathbb{R}^m$ is a *udp* group, and our strategy will be to gradually extend the supply of known *udp* groups using the well-known structure of locally compact Abelian groups.

1) $\mathbb{T}^l \oplus \mathbb{R}^m$ are *udp* groups. To see this, we have $\mathfrak{G}_{\text{int}} := \mathbb{T}^l \oplus \mathbb{R}^m \cong \mathbb{R}^{l+m}/\Lambda_{\text{int}} = \tilde{\mathfrak{G}}_{\text{int}}/\Lambda_{\text{int}}$ for some discrete subgroup Λ_{int} of $\tilde{\mathfrak{G}}_{\text{int}} := \mathbb{R}^{l+m}$. Let $\tilde{\pi} : \mathfrak{V} \oplus \tilde{\mathfrak{G}}_{\text{int}} \to \mathfrak{V} \oplus \mathfrak{G}_{\text{int}} = (\mathfrak{V} \oplus \tilde{\mathfrak{G}}_{\text{int}})/\Lambda_{\text{int}}$ be the canonical projection. If Λ is a lattice in $\mathfrak{V} \oplus \mathfrak{G}_{\text{int}}$, then, obviously, $\tilde{\Lambda} := \tilde{\pi}^{-1}(\Lambda)$ is a lattice in $\mathfrak{V} \oplus \tilde{\mathfrak{G}}_{\text{int}}$, and if Λ projects densely into $\mathfrak{G}_{\text{int}}$, then $\tilde{\Lambda}$ projects densely into $\tilde{\mathfrak{G}}_{\text{int}}$. There is a fundamental domain F of Λ_{int} in $\tilde{\mathfrak{G}}_{\text{int}}$ with almost no boundary such that $F \cap \tilde{\pi}^{-1}(W)$ is totally bounded for every totally bounded subset W of $\mathfrak{G}_{\text{int}}$. With the appropriate normalization of the Haar measure $\tilde{\mu}_{\text{int}}$ in $\tilde{\mathfrak{G}}_{\text{int}}$, we have $\mu_{\text{int}}(W) = \tilde{\mu}_{\text{int}}(F \cap \tilde{\pi}^{-1}(W))$ for every measurable subset W of $\mathfrak{G}_{\text{int}}$; especially $|\Lambda| = |\tilde{\Lambda}|$ taking the respective product measures. Furthermore, for $\tilde{W} := F \cap \tilde{\pi}^{-1}(W)$, where $W \subseteq \mathfrak{G}_{\text{int}}$, and for $\tilde{t} \in \mathfrak{V} \oplus \tilde{\mathfrak{G}}_{\text{int}}$, we obviously have $\#(\Lambda \cap (\tilde{\pi}(\tilde{t}) + B_r + W)) = \#(\tilde{\Lambda} \cap (\tilde{t} + B_r + \tilde{W}))$. If, now, $\tilde{W}$ is a totally bounded subset of $\tilde{\mathfrak{G}}_{\text{int}}$ with almost no boundary (and, hence, *udp* with respect to $\tilde{\Lambda}$), then $(\Lambda_{\text{int}} + \tilde{W}) \cap F$ is totally bounded with almost no boundary. As a consequence, it is *udp*, and as $\tilde{\pi}(\tilde{W}) = \tilde{\pi}((\Lambda_{\text{int}} + \tilde{W}) \cap F)$ and $\tilde{\pi}^{-1}\tilde{\pi}(\tilde{W}) = \Lambda_{\text{int}} + \tilde{W}$, the above remarks show that $\tilde{\pi}(\tilde{W})$ is *udp* and has almost no boundary. Let U be an open subset of $\mathfrak{G}_{\text{int}}$ and K be a compact subset of U. Then, $K = \tilde{\pi}(K')$ for $K' := \overline{F \cap \tilde{\pi}^{-1}(K)}$ which is compact. One may now choose a *udp* set $\tilde{W}$ with almost no boundary and $K' \subseteq \tilde{W} \subseteq \tilde{\pi}^{-1}(U)$. As we have shown, $W := \tilde{\pi}(\tilde{W})$ is *udp* with almost no boundary, and $K \subseteq W \subseteq U$.

2) If $\mathfrak{G}_{\text{int}}$ contains an open subgroup which is a *udp* group, then $\mathfrak{G}_{\text{int}}$ itself is a *udp* group. For this, let Λ be a lattice in $\mathfrak{V} \oplus \mathfrak{G}_{\text{int}}$ projecting densely into $\mathfrak{G}_{\text{int}}$. If we define the lattice $\tilde{\Lambda} := \Lambda \cap (\mathfrak{V} \oplus \tilde{\mathfrak{G}}_{\text{int}})$ in $\mathfrak{V} \oplus \tilde{\mathfrak{G}}_{\text{int}}$ for some open *udp* subgroup $\tilde{\mathfrak{G}}_{\text{int}}$ of $\mathfrak{G}_{\text{int}}$, $\tilde{\Lambda}$ projects densely into $\tilde{\mathfrak{G}}_{\text{int}}$. Because $\mathfrak{G}_{\text{int}} = \pi_{\text{int}}(\Lambda) + \tilde{\mathfrak{G}}_{\text{int}}$, it is immediate that every *udp* subset of $\tilde{\mathfrak{G}}_{\text{int}}$ with respect to $\tilde{\Lambda}$ is in fact *udp* with respect to Λ; and a subset of $\tilde{\mathfrak{G}}_{\text{int}}$ has almost no boundary with respect to $\tilde{\mathfrak{G}}_{\text{int}}$ iff it has almost no boundary with respect to $\mathfrak{G}_{\text{int}}$. The assertion follows if we finally observe that every totally bounded subset of $\mathfrak{G}_{\text{int}}$ is the disjoint union of finitely many translates of totally bounded subsets of $\tilde{\mathfrak{G}}_{\text{int}}$.

3) If $\mathfrak{H}$ is a compact subgroup of $\mathfrak{G}_{\text{int}}$ such that $\tilde{\mathfrak{G}}_{\text{int}} := \mathfrak{G}_{\text{int}}/\mathfrak{H}$ is a *udp* group, then, for every totally bounded $\tilde{W} \subseteq \tilde{\mathfrak{G}}_{\text{int}}$ with almost no boundary, the set $\tilde{\pi}^{-1}(\tilde{W})$ is *udp* for every lattice Λ in $\mathfrak{V} \oplus \mathfrak{G}_{\text{int}}$ projecting densely into $\mathfrak{G}_{\text{int}}$ and has almost no boundary (where we have $\tilde{\pi} : \mathfrak{V} \oplus \mathfrak{G}_{\text{int}} \to \mathfrak{V} \oplus \tilde{\mathfrak{G}}_{\text{int}}$ canonically). Namely, we observe that $\partial \tilde{\pi}^{-1}(\tilde{W}) = \tilde{\pi}^{-1}(\partial \tilde{W})$, and, with the appropriate normalization of the

Haar measure $\tilde{\mu}_{\text{int}}$ on $\tilde{\mathfrak{G}}_{\text{int}}$, we have $\tilde{\mu}_{\text{int}}(\partial\tilde{\pi}^{-1}(\tilde{W})) = \tilde{\mu}_{\text{int}}(\tilde{\pi}^{-1}(\partial\tilde{W})) = \mu_{\text{int}}(\partial\tilde{W})$ for $\tilde{W} \subseteq \tilde{\mathfrak{G}}_{\text{int}}$. If we set $\tilde{\Lambda} := \tilde{\pi}(\Lambda)$, then $\tilde{\Lambda}$ projects densely into $\tilde{\mathfrak{G}}_{\text{int}}$, if Λ projects densely into $\mathfrak{G}_{\text{int}}$. Furthermore, $|\tilde{\Lambda}| = \#(\mathfrak{H} \cap \Lambda) \cdot |\Lambda|$. Finally, one calculates for each $t \in \mathfrak{G}_{\text{int}}$: $\#(\mathfrak{H} \cap \Lambda) \cdot \#(\tilde{\Lambda} \cap (\tilde{\pi}(t) + B_r + \tilde{W})) = \#(\Lambda \cap (t + B_r + \tilde{\pi}^{-1}(\tilde{W})))$, from which the assertion follows.

