# Self-Similar Measures for Quasicrystals

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*Dedicated to Peter A. B. Pleasants on the occasion of his 60th birthday.*

Abstract. We study self-similar measures of Hutchinson type, defined by compact families of contractions, both in a single and multi-component setting. The results are applied in the context of general model sets to infer, via a generalized version of Weyl's Theorem on uniform distribution, the existence of invariant measures for families of self-similarities of regular model sets.

## Introduction

There have been two very successful approaches to generate aperiodic sets with features of long-range internal order. The first is by creating tilings by the method of inflation followed by decomposition using a finite set of proto-tiles. The second is by creating point sets through the method of cut and project sets, or model sets as we call them here. Neither theory subsumes the other and they both have their own particular virtues. However, they have a considerable overlap. It is easy to replace a tiling by an equivalent set of points (by selecting suitable points from each type of tile) and in many cases the result is a model set or, more generally, a union of several model sets, one for each type of point. Conversely, there are many ways to obtain a tiling from a point set (for instance by using the Voronoi cells, or the dual Delone cells determined by them), and the equivalence concept of mutual local derivability, see [**B, P**] and references given there, is an adequate tool to make this connection precise. So it is natural to study the most notable feature of inflation tilings, namely their self-similarity, in the context of model sets, and indeed, even when no simple tiling is in sight, many interesting model sets have striking self-similarity.

The objective of this paper is to set up some of the machinery that makes such a study possible and to show how naturally it can be associated with families of self-similar measures on locally compact Abelian groups.

There is one initial hurdle which is not usually considered in the study of model sets. In order to have any sort of reasonable correspondence with the tiling world, and to create a useful theory, we need to have a *multi-component* model set context in which there are a finite number of model sets, all based on the same cut and project scheme, that are mutually coupled by the self-similarities. After all, almost

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all inflation tiling systems have several types of tiles, and the decomposition of inflated tiles typically involves all of these various types simultaneously.

Thus the situation that we envision consists of a *family* of model sets $\Lambda_1, \ldots, \Lambda_n$ in some real space $\mathbb{R}^m$, all based on the same cut and project scheme, and a set of families of inflationary mappings $F_{ij}$ on $\mathbb{R}^m$ with the property that

$$\Lambda_i \;=\; \bigcup_{j=1}^{n} \bigcup_{f\in F_{ij}} f(\Lambda_j)\,, \qquad 1 \le i \le n\,. \tag{0.1}$$

There are two new features that are different from the tiling situation. In inflation tilings, the usual idea is that the tiles only overlap on their boundaries which are of measure 0. That would be equivalent to some form of disjointness in Eq. (0.1) which we do *not* wish to assume. Secondly, in tilings, the sets $F_{ij}$ of mappings are assumed finite. Again, we do *not* wish to make this assumption. In fact, it is quite useful to make $F_{ij}$ consist of *all* possible mappings of a certain type (for example all affine mappings) that are consistent with Eq. (0.1), and this is in general an infinite set.

As soon as overlapping is allowed, there naturally arises the question of whether there is implicit in Eq. (0.1) a corresponding set of relative weights which measure the "frequency" of occurrence of the points of $\Lambda_i$ coming from the substitution process (0.1). If we were counting, this could be loosely construed as noting the occurrences of points of $\Lambda_i$ with their multiplicities as they appear in the righthand side of (0.1). This links up to a recent approach of Lagarias and Wang [**LW**]. However, in our situation, there is no reason to assume that our weights can be normalized to be integral nor that there are only a finite number of contributions to $\Lambda_i$ in the right-hand side of (0.1).

This then is the primary goal of this contribution: to discuss the existence and nature of self-similar densities on systems of model sets which are coupled together by a substitution system of type (0.1). Our definition of model sets is based on arbitrary locally compact Abelian groups as internal spaces, and so is more flexible and handles more situations (see for instance [**BMS, LM**]) than the usual method of projection from Euclidean spaces. This extra generality requires a little more care than usual but it is remarkable how much of the theory is natural in this context.