4) Compactly generated locally compact Abelian groups are *udp* groups. (A topological group is called compactly generated if it is algebraically generated by a compact subset.) Let U be an open subset of $\mathfrak{G}_{\text{int}}$ and K be a compact subset of U. There is an open subset U' of U and a neighbourhood V of 0 such that $K \subseteq U'$ and $U' + V \subseteq U$. According to the structure theorem cited in the Appendix we find a compact subgroup $\mathfrak{H}$ of $\mathfrak{G}_{\text{int}}$ which is contained in V such that $\tilde{\mathfrak{G}}_{\text{int}} := \mathfrak{G}_{\text{int}}/\mathfrak{H} \cong \mathbb{T}^l \oplus \mathbb{R}^m \oplus D$ where D is a discrete group. Then, $\tilde{\pi}(K)$ is compact and $\tilde{\pi}(U')$ is open in $\tilde{\mathfrak{G}}_{\text{int}}$. Because of 1) and 2), $\tilde{\mathfrak{G}}_{\text{int}}$ is an *udp* group, and we find a *udp* set $\tilde{W}$ with almost no boundary and $\tilde{\pi}(K) \subseteq \tilde{W} \subseteq \tilde{\pi}(U')$. Because of 3), $W := \tilde{\pi}^{-1}(\tilde{W})$ is *udp* with almost no boundary, and, clearly, $K \subseteq W \subseteq U$.

5) Every locally compact Abelian group is a *udp* group. We only have to observe that the subgroup of $\mathfrak{G}_{\text{int}}$ generated by any compact neighbourhood of the identity is an open subgroup of $\mathfrak{G}_{\text{int}}$. One applies 4) and 2) to get the final result. $\square$

This completes the proof of Theorem 1.

5 Regular Cut-and-Project sets

After having established the density formula we now start to investigate the question of existence and uniqueness of cut-and-projection schemes for discrete point sets in a locally compact Abelian group $\mathfrak{G}$. We have no really striking motivation also to generalize the real space to be a more general group, but, for the further development, the linear structure of $\mathfrak{V}$ does not provide any advantage anymore, and, as usual, the structures are more transparent if one frees oneself from unnecessary restrictions. Even the density formula allows for a generalization to $\mathfrak{G}$ instead of $\mathfrak{V}$, but that would involve a thorough discussion of the definition of densities of point sets in $\mathfrak{G}$ which we don't want to go into here.

The formulation of the cut-and-project scheme in this more general context is trivially re-writing the definitions of the first section, but, for the sake of completeness and seizing the opportunity to reshape our notation, we expose it again. Given two locally compact Abelian groups $\mathfrak{G}$ (the real space) and $\mathfrak{G}_{\text{int}}$ (the internal space) and given a subgroup Λ of $\mathfrak{G} \oplus \mathfrak{G}_{\text{int}}$, for each subset W of $\mathfrak{G}_{\text{int}}$ we define the cut-and-project set

$$\begin{aligned} P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W) &:= \{\, \pi(q) \mid q \in \Lambda,\ \pi_{\text{int}}(q) \in W \,\} && (5.1)\\ &= \pi(\Lambda \cap (\mathfrak{G} + W)) && (5.2)\\ &= \mathfrak{G} \cap (\Lambda - W) && (5.3) \end{aligned}$$

where $\pi : \mathfrak{G} \oplus \mathfrak{G}_{\text{int}} \to \mathfrak{G}$ and $\pi_{\text{int}} : \mathfrak{G} \oplus \mathfrak{G}_{\text{int}} \to \mathfrak{G}_{\text{int}}$ are again the canonical projection homomorphisms.

Of course, we are not interested in cut-and-project sets without imposing some additional conditions on Λ and W, but it will be convenient to have some handy notation which covers more general situations. We will need the following properties of cut-and-project sets later on.

Lemma 5.1 *Let W be totally bounded.*

a): *If Λ is a lattice in $\mathfrak{G} \oplus \mathfrak{G}_{int}$ and $W^\circ \cap \pi_{int}(\Lambda) \neq \emptyset$, then $P(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$ is discrete and relatively dense in $\mathfrak{G}$.*

b): *If $\Lambda \cap \mathfrak{G}_{int} = \{0\}$, $\overline{\pi_{int}(\Lambda)} = \mathfrak{G}_{int}$, $W^\circ \neq \emptyset$, and $P(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$ is discrete and relatively dense in $\mathfrak{G}$, then Λ is a lattice in $\mathfrak{G} \oplus \mathfrak{G}_{int}$.*

Proof: We set $P := P(\mathfrak{G}, \mathfrak{G}_{\mathrm{int}}, \Lambda, W)$.
a) If $K \subseteq \mathfrak{G}$ is compact, then $P \cap K = K \cap (\Lambda - W) = \pi(\Lambda \cap (K + W))$ is finite. Therefore, P is discrete. Let $\tilde{K}$ be a compact subset of $\mathfrak{G} \oplus \mathfrak{G}_{\mathrm{int}}$ such that $\mathfrak{G} \oplus \mathfrak{G}_{\mathrm{int}} = \Lambda + \tilde{K}$. If we set $K_1 := \pi(\tilde{K})$ and $K_2 := \pi_{\mathrm{int}}(\tilde{K})$, obviously $\mathfrak{G} \oplus \mathfrak{G}_{\mathrm{int}} = \Lambda + K_1 + K_2$. Now, $\mathfrak{G} \subseteq \mathfrak{G} + \pi_{\mathrm{int}}(\Lambda) \subseteq \Lambda + K_1 + (K_2 \cap \pi_{\mathrm{int}}(\Lambda))$. Because $K_2 \cap \pi_{\mathrm{int}}(\Lambda)$ is totally bounded and $W^\circ \cap \pi_{\mathrm{int}}(\Lambda) \neq \emptyset$, there is a finite $F \subseteq \Lambda$ such that $K_2 \cap \pi_{\mathrm{int}}(\Lambda) \subseteq \pi_{\mathrm{int}}(F) - W$. We conclude that $\mathfrak{G} \subseteq \Lambda + K_1 + \pi_{\mathrm{int}}(F) - W = (\Lambda - W) + \pi(F) + K_1$, hence, $\mathfrak{G} \subseteq \mathfrak{G} \cap (\Lambda - W) + \pi(F) + K_1 = P + K$ for $K := \pi(F) + K_1$. So, P is relatively dense in $\mathfrak{G}$.
b) We have to show that Λ is discrete and relatively dense in $\mathfrak{G} \oplus \mathfrak{G}_{\mathrm{int}}$. There is a point $q \in \Lambda$ with $\pi_{\mathrm{int}}(q) \in W^\circ$, in particular $\pi(q) \in P$. If U is an open subset of $\mathfrak{G}$ such that $U \cap P = \{\pi(q)\}$, then $\Lambda \cap (U + W^\circ) = \{q\}$, so, Λ is discrete. Let, on the other hand, K be a compact subset of $\mathfrak{G}$ with $\mathfrak{G} = K + P$. Then, obviously, $\mathfrak{G} \subseteq \Lambda + K - W$. Because $W^\circ \neq \emptyset$ and $\overline{\pi_{\mathrm{int}}(\Lambda)} = \mathfrak{G}_{\mathrm{int}}$, we have $\mathfrak{G} \oplus \mathfrak{G}_{\mathrm{int}} = \Lambda + \mathfrak{G} + W$, therefore, $\mathfrak{G} \oplus \mathfrak{G}_{\mathrm{int}} = \Lambda + K + W - W$ which means that Λ is relatively dense in $\mathfrak{G} \oplus \mathfrak{G}_{\mathrm{int}}$. □

We now define a convenient normal form of projection schemes. Here, we write $\langle M \rangle$ for the subgroup algebraically generated by a subset M of a group.

Definition 5.2 *A* standard projection strip *over $\mathfrak{G}$ with internal space $\mathfrak{G}_{int}$, lattice Λ and window W is a quadruple $(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$ such that*

a): *$\mathfrak{G}$ and $\mathfrak{G}_{int}$ are locally compact Abelian groups,*

b): *Λ is a lattice in $\mathfrak{G} \oplus \mathfrak{G}_{int}$ with $\Lambda \cap \mathfrak{G}_{int} = \{0\}$ and $\overline{\pi_{int}(\Lambda)} = \mathfrak{G}_{int}$ ($\pi : \mathfrak{G} \oplus \mathfrak{G}_{int} \to \mathfrak{G}$, $\pi_{int} : \mathfrak{G} \oplus \mathfrak{G}_{int} \to \mathfrak{G}_{int}$ canonical),*

c): *W is a non-void compact subset of $\mathfrak{G}_{int}$ with $\overline{W^\circ} = W$, $\pi_{int}(\Lambda) \cap \partial W = \emptyset$, $\langle W - W \rangle = \mathfrak{G}_{int}$, and $\{ c \in \mathfrak{G}_{int} \mid c + W = W \} = \{0\}$.*

A subset P of the locally compact Abelian group $\mathfrak{G}$ is a regular cut-and-project set *(in $\mathfrak{G}$) iff there is a standard projection strip $(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$ and a $p \in \mathfrak{G}$ such that $P = p + P(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$.*

In order to show that the class of regular cut-and-project sets covers those sets which are obtained by certain variants of the cut-and-projection method we give the following proposition large parts of which can also be found in [**9**]. Because we rely on this proposition at some point later on, we write down the proof in some detail (borrowing certain ideas from [**9**]).

Proposition 5.3 *Let $\mathfrak{G}$ and $\mathfrak{G}_{int}$ be locally compact Abelian groups and Λ a lattice in $\mathfrak{G} \oplus \mathfrak{G}_{int}$. If $W_1, \ldots, W_l$ are totally bounded subsets of $\mathfrak{G}_{int}$ with $\pi_{int}(\Lambda) \cap \overline{W_i} = \pi_{int}(\Lambda) \cap W_i^\circ \neq \emptyset$ for $i = 1, \ldots, l$, and if $p_1, \ldots, p_l \in \mathfrak{G}$, then*

$$P := \bigcup_{i=1}^{l} (p_i + P(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W_i)) \tag{5.4}$$

is a regular cut-and-project set.

Proof: We observe that P is discrete and relatively dense in $\mathfrak{G}$ because of Lemma 5.1.

Without loss of generality we may assume that $l = 1$, $p_1 = 0$, and $0 \in W^\circ$ for $W := W_1$. Otherwise, for some $q \in \Lambda$ with $\pi_{\text{int}}(q) \in W_1^\circ$, we would replace $\mathfrak{G}_{\text{int}}$ by $\mathfrak{G}'_{\text{int}} := \mathbb{Z}^l \oplus \mathfrak{G}_{\text{int}}$, Λ by $\Lambda' := \Lambda + \{\, \vec{m} + \sum_{i=1}^l m_i p_i \mid \vec{m} \in \mathbb{Z}^l \,\} \subseteq \mathfrak{G} \oplus \mathbb{Z}^l \oplus \mathfrak{G}_{\text{int}}$, and $W_1 \ldots, W_l$ by $W' := -\pi_{\text{int}}(q) - \vec{e}_1 + \bigcup_{i=1}^l (\vec{e}_i + W_i)$ (where $\vec{e}_1, \ldots, \vec{e}_l$ is the canonical basis of $\mathbb{Z}^l$). It is clear that $P = p_1 + \pi(q) + P(\mathfrak{G}, \mathfrak{G}'_{\text{int}}, \Lambda', W')$, and in order to keep the notation readable, we suppress the primes again.

We now define $W' := \overline{\pi_{\text{int}}(\Lambda) \cap W}$, $\mathfrak{G}'_{\text{int}} := \langle W' \rangle$, and $\Lambda' := \Lambda \cap (\mathfrak{G} + \mathfrak{G}'_{\text{int}})$. Then, as W' is a neighbourhood of 0 in $\overline{\pi_{\text{int}}(\Lambda)}$, $\mathfrak{G}'_{\text{int}}$ is an open and, therefore, closed subgroup of $\overline{\pi_{\text{int}}(\Lambda)}$.

From the series of the obvious relations $\pi_{\text{int}}(\Lambda') \cap W' = \pi_{\text{int}}(\Lambda) \cap W' = \pi_{\text{int}}(\Lambda) \cap W = \pi_{\text{int}}(\Lambda) \cap W^\circ \subseteq \pi_{\text{int}}(\Lambda) \cap W'^{\circ'} = \pi_{\text{int}}(\Lambda') \cap W'^{\circ'}$, where $^{\circ'}$ means the interior relative to $\mathfrak{G}'_{\text{int}}$, we conclude that $\overline{W'^{\circ'}} = W'$, $\overline{\pi_{\text{int}}(\Lambda')} = \mathfrak{G}'_{\text{int}} = \langle W' \rangle$, $\pi_{\text{int}}(\Lambda') \cap W' = \pi_{\text{int}}(\Lambda') \cap W'^{\circ'}$, and $P = P(\mathfrak{G}, \mathfrak{G}'_{\text{int}}, \Lambda', W')$. Again, we suppress the primes.