The method by which we attack the problem is to use the formalism of the underlying cut and project scheme to pass everything over to the internal side of the picture, i.e. onto the locally compact Abelian group that is controlling the projection. The advantage of doing this is that the system of inflationary mappings turns into a family of contractions, and what has started off as a problem in the domain of discrete mathematics turns into one of analysis. A primary virtue of systems of contractions is the ready-built Hutchinson theory of iterated function systems with their attractors and self-similar measures. In our multi-component setting with infinite families of mappings, we need a slight variation on this theme, which occupies the first five Sections of the paper. These parts of the paper have nothing in particular to do with model sets, but rather are a development of Hutchinson's theory in the multi-component situation where the coupling is by *compact* families of contractions each of which has its own, essentially arbitrarily pre-given positive Borel measure. An important part of this is determining some conditions under which the self-similar measures are in fact absolutely continuous and, more importantly, when the representing $L^1$-functions, the Radon-Nikodym densities, are actually continuous (continuously representable self-similar measures), for it is only then that we can bring back information to the discrete side again.

Section 6 of this article brings in the model sets and develops the mathematics that allows us to pass information back from the internal side to the model set side in $\mathbb{R}^m$. The primary tool here is Weyl's theory of uniform projection, but we have to redevelop this in the context of locally compact Abelian groups and model sets.

In Section 7 we are, at last, set to tackle the problem of determining self-similar distribution of weights (also called self-similar densities from now on) on the model sets themselves. The Weyl theory applies only to continuous functions, so we can only refer to it when our self-similar measures have continuously representable Radon-Nikodym densities. Fortunately, this is the case in a number of interesting situations. We provide a general description of the situations in which such self-similar densities exist, and then, in Section 8, we offer a number of examples which illustrate what we have achieved.

The results obtained here extend those of two previous papers [**BM1, BM2**]. There, we considered only model sets based on Euclidean internal spaces, and the context was not primarily measure-theoretical as it is here. The method of dealing with multi-component model sets also differed from the product approach that we have adopted in this article. There are nonetheless several examples of self-similar densities (called invariant densities there) in those papers which the reader may find of interest.

## 1. Compact families of contractions and attractors

Let us first review some basic facts from the theory of iterated function systems, both finite and compact, in a way that is adequate for our needs.

**1.1. Hutchinson's contraction principle.** Let $X$ be a complete metric space with metric $d$. We denote by $\mathcal{K}X$ the set of all non-empty compact subsets of $X$. Let $d(x,U) := \inf\{d(x,u) \mid u \in U\}$ be the distance of $x$ from $U$. Note that $d(x,U)=0$ implies $x \in \overline{U}$. For $\varepsilon > 0$, the $\varepsilon$-*fringe* of a subset $U \in \mathcal{K}X$ is

$$[U]_\varepsilon \;:=\; \{x \in X \mid d(x,U) < \varepsilon\}\,. \tag{1.1}$$

In view of the set $\mathcal{K}X$, we introduce the *Hausdorff metric*: for $U,V \in \mathcal{K}X$, it is defined by

$$d_H(U,V) \;:=\; \inf\{\varepsilon > 0 \mid U \subset [V]_\varepsilon \text{ and } V \subset [U]_\varepsilon\}\,. \tag{1.2}$$

Note that for singletons $U=\{u\}$ and $V=\{v\}$, one has $d_H(\{u\},\{v\}) = d(u,v)$. An alternative way to determine $d_H(U,V)$ is

$$d_H(U,V) \;=\; \sup\{d(u,V),\, d(v,U) \mid u \in U,\, v \in V\}\,.$$

Relative to the Hausdorff metric, $\mathcal{K}X$ is again a complete metric space. If now $U_j, V_j \in \mathcal{K}X$, $j \in J$, are two sets of compact subsets of $X$ then it is easy to see [**Wi**, Note 2.1.6] that

$$d_H\Big(\textstyle\bigcup_{j\in J} U_j, \bigcup_{j\in J} V_j\Big) \;\le\; \sup_{j\in J} d_H(U_j,V_j)\,. \tag{1.3}$$

Given complete metric spaces $X, Y$, we consider the space $C(X,Y)$ of all continuous mappings of $X$ into $Y$, endowed with the compact-open topology. This topology has the property of making the evaluation maps

$$\mathrm{eval}_x : C(X,Y) \longrightarrow Y \; ; \quad f \mapsto f(x) \tag{1.4}$$

continuous [**Kel**, Thm. 7.4]. There is a natural extension of mappings that leads to

$$\mathcal{K}(\centerdot) : C(X,Y) \longrightarrow C(\mathcal{K}X, \mathcal{K}Y) \tag{1.5}$$

with $(\mathcal{K}(f))(U) := f(U)$. This mapping is continuous [**Wi**, Prop. 2.5.1][1].

A mapping $f : X \longrightarrow Y$ of metric spaces is *Lipschitz* if there is an $r > 0$ with $d_Y(f(x_1), f(x_2)) \leq r\, d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. If $f$ is Lipschitz then the infimum of all such $r$ is the *Lipschitz constant* $r_f$ of $f$. If $r_f < 1$ then $f$ is called a *contraction.* The set of all Lipschitz maps from $X$ to $Y$ with Lipschitz constant equal to $r$ (resp. at most $r$) is denoted by $\mathrm{Lip}(r, X, Y)$ (resp. $\mathrm{Lip}(\leq r, X, Y)$). Lipschitz functions are clearly uniformly continuous mappings. We observe:

$$f \in \mathrm{Lip}(r, X, Y) \quad \Longrightarrow \quad \mathcal{K}(f) \in \mathrm{Lip}(r, \mathcal{K}X, \mathcal{K}Y)\,. \tag{1.6}$$

Note in particular that $r_f = r_{\mathcal{K}(f)}$, as is seen by looking at singleton sets.

We are interested in *union maps.* Let $X, Y$ be complete metric spaces and let $F = \{f_1, f_2, \ldots, f_N\}$ be a set of continuous maps from $X$ to $Y$. We define two mappings, both given the same name:

$$\begin{array}{llcl} {}^{\cup}F : & X \to \mathcal{K}Y \; ; & {}^{\cup}F(x) & := \; \bigcup_{i=1}^{N} \{f_i(x)\} \\ {}^{\cup}F : & \mathcal{K}X \to \mathcal{K}Y \; ; & {}^{\cup}F(U) & := \; \bigcup_{i=1}^{N} f_i(U)\,. \end{array} \tag{1.7}$$

In view of (1.3), the union map ${}^{\cup}F$ is Lipschitz if all mappings in $F$ are Lipschitz, and we have $r_{\cup F} \leq \sup\{r_f \mid f \in F\}$.

When $X = Y$ and $F$ consists of contractions, then $F$ is called an *iterated function system* (IFS). The principal result, see [**Hut**, § 3.1] or [**Wi**, Prop. 3.1.1], is based on the general Banach contraction principle and reads as follows.

THEOREM 1.1. (Hutchinson's contraction principle) *Let $F$ be an* IFS *on a complete metric space $X$. Then there is a unique $W \in \mathcal{K}X$ which is a fixed point of ${}^{\cup}F$, i.e. $W = \bigcup_{f \in F} f(W)$. Furthermore, for any $Z \in \mathcal{K}X$, $({}^{\cup}F)^{\ell}(Z)$ converges to $W \in \mathcal{K}X$ in the Hausdorff metric, as $\ell \to \infty$.* □

$W$ is the *attractor* of the IFS. Our aim, in this and in the following Section, is to generalize this result in two directions:

1. to compact sets of contractions (which is also well known);
2. to products of metric spaces (in what we call the multi-component situation).

**1.2. Compact sets of contractions.** Let $X, Y$ be complete metric spaces. We form $\mathcal{K}X$ as above. We remark that the union of any compact subset of $\mathcal{K}X$ is a compact subset of $X$, i.e.

$$C \in \mathcal{K}\mathcal{K}X \quad \Longrightarrow \quad \bigcup C := \bigcup_{U \in C} U \;\in\; \mathcal{K}X\,, \tag{1.8}$$

and that the union map $\bigcup : \mathcal{K}\mathcal{K}X \to \mathcal{K}X$ is continuous, cf. [**Wi**, Ch. 1.5].

[1] Wicks' book [**Wi**] is a great place to look for information on spaces of compact sets, and it was an important source for the part of this paper on compact spaces of maps. Unfortunately, or fortunately, according to one's taste, Wicks uses non-standard analysis to streamline his presentation, so 'standard' readers need to look elsewhere for the proofs, or to adapt them.