Because $\Lambda \cap \mathfrak{G}_{\text{int}}$ is closed and W is compact, the set $\tilde{W} := \Lambda \cap \mathfrak{G}_{\text{int}} + W$ is closed. Therefore, the subgroup $\mathfrak{H} := \{\, c \in \mathfrak{G}_{\text{int}} \mid c + \tilde{W} = \tilde{W} \,\}$ of $\mathfrak{G}_{\text{int}}$ is closed. Furthermore, $P = P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, \tilde{W})$. Let $\tilde{\pi}$ be the canonical projection $\mathfrak{G} \oplus \mathfrak{G}_{\text{int}} \to (\mathfrak{G} \oplus \mathfrak{G}_{\text{int}})/\mathfrak{H} = \mathfrak{G} \oplus (\mathfrak{G}_{\text{int}}/\mathfrak{H})$. We define $\mathfrak{G}'_{\text{int}} := \mathfrak{G}_{\text{int}}/\mathfrak{H}$, $\Lambda' := \tilde{\pi}(\Lambda)$, and $W' := \tilde{\pi}(W) = \tilde{\pi}(\tilde{W})$. By definition, $P = P(\mathfrak{G}, \mathfrak{G}'_{\text{int}}, \Lambda', W')$, $\Lambda' \cap \mathfrak{G}'_{\text{int}} = \{0\}$, $\overline{\tilde{\pi}_{\text{int}}(\Lambda')} = \mathfrak{G}'_{\text{int}}$ (for the canonical projection $\tilde{\pi}_{\text{int}} : \mathfrak{G} \oplus (\mathfrak{G}_{\text{int}}/\mathfrak{H}) \to \mathfrak{G}_{\text{int}}/\mathfrak{H}$), and $W'^\circ \neq \emptyset$. By Lemma 5.1b), Λ' is a lattice in $\mathfrak{G} \oplus \mathfrak{G}'_{\text{int}}$, and it is now obvious that $(\mathfrak{G}, \mathfrak{G}'_{\text{int}}, \Lambda', W')$ is a standard projection strip. $\square$

For an *arbitrary* non-void subset P of $\mathfrak{G}$ we define $\Gamma(P) := \langle P - P \rangle$ and, for compact subsets K of $\mathfrak{G}$, $T_K(P) := \{\, t \in \Gamma(P) \mid t + (P \cap K) = P \cap (t + K) \,\}$.

From the definition, $T_K(P)$ is the set of all translations in $\mathfrak{G}$ which map the patch $P \cap K$ onto another patch of P (if $P \cap K \neq \emptyset$). In the case of regular cut-and-projection sets, these sets are rather non-trivial, as shown in

Lemma 5.4 *Assume that $(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$ is a standard projection strip and let $P := P(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$. Then, for compact $K \subseteq \mathfrak{G}$, we have*

$$T_K(P) = P(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W_K),$$

where

$$\begin{aligned} W_K \;\; := \;\; & \bigcap \{\, -\pi_{int}(q) + W \mid q \in \Lambda \cap (K + W) \,\} \\ & \setminus \bigcup \{\, -\pi_{int}(q) + W \mid q \in \Lambda \cap (K + \mathfrak{G}_{int} \setminus W) \,\} \end{aligned} \tag{5.5}$$

is a neighbourhood of 0 *in $\mathfrak{G}_{int}$. Furthermore, every neighbourhood of* 0 *in $\mathfrak{G}_{int}$ contains such a set W_K for some compact $K \subseteq \mathfrak{G}$.*

Proof: From the definitions it follows that $T_K(P) \subseteq \pi(\Lambda)$. Obviously, for $q_0 \in \Lambda$, we have $\pi(q_0) \in T_K(P)$ if and only if $\pi_{\text{int}}(q_0) \in -\pi_{\text{int}}(q) + W$ for all $q \in \Lambda \cap (K + W)$ and $\pi_{\text{int}}(q_0) \notin -\pi_{\text{int}}(q) + W$ for all $q \in \Lambda \cap (K + \mathfrak{G}_{\text{int}} \setminus W)$. This immediatly results in $T_K(P) = P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W_K)$ for W_K given by (5.5).

Now, because $\pi_{\text{int}}(\Lambda) \cap \partial W = \emptyset$, 0 is in the interior of every $-\pi_{\text{int}}(q) + W$ with $q \in \Lambda \cap (K + W)$ and in the interior of every $-\pi_{\text{int}}(q) + \mathfrak{G}_{\text{int}} \setminus W$ with $q \in \Lambda \cap (K + \mathfrak{G}_{\text{int}} \setminus W)$. As each compact neighbourhood of 0 in $\mathfrak{G}_{\text{int}}$ intersects only finitely many $-\pi_{\text{int}}(q) + W$ with $q \in \Lambda \cap (K + \mathfrak{G}_{\text{int}})$, because Λ is discrete, we conclude that W_K is a neighbourhood of 0.

Let, now, U be an arbitrary open neighbourhood of 0 in $\mathfrak{G}_{\text{int}}$. In a moment we will show that the intersection of all $-\pi_{\text{int}}(q) + W$ with $\pi_{\text{int}}(q) \in W$, $q \in \Lambda$ contains only 0. Then, the set $\bigcap \{\, (-\pi_{\text{int}}(q) + W) \setminus U \mid q \in \Lambda,\ \pi_{\text{int}}(q) \in W \,\}$ is void, and, because all $(-\pi_{\text{int}}(q)+W)\backslash U$ are compact, there is a finite subset F of Λ such that $\pi_{\text{int}}(F) \subseteq W$ and $U \supseteq \bigcap\{\,(-\pi_{\text{int}}(q)+W)\backslash U \mid q \in F\,\} \supseteq W_K$ for $K := \pi(F)$.

In order to complete the proof, let us assume that $c \in \bigcap\{\,-\pi_{\text{int}}(q)+W) \mid q \in \Lambda,\ \pi_{\text{int}}(q) \in W\,\}$. This amounts to $W \cap \pi_{\text{int}}(\Lambda) \subseteq -c+W$, hence, $W \subseteq -c+W$, because $W = \overline{W^\circ}$ and $\overline{\pi_{\text{int}}(\Lambda)} = \mathfrak{G}_{\text{int}}$. One concludes for every strictly positive $n \in \mathbb{N}$ that $nc+W \subseteq c+W \subseteq W$. As W is compact, the sequence $(nc)_{n\in\mathbb{N}}$ has adherence points, and one finds for each neighbourhood V of 0 in $\mathfrak{G}_{\text{int}}$ some strictly positive $n \in \mathbb{N}$ such that $nc \in V$. It follows that $W \subseteq c+W-V \subseteq W-V$ for each such V, therefore, $c+W=W$, and this means that $c=0$ according to our definition of standard projection strips. □

6 Existence of Standard Projection Strips

The last lemma of the previous section provides us with a major clue how a criterion for the existence of a cut-and-project scheme for a given set could possibly look like. Namely, the only thing to do is to study the topology of $\mathfrak{G}_{\text{int}}$ as it shows up in the cut-and-project set itself.