Consider the space $C(X, \mathcal{K}Y)$ equipped with the compact-open topology. Let $F \in \mathcal{K}C(X, \mathcal{K}Y)$, i.e. $F$ is a compact subset of contiuous mappings from $X$ into the space of compact subsets of $Y$. In view of (1.4), we have, for all $x \in X$,

$$F(x) \;:=\; \{f(x) \mid f \in F\} \;\in\; \mathcal{K}\mathcal{K}Y\,, \tag{1.9}$$

i.e. $F(x)$ is compact. From (1.8), we deduce that $\bigcup_{f\in F} f(x)$ is also compact. Thus, as in (1.7), we have a new mapping

$$\begin{array}{rl} {}^{\cup}F : & X \longrightarrow \mathcal{K}Y \\ & x \;\mapsto\; \bigcup_{f\in F} f(x)\,. \end{array} \tag{1.10}$$

With these preliminary definitions out of the way, we define $G$ to be a *compact admissible family of Lipschitz mappings* [**Wi**, Sec. 3.1] from $X$ to $Y$ if

**C1** $G$ is a compact set of Lipschitz mappings from $X$ to $Y$;
**C2** there exists $r > 0$ such that $r_g \le r$ for all $g \in G$.

If $r < 1$ then we call $G$ a *compact admissible family of contractions.*

Let $G$ be a compact admissible family of Lipschitz mappings from $X$ to $Y$. From Eq. (1.5), $\mathcal{K}(G) \subset C(\mathcal{K}X, \mathcal{K}Y)$ is compact, and since the $\mathcal{K}$ operator preserves Lipschitz constants, $\mathcal{K}(G)$ is itself a compact admissible family of Lipschitz mappings. Thus, from (1.10) we obtain

$${}^{\cup}\mathcal{K}G := {}^{\cup}(\mathcal{K}(G)) \;\in\; C(\mathcal{K}X, \mathcal{K}Y) \;: \qquad U \mapsto \bigcup_{g\in G} g(U)\,. \tag{1.11}$$

Proposition 1.2. *${}^{\cup}\mathcal{K}G \in C(\mathcal{K}X, \mathcal{K}Y)$ is Lipschitz with Lipschitz constant*

$$r_{{}^{\cup}\mathcal{K}G} \;\le\; \sup\{r_{\mathcal{K}(g)} \mid g \in G\} \;=\; \sup\{r_g \mid g \in G\} \;\le\; r\,. \qquad \square$$

If $G$ consists of contractions and $X = Y$ then we call $G$ a *compact iterated function system* (IFS). This generalizes the previous definition which will now be referred to as a *finite* IFS. Hutchinson's theorem evidently generalizes:

Proposition 1.3. *Let $G$ be a compact admissible family of contractions from $X$ to $Y$ with uniform Lipschitz bound $r < 1$. Then ${}^{\cup}\mathcal{K}G : \mathcal{K}X \longrightarrow \mathcal{K}Y$ is a contraction with Lipschitz constant at most $r$. If, in addition, $X = Y$ (so $G$ is a compact* IFS*), then there is a unique $W \in \mathcal{K}X$ (the attractor of the* IFS*) which is invariant under $G$ : $W = \bigcup_{g\in G} g(W)$. For arbitrary $Z \in \mathcal{K}X$, the iterates $({}^{\cup}\mathcal{K}G)^{\ell}(Z)$ converge to $W \in \mathcal{K}X$ as $\ell \to \infty$.*

Proof: For $Z \in \mathcal{K}X$, $\big(({}^{\cup}\mathcal{K}G)^{\ell}(Z)\big)_{\ell\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{K}X$ since ${}^{\cup}\mathcal{K}G$ is a contraction. Completeness shows the existence of a limit, which is a fixed point, and uniqueness follows immediately from contractivity. $\square$

## 2. Self-similar measures for compact sets of contractions

One of the important contributions of Hutchinson [**Hut**] was the realization that the attractor of an IFS carries a *measure* that is likewise an invariant of the IFS, and indeed a far finer one than the attractor itself. In this Section, we re-establish this result in the context of compact iterated function systems. The basic assumption that we need is that our compact space of contractions carries a measure of its own. We do not need to specify in advance what this measure is. In the original case studied by Hutchinson, where the IFS was finite, this supplementary measure was (effectively) counting measure.

Let $X$ be a compact metric space. We denote by $\mathcal{P}(X)$ the space of all probability measures on $X$ — that is, positive regular Borel measures $\mu$ with total measure $\mu(X) = 1$, see [**RS**, Ch. IV.4] for background material. Note that

$$\mathcal{P}(X) \;\subset\; \mathcal{M}_{+}(X) \;\subset\; \mathcal{M}(X) \;\subset\; \mathcal{M}_{\mathbb{C}}(X)\,,$$

where $\mathcal{M}_{+}(X)$, $\mathcal{M}(X)$, and $\mathcal{M}_{\mathbb{C}}(X)$ denote the spaces of positive, signed (or real), and complex regular Borel measures, respectively. For later use, we also define

$$\mathcal{M}^{m}_{+}(X) \;:=\; \{\mu \in \mathcal{M}_{+}(X) \mid \mu(X) = m\}$$

so that $\mathcal{P}(X) = \mathcal{M}^{1}_{+}(X)$. Since the Riesz-Markov theorem [**RS**, Thms. IV.14 and IV.18] states that regular Borel measures are in one-to-one correspondence with linear functionals on the space $C(X,\mathbb{R})$ (resp. $C(X,\mathbb{C})$), equipped with the compact-open topology, we shall usually identify these pictures. So, we shall write $\mu(E)$ for the measure of a Borel set $E$, but often also $\mu(g)$ instead of $\int_X g \,\mathrm{d}\mu$ for the measure (= integral) of a function.

In view of this, it is natural to equip $\mathcal{M}(X)$ (resp. $\mathcal{M}_{\mathbb{C}}(X)$) with the weak-$*$ topology, called the *vague topology* in this context [**RS**, p. 114], the weakest topology that makes all the mappings $\mu \mapsto \mu(g)$ of $\mathcal{M}(X) \to \mathbb{R}$ (resp. of $\mathcal{M}_{\mathbb{C}}(X) \to \mathbb{C}$) continuous, where $g \in C(X,\mathbb{R})$ (resp. $g \in C(X,\mathbb{C})$). Let us mention that, since $X$ is compact, $C(X,\mathbb{C})$ is actually a Banach space with the sup-norm $\|.\|_\infty$ (which induces the compact-open topology), hence $\mathcal{M}_{\mathbb{C}}(X) = C(X,\mathbb{C})^*$ can also be viewed as a Banach space, with induced norm

$$\|\mu\| \;:=\; \sup\{\, |\mu(g)| \mid g \in C(X,\mathbb{C}),\; \|g\|_\infty = 1\}$$

(see [**RS**, Thm. III.2]). With Hahn's decomposition theorem [**RS**, Thm. IV.16], one gets $\|\mu\| = |\mu|(X)$, where $|\mu| \in \mathcal{M}_{+}(X)$ denotes the total variation measure of $\mu$. The analogous statement also holds for $\mathcal{M}(X) = C(X,\mathbb{R})^*$. We shall need both ways to look at $\mathcal{M}(X)$. Another result in this context, using the Banach-Alaoglu Theorem [**RS**, Thm. IV.21], is that the unit balls in $\mathcal{M}_{\mathbb{C}}(X)$ and $\mathcal{M}(X)$ are compact in the vague topology. The closed subspace $\mathcal{P}(X) = \{\mu \in \mathcal{M}_{+}(X) \mid \|\mu\| = 1\}$ is then also compact. For an alternative derivation of the last statement, without reference to the Banach-Alaoglu Theorem, see [**A**, Prop. 8.1].

Following Hutchinson, we now define a *metric* on $\mathcal{P}(X)$:

$$L(\mu,\nu) \;:=\; \sup\{\, |\mu(\phi) - \nu(\phi)| \mid \phi \in \mathrm{Lip}(1,X,\mathbb{R})\}\,. \tag{2.1}$$

In fact, it is hardly clear that this *is* a metric[2], but we shall show that below. It is useful to observe that, in this definition, we can also replace $\mathrm{Lip}(1,X,\mathbb{R})$ by $\mathrm{Lip}(\leq 1,X,\mathbb{R})$ without altering the resulting function $L$. We often make use of this in the sequel. Since $X$ is compact, $\mathrm{diam}(X) := \sup\{d(x,y) \mid x,y \in X\}$, the *diameter* of $X$, is finite, and we can state another property explicitly.