Let, for this purpose, P be a non-void subset of $\mathfrak{G}$. We define a topology on $\Gamma(P)$ by declaring

$$\{\,t+T_K(P) \mid K \subseteq \mathfrak{G} \text{ compact}\,\} \tag{6.1}$$

to be a basis of neighbourhoods of the point $t \in \Gamma(P)$. In fact, one has $0 \in T_K(P)$ for every compact $K \subseteq \mathfrak{G}$, furthermore, $T_{K_1}(P) \cap T_{K_2}(P) = T_{K_1\cup K_2}(P)$ for all compact $K_1, K_2 \subseteq \mathfrak{G}$, and one calculates for compact $K \subseteq \mathfrak{G}$ and $t \in T_K(P)$ that $T_K(P) = t + T_{t+K}(P)$. This topology is obviously translationally invariant, and we just have calculated that the elements of the sets (6.1) are *open* neighbourhoods of the point $t \in \Gamma(P)$. Let us call this topology the P-topology on $\Gamma(P)$.

In general, the P-topology does not make a topological group out of $\Gamma(P)$. On the contrary, we have the following

Theorem 2 *Let P be a non-void subset of the locally compact Abelian group $\mathfrak{G}$. Then, P is a regular cut-and-project set in $\mathfrak{G}$ if and only if the following conditions hold:*

a): *$P-P$ is discrete,*

b): *$T_K(P)$ is relatively dense in $\mathfrak{G}$, for every compact $K \subseteq \mathfrak{G}$,*

c): *For every compact $K \subseteq \mathfrak{G}$, there is a compact $K' \subseteq \mathfrak{G}$ such that*

$$T_{K'}(P) - T_{K'}(P) \subseteq T_K(P). \tag{6.2}$$

If $P = t_0 + P(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$ for some $t_0 \in \mathfrak{G}$ and a standard projection strip $(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$, then $\Gamma(P) = \pi(\Lambda)$, the P-topology turns $\Gamma(P)$ into a topological group, and $\mathfrak{G}_{int}$ is a completion of $\Gamma(P)$ via the canonical homomorphism

$$\varphi : \Gamma(P) \to \mathfrak{G}_{int},\ \ \pi(q) \mapsto \pi_{int}(q)\ \ (q \in \Lambda). \tag{6.3}$$

For every $t \in P(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$, the window W is determined by

$$W = \overline{\varphi(t + T_t(P))}. \tag{6.4}$$

Proof: Let P be a regular cut-and-project set in $\mathfrak{G}$, without loss of generality $P = P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W)$ for some standard projection strip $(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W)$. With the help of Lemma 5.1a), we conclude that $P - P \subseteq P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W-W)$ is discrete, and that $T_K(P) = P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W_K)$ is relatively dense in $\mathfrak{G}$ (W_K being given by (5.5)). If $K \subseteq \mathfrak{G}$ is compact, then, according to Lemma 5.4, there is a neighbourhood U of 0 in $\mathfrak{G}_{\text{int}}$ with $U - U \subseteq W_K$, and, again with Lemma 5.4, we find a compact $K' \subseteq \mathfrak{G}$ such that $W_{K'} \subseteq U$. Then, $T_{K'}(P) - T_{K'}(P) \subseteq P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W_{K'} - W_{K'}) \subseteq P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W_K) = T_K(P)$. We have seen that P fulfils conditions a), b), and c).

Let $q \in \Lambda$ such that $\pi(q) \in P$, i.e., $\pi_{\text{int}}(q) \in W$. We have $\langle P - P \rangle = \langle P - \pi(q) \rangle$ and $\mathfrak{G}_{\text{int}} = \langle W - W \rangle = \langle W - \pi_{\text{int}}(q) \rangle$. Furthermore, $P - \pi(q) = \pi(\Lambda \cap (\mathfrak{G} + W - \pi_{\text{int}}(q)))$. If we define $\tilde{\Lambda}$ to be the subgroup of Λ generated by $\Lambda \cap (\mathfrak{G} + W - \pi_{\text{int}}(q))$, then $\pi_{\text{int}}(\Lambda) \cap (W - \pi_{\text{int}}(q)) = \pi_{\text{int}}(\tilde{\Lambda}) \cap (W - \pi_{\text{int}}(q))$, and, therefore, $\overline{\pi_{\text{int}}(\tilde{\Lambda})} = \langle W - W \rangle = \mathfrak{G}_{\text{int}}$, from which we conclude that $\pi_{\text{int}}(\Lambda) \subseteq W + \pi_{\text{int}}(\tilde{\Lambda})$, hence, $\pi_{\text{int}}(\tilde{\Lambda}) = \pi_{\text{int}}(\Lambda)$. From the definition of $\tilde{\Lambda}$, we have $\Lambda \cap \mathfrak{G} \subseteq \tilde{\Lambda}$. Therefore, the subgroup $\tilde{\Lambda}$ of Λ contains the kernel of π_{int} restricted to Λ. Therefore, from $\pi_{\text{int}}(\tilde{\Lambda}) = \pi_{\text{int}}(\Lambda)$ we may conclude that $\Lambda = \tilde{\Lambda}$, and, from the definition of $\tilde{\Lambda}$, we see that indeed $\langle P - P \rangle = \pi(\Lambda)$.

Precisely by condition c), $\Gamma(P)$ with the P-topology becomes a topological group. The kernel of φ as given by (6.3) is $\Lambda \cap \mathfrak{G}$, and this, in turn, is the translation group of P, namely, $\{ t \in \mathfrak{G} \mid t + P = P \}$, because, if $t + P = P$, then $t = \pi(q)$ for some $q \in \Lambda$ for which $\pi_{\text{int}}(q) + W = W$ must hold, i.e., $\pi_{\text{int}}(q) = 0$ as $(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W)$ is a standard projection strip. It is immediate that $\{ t \in \mathfrak{G} \mid t + P = P \}$ is the intersection of all $T_K(P)$ ($K \subseteq \mathfrak{G}$ compact) which is nothing but the closure of 0 in $\Gamma(P)$ with respect to the P-topology. If we now observe with the help of Lemma 5.4 that φ induces a continuous and open homomorphism of $\Gamma(P)$ with kernel $\overline{\{0\}}$ (closure relative to $\Gamma(P)$) onto the dense subgroup $\pi_{\text{int}}(\Lambda)$ of $\mathfrak{G}_{\text{int}}$, we see that indeed $\varphi : \Gamma(P) \to \mathfrak{G}_{\text{int}}$ is a completion of $\Gamma(P)$.

Equation (6.4) is an immediate consequence of the definitions.

Let, on the other hand, P be an arbitrary non-void subset of $\mathfrak{G}$ fulfilling conditions a), b), and c). As we already noted, condition c) says that $\Gamma(P)$ is a topological group when equipped with the P-topology. In the sequel, every topological notion in connection with $\Gamma(P)$ is to be understood relative to the P-topology.

For $p \in P$, the set $T_{\{p\}}(P)$ is open, closed, and totally bounded: We already know that $T_{\{p\}}(P)$ is open. If $t \in \Gamma(P)$, then $t \in \Gamma(P) \backslash T_{\{p\}}(P)$ if and only if $t + p \notin P$. Either $\Gamma \backslash T_{\{p\}}(P) = \emptyset$, and then $T_{\{p\}}(P)$ is trivially closed, or, e.g., $t_0 \in \Gamma(P)$ such that $t_0 + p \notin P$, and one calculates that $\Gamma \backslash T_{\{p\}}(P) = t_0 + T_{\{t_0 + p\}}(P)$ is open, i.e., $T_{\{p\}}(P)$ is closed.

In order to show that $T_{\{p\}}(P)$ is totally bounded, let K be a compact subset of $\mathfrak{G}$ and set $K_0 := \{p\} \cup K$. Because of condition b), we find a compact $K' \subseteq \mathfrak{G}$ such that $\mathfrak{G} = K' + T_{K_0}(P)$. The set $F := (P - P) \cap (-p + K')$ is finite according to condition a), and we will show that $T_{\{p\}}(P) \subseteq F + T_{K_0}(P)$ which implies $T_{\{p\}}(P) \subseteq F + T_K(P)$ and, therefore, that $T_{\{p\}}(P)$ is totally bounded. Let $t_0 \in T_{\{p\}}(P)$ and, hence, $t_0 + p \in P$, let, furthermore, $t \in T_{K_0}(P)$ with $t_0 + p \in t + K'$. Then, $t_0 - t \in -p + K'$ and $t + (P \cap K_0) = P \cap (t + K_0)$, especially, $t + p \in P$. As a consequence, $t_0 - t = t_0 + p - (t + p) \in P - P$, so $t_0 - t \in F$, and it follows that $t_0 = t_0 - t + t \in F + T_{K_0}(P)$ which was the goal.

Now, there is a complete Hausdorff group which we call $\mathfrak{G}_{\text{int}}$ and a continuous homomorphism $\Gamma(P) \to \mathfrak{G}_{\text{int}}$ which we call φ such that $(\mathfrak{G}_{\text{int}}, \varphi)$ is a completion of $\Gamma(P)$. That is, the kernel of φ is the closure of $\{0\}$ in $\Gamma(P)$ which is the intersection of all $T_K(P)$ for compact $K \subseteq \mathfrak{G}$ which, in turn, is the set $\{ t \in \mathfrak{G} \mid t + P = P \}$. Furthermore, φ induces an open mapping of $\Gamma(P)$ onto the dense subgroup $\varphi(\Gamma(P))$ of $\mathfrak{G}_{\text{int}}$.

The first observation we conclude from this is that, for $p \in P$ and $t \in \Gamma(P)$, we have $\varphi(t) \in \varphi(T_{\{p\}}(P))$ if and only if $t \in T_{\{p\}}(P)$. The second observation is that, if we set $W := \overline{\varphi(T_{\{p\}}(P))}$ (meaning the closure in $\mathfrak{G}_{\mathrm{int}}$), then, $W \cap \varphi(T_{\{p\}}(P)) = W^\circ \cap \varphi(T_{\{p\}}(P))$, because $T_{\{p\}}(P)$ is open and closed in $\Gamma(P)$. Especially, $\partial W \cap \varphi(T_{\{p\}}(P)) = \emptyset$. Thirdly, W is a compact subset of $\mathfrak{G}_{\mathrm{int}}$ with non-void interior; it is compact, because it is the closure of the totally bounded set $\varphi(T_{\{p\}}(P))$ in a complete Hausdorff space. Therefore, $\mathfrak{G}_{\mathrm{int}}$ is a locally compact Abelian group.

Let us fix some $p \in P$ and define W as in the preceding paragraph. We define Λ to be the subgroup $\{\, t+\varphi(t) \in \mathfrak{G} \oplus \mathfrak{G}_{\mathrm{int}} \mid t \in \Gamma(P)\,\}$ of $\mathfrak{G} \oplus \mathfrak{G}_{\mathrm{int}}$. Then, $-p+P = T_{\{p\}}(P) = P(\mathfrak{G}, \mathfrak{G}_{\mathrm{int}}, \Lambda, W)$. Because of conditions a) and b), $P(\mathfrak{G}, \mathfrak{G}_{\mathrm{int}}, \Lambda, W)$ is discrete and relatively dense in $\mathfrak{G}$, and Lemma 5.1b) now guarantees that Λ is a lattice in $\mathfrak{G} \oplus \mathfrak{G}_{\mathrm{int}}$. Therefore, all conditions of Proposition 5.3 are fulfilled, and we see that P is in fact a regular cut-and-project set (by the way, it could be calculated directly that $(\mathfrak{G}, \mathfrak{G}_{\mathrm{int}}, \Lambda, W)$ is a standard projection strip).

This concludes the proof of the theorem. □

It is now easy to give a preliminary answer to the question of the uniqueness of cut-and-project schemes.

Corollary 6.1 *If $(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$ and $(\mathfrak{G}, \mathfrak{G}'_{int}, \Lambda', W')$ are standard projection strips such that $P(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W) = P(\mathfrak{G}, \mathfrak{G}'_{int}, \Lambda', W')$, then there is a unique topological isomorphism $\psi : \mathfrak{G}_{int} \to \mathfrak{G}'_{int}$ such that $\psi(\pi_{int}(q)) = \pi'_{int}(q')$ for all $q \in \Lambda$, $q \in \Lambda'$ with $\pi(q) = \pi'(q')$ and $\psi(W) = W'$.*

Proof: According to the previous theorem, $\varphi : \Gamma(P) \to \mathfrak{G}_{\mathrm{int}}$ and $\varphi' : \Gamma(P) \to \mathfrak{G}'_{\mathrm{int}}$ given by (6.3) and its primed analogue are completions of $\Gamma(P)$ with respect to the P-topology, where we have set $P := P(\mathfrak{G}, \mathfrak{G}_{\mathrm{int}}, \Lambda, W)$. The assertions of the corollary are just a restatement of the uniqueness of the completion of $\Gamma(P)$ up to a unique topological isomorphism and of Equation (6.4). □

7 Local Uniqueness of Standard Projection Strips

Now we want to refine the last result of the previous section. What we have shown so far is that a standard projection strip is (essentially) uniquely determined by its regular cut-and-project set. This statement may be considerably sharpened to the effect that already the finite patches of a cut-and-project set suffice to reconstruct its standard projection strip. Put in a different way, we will show that a standard projection strip is uniquely determined not only by its regular cut-and-project set, but rather by the local isomorphism class of the latter. Recall that two subsets P_1, P_2 of $\mathfrak{G}$ are called locally isomorphic iff, for each compact subset K of $\mathfrak{G}$, there are elements t_1 and t_2 of $\mathfrak{G}$ such that $P_1 \cap K = (t_2 + P_2) \cap K$ and $P_2 \cap K = (t_1 + P_1) \cap K$. The equivalence class of a set P with respect to this relation is dubbed the local isomorphism (LI) class of P [**7**].

The first thing we have to investigate is how the LI class of a regular cut-and-project set $P(\mathfrak{G}, \mathfrak{G}_{\mathrm{int}}, \Lambda, W)$ shows up in the projection scheme. A glance at Theorem 2 reveals that it is not a priori clear that every element of the LI class of $P(\mathfrak{G}, \mathfrak{G}_{\mathrm{int}}, \Lambda, W)$ is itself a regular cut-and-project set. In fact, it turns out that this is not true in general (this we will not show in the present paper but we have to be aware of this situation).