Lemma 2.1. *Let $L$ be defined, on $\mathcal{P}(X)$, by Eq.* (2.1)*. Then we have*

$$L(\mu,\nu) \;=\; \sup\{\, |\mu(\psi) - \nu(\psi)| \mid \psi \in \mathcal{L}\}$$

*where $\mathcal{L} := \{\psi \in \mathrm{Lip}(1,X,\mathbb{R}) \mid \|\psi\|_\infty \leq \mathrm{diam}(X)\}$. Furthermore, $\mathcal{L} \subset C(X,\mathbb{R})$ is compact (in the compact-open topology).*

[2]This is well-known among experts, see [**Hut**, § 4.3], but we could not find an explicit proof in the literature. Since it is an important part of our argument and the proof is not entirely trivial, we include it here. Note that the restriction to $\mathcal{P}(X)$ (or to $\mathcal{M}^{m}_{+}(X)$ for some $m > 0$) is vital.

PROOF: Let $\mu, \nu \in \mathcal{P}(X)$, $\phi \in \mathrm{Lip}(1, X, \mathbb{R})$. Then, for any $c \in \mathbb{R}$, we have $|\mu(\phi - c) - \nu(\phi - c)| = |\mu(\phi) - \nu(\phi)|$ since $\mu(c) = \nu(c)\ (= c)$. Let $a \in X$. Choosing $c = \phi(a)$ we obtain $|\phi(x) - \phi(a)| \leq r_\phi d(x, a) \leq \mathrm{diam}(X)$. So, $\psi(x) := \phi(x) - \phi(a)$ is a function in $\mathcal{L}$, and the restriction to $\mathcal{L}$ does not change the supremum value of $|\mu(\phi) - \nu(\phi)|$. This establishes the first assertion.

Note that $\mathcal{L}$ is clearly closed in $C(X, \mathbb{R})$. Also, since the $\psi \in \mathcal{L}$ are uniformly bounded, we have $\overline{\mathcal{L}(x)} = \overline{\{\psi(x) \mid \psi \in \mathcal{L}\}} \subset [-\mathrm{diam}(X), \mathrm{diam}(X)]$ for every $x \in X$, so each $\overline{\mathcal{L}(x)}$ is compact. Finally, $\mathcal{L}$ is equi-continuous since it consists of Lipschitz functions with uniformly bounded Lipschitz constants. By Ascoli's theorem, see [**Kel**, Thms. 7.21 and 7.22], $\mathcal{L}$ itself is then compact in $C(X, \mathbb{R})$ (in the compact-open topology). □

PROPOSITION 2.2. *$L$ is a metric on $\mathcal{P}(X)$ and induces the vague topology on $\mathcal{P}(X)$. In particular, $\mathcal{P}(X)$ is a complete metric space.*

PROOF: That $\mathcal{P}(X)$ is a complete metric space follows from its compactness (see above) as soon as we have shown $L$ to be a metric.

We use Lemma 2.1. If $\mu, \nu \in \mathcal{P}(X)$, then $\mu - \nu : \mathcal{L} \longrightarrow \mathbb{R}$ is continuous and so has compact image. This shows that $L(\mu, \nu)$ is finite. Non-negativity and symmetry are obvious, as is the triangle inequality. Thus $L$ is certainly a pseudo-metric. It remains to be shown that $L(\mu, \nu) = 0$ implies $\mu = \nu$. Assume the converse and set $\omega = \mu - \nu$. Then $c := \|\omega\| > 0$ and there is a $\delta > 0$ and a function $g \in C(X, \mathbb{R})$ such that $|\omega(g)| \geq \delta > 0$.

Since Lipschitz functions are dense in $C(X, \mathbb{R})$, see Lemma A.2 in the Appendix, we can choose $\phi$ Lipschitz with $\|g - \phi\|_\infty < \delta / 2c$. Then,

$$|\omega(g - \phi)| \ \leq \ \|\omega\| \cdot \|g - \phi\|_\infty \ < \ \frac{\delta}{2}$$

and thus $|\omega(\phi)| \geq \delta/2 > 0$. Now, we don't know the Lipschitz constant $r_\phi$ of $\phi$, but $\phi' := \phi / r_\phi$ is in $\mathrm{Lip}(1, X, \mathbb{R})$ and still $|\omega(\phi')| > 0$, so $L(\omega, 0) = L(\mu, \nu) > 0$, which contradicts the assumption. This shows that $L$ is a metric.