The next lemma points to the possible failure of the criterion given in Theorem 2.

Lemma 7.1 *Let P_1 and P_2 be locally isomorphic subsets of $\mathfrak{G}$. Then, for every compact subset K of $\mathfrak{G}$, there is a point $t \in \mathfrak{G}$ such that*

$$T_K(P_1) - T_K(P_1) = T_{t+K}(P_2) - T_{t+K}(P_2). \tag{7.1}$$

Proof: Let $t \in \mathfrak{G}$ such that $P_1 \cap K = (-t + P_2) \cap K$. If t_1, t_2 are in $T_K(P_1)$, then we find some compact $K' \subseteq \mathfrak{G}$ which encloses $t_1 + K$ and $t_2 + K$, and, then some $t' \in \mathfrak{G}$ such that $P_1 \cap K' = (t' + P_2) \cap K'$. We calculate that for $i = 1, 2$: $t_i - t + P_2 \cap (t+K) = t_i + (-t+P_2) \cap K = t_i + P_1 \cap K = P_1 \cap (t_i + K) = (t' + P_2) \cap K = t' + P_2 \cap (t_i - t - t' + t + K)$, and it follows that $t_i - t - t' \in T_{t+K}(P_2)$. Therefore, $T_K(P_1) - T_K(P_1) \subseteq T_{t+K}(P_2) - T_{t+K}(P_2)$, and the reverse relation follows by symmetry. □

From this lemma, it is clear that, if P is a regular cut-and-project set, and P' is locally isomorphic to P, then P' automatically fulfils condition a) of Theorem 2; furthermore, from the definition of local isomorphism, P' must also fulfil b). We call those elements of the LI class of a regular cut-and-project set which fulfil condition c) of Theorem 2 and which are, therefore, regular cut-and-project sets, the *regular* elements of the LI class, whereas the remaining ones will be called *singular*.

The desired uniqueness theorem will be a consequence of

Lemma 7.2 *Let $(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$ be a standard projection strip and let $P \subseteq \mathfrak{G}$ be locally isomorphic to $P(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$. Then, P is a regular element of its LI class and $P \subseteq \langle P - P \rangle$ if and only if there is a point $c \in \mathfrak{G}_{int}$ such that $(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, c + W)$ is a standard projection strip and $P = P(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, c + W)$. In this case, c is uniquely determined by P.*

Proof: Assume that $P \subseteq \langle P - P \rangle$ is a regular element of its LI class. As a consequence of Lemma 7.1, it follows that the P-topology on $\Gamma(P) = \langle P - P \rangle = \langle P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W) - P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W) \rangle = \Gamma(P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W))$ coincides with the $P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W)$-topology. The uniqueness of the completion of $\Gamma(P)$ up to topological isomorphism provides us with a standard projection strip $(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W')$ such that $P = P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W')$. Because $P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W')$ and $P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W)$ are locally isomorphic, for every finite $F \subseteq \pi_{\text{int}}(\Lambda) \cap W$ there is a $q_F \in \Lambda$ such that $\pi_{\text{int}}(q_F) + F \subseteq W'$. If $F \neq \emptyset$, then $\pi_{\text{int}}(q_F) \in W' - W$ which is a compact set. Let c be an adherence point of the net $(\pi_{\text{int}}(q_F))_F$, where F runs over all finite subsets of $\pi_{\text{int}}(\Lambda) \cap W$. Then, for every neighbourhood U of 0 in $\mathfrak{G}_{\text{int}}$ and every F, there is an $F' \supseteq F$ such that $\pi_{\text{int}}(q_{F'}) - c \in U$. This results in $c + F \subseteq W' + U$. Because W and W' are compact, we have $c + W \subseteq W'$. Symmetrically, we find a $c' \in \mathfrak{G}_{\text{int}}$ such that $c' + W' \subseteq W$, and we would have $c + c' + W \subseteq W$ which is possible only for $c = -c'$, according to the proof of Lemma 5.4.

Let, on the other hand, $c \in \mathfrak{G}_{\text{int}}$ be such that $(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, c + W)$ is a standard projection strip and $P = P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, c + W)$. For arbitrary compact $K \subseteq \mathfrak{G}$, we know from Lemma 5.4 that W_K given by (5.5) is a neighbourhood of 0 in $\mathfrak{G}_{\text{int}}$. If we choose $q_0 \in \Lambda$ such that $\pi_{\text{int}}(q_0) - c \in W_K$, from the expression (5.5) we calculate that, if $q \in \Lambda$ and $\pi(q) \in P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W) \cap K$, then $\pi_{\text{int}}(q) \in c - \pi_{\text{int}}(q_0) + W$, and, if $q \in \Lambda$ and $\pi(q) \in K \backslash P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W)$, then $\pi_{\text{int}}(q) \notin c - \pi_{\text{int}}(q_0) + W$. This means that $P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W) \cap K = P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, c - \pi_{\text{int}}(q_0) + W) \cap K = (-\pi(q_0) + P) \cap K$. For reasons of symmetry we conclude that P is locally isomorphic to $P(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \Lambda, W)$, and it is clear that P is a regular element of its LI class.

The uniqueness of c follows directly from the expression for the window (6.2) in Theorem 2. □

The announced uniqueness result is now an immediate consequence of combining the previous lemma with Corollary 6.1 of Theorem 2:

Theorem 3 *Let $(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$ and $(\mathfrak{G}, \mathfrak{G}'_{int}, \Lambda', W')$ be standard projection strips. Then, $P(\mathfrak{G}, \mathfrak{G}_{int}, \Lambda, W)$ and $P(\mathfrak{G}, \mathfrak{G}'_{int}, \Lambda', W')$ are locally isomorphic if and only if there is a topological isomorphism $\psi : \mathfrak{G}_{int} \to \mathfrak{G}'_{int}$ and a point $c \in \mathfrak{G}_{int}$*

such that $\psi(\pi_{int}(q)) = \pi'_{int}(q')$ *for all* $q \in \Lambda$, $q \in \Lambda'$ *with* $\pi(q) = \pi'(q')$ *and* $\psi(c + W) = W'$. *In this case,* ψ *and* c *are uniquely determined.*

□

Without going into details, let us remark that this uniqueness result provides the one-to-one correspondence of (generalized) point symmetries of regular cut-and-project sets with point symmetries of the lattice which leave the internal space and the window fixed.

One might ask the question how the singular elements of the LI class of a regular cut-and-project set may be incorporated in the projection formalism. In fact, Lemma 7.2 promises that they cannot be generated without either relaxing the conditions on standard projection strips or altering the projection formalism itself. It turns out [**11**] that indeed a reformulation of the formalism provides a clean way to handle the singular sets, but, as this does not really change the results we have presented and gives rise to rather more complicated formulas instead, we have refrained from exposing it here.

8 Conclusion

Of course, the material presented in this article leaves a lot of interesting questions open. Concerning the density formula, it would be of value to know if the condition that the window has almost no boundary turns out to be a necessary one at least for the uniformity of the limit in Equation (4.1). Trivially, there are subsets of the internal space which would violate the density formula when taken as windows (e.g., the internal projection of the lift of a cut-and-project set in its lattice). Less trivially, it is possible to provide examples of nowhere dense compact windows of strictly positive volume. Then, the resulting cut-and-project sets have arbitrarily large empty regions. So, the question of interest would be: given a compact window W in the internal space such that $\overline{W^\circ} = W$ and $\mu_{\rm int}(\partial W) > 0$, does the density formula (2.4) *always* fail in this situation?

Concerning the criterion on the existence of cut-and-project schemes, it is clear that the condition c) in Theorem 2 is not a very handy one. As a matter of fact, it has to be as non-local as it stands, because it singles out the *regular* elements of an LI class thereby neclecting what could be, in a certain sense and under some weird conditions, the vast majority in this class. The task, therefore, would be to find a criterion for the LI class as a whole to be generated by a standard projection strip. As a result, one would obtain a more direct approach to the question whether certain point sets which are generated by substitution rules are cut-and-project sets, avoiding some difficulties in the approach via Fourier transforms.

Finally, it would be important to characterize those sets where the internal space is sufficiently well-behaved, i.e., a finite product of vector spaces, tori, and discrete goups. At least, from Theorem 2, one easily concludes that, for σ-compact direct spaces, the internal space can always be chosen to be second countable, esp., metrizable. Furthermore, an application of the structure theorem on locally compact Abelian groups shows that regular cut-and-project sets may be approximated from by regular cut-and-project sets with well-behaved internal spaces (see [**9**]).

Acknowledgement

It is a pleasure to thank M. Baake, A. Hof, and R. V. Moody for a series of inspiring discussions on the subject exposed in the present paper. The hospitality at The Fields Institute, Toronto, is gratefully acknowledged.

Appendix A Appendix Topological Abelian Groups

For the reader's convenience, we collect some definitions and facts on topological Abelian groups.

A topological Abelian group is a commutative group $\mathfrak{G}$ equipped with a (not necessarily Hausdorff) topology such that the map $\mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$, $(x, y) \mapsto x - y$ becomes continuous. Such a topology on $\mathfrak{G}$ may always be defined by taking as a basis of neighbourhoods of 0 a non-void set $\mathfrak{U}$ of non-void subsets of $\mathfrak{G}$ with the property that for every $U_1, U_2 \in \mathfrak{U}$, there is some $U_3 \in \mathfrak{U}$ with $U_3 \subseteq U_1 \cap U_2$, and for every $U \in \mathfrak{U}$, there is a $V \in \mathfrak{U}$ such that $V - V \subseteq U$.

The quotient of a topological Abelian group $\mathfrak{G}$ with respect to a subgroup $\mathfrak{H}$ is topologized by taking as neighbourhoods of 0 in $\mathfrak{G}/\mathfrak{H}$ the images $\pi(U)$ of all neighbourhoods U of 0 in $\mathfrak{G}$ under the canonical projection $\pi : \mathfrak{G} \to \mathfrak{G}/\mathfrak{H}$. The projection homomorphism π then becomes a continuous open mapping.

Observing that the topology of a topological Abelian group is in fact created by a uniform structure, there exist the notions of Cauchy nets (and Cauchy filters) and completeness. We did not need the actual definitions of these terms but only the following facts which have familiar analogues in elementary analysis. Every topological Abelian group can be "completed" (see, e.g., [**1**], chap. II,§ 3, no. 7&8, and chap. III, § 3 no. 4&5):

Completion of Topological Abelian Groups *For an arbitrary topological Abelian group $\mathfrak{G}$ there is a topological Abelian group $\hat{\mathfrak{G}}$ which is Hausdorff and complete, and a continuous homomorphism $\varphi : \mathfrak{G} \to \hat{\mathfrak{G}}$ whose image is dense in $\hat{\mathfrak{G}}$ and whose kernel is the closure of $\{0\}$ in $\mathfrak{G}$, such that φ induces an open mapping of $\mathfrak{G}$ onto $\varphi(\mathfrak{G})$.*

If $\varphi' : \mathfrak{G} \to \hat{\mathfrak{G}}'$ is another homomorphism of $\mathfrak{G}$ into a complete Hausdorff Abelian goup with the same properties, then there exists a unique topological isomorphism $\psi : \hat{\mathfrak{G}} \to \hat{\mathfrak{G}}'$ such that $\varphi' = \psi \circ \varphi$.

The pair $(\hat{\mathfrak{G}}, \varphi)$ is called a completion of $\mathfrak{G}$.

A subset W of a topological Abelian group $\mathfrak{G}$ is totally bounded iff, for every neighbourhood U of 0 in $\mathfrak{G}$, there is a finite subset F of $\mathfrak{G}$ such that $W \subseteq F + U$. The closure of a totally bounded subset of a complete Hausdorff Abelian group is compact. If a topological group possesses a totally bounded neighbourhood of 0, then its completion is a locally compact group. Locally compact Abelian groups are complete (we understand the term "locally compact" as including the Hausdorff separation property).

The structure of locally compact Abelian groups is very well investigated, e.g., one has the following theorem which we cite from ([**3**], Thm. 9.6, p. 89):

Structure Theorem on Locally Compact Abelian Groups *Let $\mathfrak{G}$ be a locally compact, compactly generated Abelian group. Every neighbourhood U of 0 contains a compact subgroup $\mathfrak{H}$ such that $\mathfrak{G}/\mathfrak{H}$ is topologically isomorphic with $\mathbb{T}^a \times \mathbb{R}^b \times \mathbb{Z}^c \times F_1$, where a, b, and c are non-negative integers and F_1 is a finite Abelian group.*

References

[1] Bourbaki, N. [1966], *General Topology I*. Addison-Wesley, Reading.

[2] Duneau, M. and Katz, A. [1985], *Quasiperiodic Patterns*, Phys. Rev. Lett., **54**, 2688–2691.

[3] Hewitt, E. and Ross, K. A. [1979], *Abstract Harmonic Analysis I*, Springer, New York.

[4] Hof, A. [1995], *Uniform distribution and the projection method*, preprint.

[5] Kalugin, P. A., Kitayev, A. Y. and Levitov, L. S. [1985], *6-Dimensional Properies of AlMn alloy*, J. Physique Lett., **46**, L601–L607.

[6] Kramer, P. and Neri, R. [1984], *On Periodic and Non-Periodic Space Fillings of E^n Obtained by Projection*, Acta Cryst., **A40**, 580–587.
[7] Levine, D. and Steinhardt, P. J. [1986], *Quasicrystals. I. Definition and Structure*, Phys. Rev., **B34**, 596–616.
[8] Meyer, Y. [1972], *Algebraic Numbers and Harmonic Analysis*, North-Holland, Amsterdam.
[9] Moody, R. V. [1997], *Meyer Sets and their Duals*, in: Moody, R. V. (ed), *The Mathematics of Long-Range Aperiodic Order*, Kluwer Academic Publishers, Dordrecht.
[10] Oguey, C., Duneau, M. and Katz, A. [1988] *A Geometrical Approach of Quasiperiodic Tilings*, Commun. Math. Phys., **118**, 99–118.
[11] Schlottmann, M. [1993] *Geometrische Eigenschaften quasiperiodischer Strukturen*, Dissertation, Tübingen.