Finally, we compare the topologies. Let $\mu_n \to \mu$ vaguely as $n \to \infty$. Since $\mu_n \in \mathcal{P}(X)$, we get $|\mu_n(\phi_1) - \mu_n(\phi_2)| \leq \|\phi_1 - \phi_2\|_\infty$ independently of $n$, so the $\mu_n$ constitute a family of equi-continuous mappings in $C(C(X, \mathbb{R}), \mathbb{R})$. Consequently, by Lemma A.3 of the Appendix, $\mu_n(\phi) \to \mu(\phi)$ uniformly on $\mathcal{L}$ because $\mathcal{L}$ is compact (Lemma 2.1), and hence $L(\mu_n, \mu) \to 0$. Conversely, observe that $\mathcal{L}$ is compact in the vague topology and Hausdorff under the metric $L$. But the identity is a one-to-one mapping, and (due to the previous argument) also continuous, when viewed as a mapping from $\mathcal{L}$ with the vague topology to $\mathcal{L}$ with the metric topology. Therefore, it is a homeomorphism [**Kel**, Thm. 5.8], and the topologies coincide. □

Let $F$ be a compact IFS on the compact metric space $X$ and let $W \in \mathcal{K}X$ be its attractor. Let $r_F := \sup\{r_f \mid f \in F\} < 1$.

For each $f \in F$, $f(W) \subset W$, and we obtain a bounded linear operator $f.(\,\boldsymbol{.}\,)$ on the space of all signed Borel measures $\mathcal{M}(W)$ of $W$ by

$$\mu \mapsto f.\mu \ ; \quad f.\mu(\phi) \ := \ \mu(\phi \circ f) \tag{2.2}$$

for all $\phi \in C(W, \mathbb{R})$. Evidently, $f.(\,\boldsymbol{.}\,) : \ \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$, i.e. if $\mu$ is a probability measure, so is $f.\mu$. Note that the matching definition for Borel sets $E$ reads $f.\mu(E) = \mu(f^{-1}(E))$ where $f^{-1}(E)$ is the preimage of $E$ under $f$.

PROPOSITION 2.3. *The mapping $F \times \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ defined by $(f,\mu) \mapsto f.\mu$ is continuous.*

PROOF: $\mathcal{P}(W)$ is a compact metric space, so certainly Hausdorff. Fix $f \in F$ and consider the mapping $\mu \mapsto f.\mu$. Then, for $\phi \in \mathrm{Lip}(1,W,\mathbb{R})$, we clearly have $|f.\mu(\phi) - f.\nu(\phi)| = |(\mu-\nu)(\phi \circ f)| \leq L(\mu,\nu)$ because $\phi \circ f$ has Lipschitz constant $\leq 1$ due to the definition of $F$. So $\mu \mapsto f.\mu$ is Lipschitz and thus (uniformly) continuous on $\mathcal{P}(W)$. It follows that $F \times \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ is jointly continuous [**Kel**, Thm. 7.5]. $\square$

PROPOSITION 2.4. *Let $F$ be a compact IFS on the compact metric space $X$, with attractor $W \in \mathcal{K}X$. Let $\nu \in \mathcal{P}(F)$. Then the $\nu$-averaging mapping*

$$\begin{array}{rccc} \mathcal{A}_\nu : & \mathcal{P}(W) & \longrightarrow & \mathcal{P}(W) \\ & \mu & \mapsto & \int_F (f.\mu)\, \mathrm{d}\nu(f) \end{array} \tag{2.3}$$

*is a contraction relative to the $L$-metric on $\mathcal{P}(W)$, with contraction constant at most $r := r_F = \sup\{r_f \mid f \in F\}$. In particular, there is a unique measure $\rho^{(\nu)} \in \mathcal{P}(W)$ which satisfies*

$$\rho^{(\nu)} \;=\; \int_F (f.\rho^{(\nu)})\, \mathrm{d}\nu(f). \tag{2.4}$$

PROOF: Let $\omega_1, \omega_2 \in \mathcal{P}(W)$ and let $\phi \in \mathrm{Lip}(1,W,\mathbb{R})$. Then

$$\begin{aligned} \big|(\mathcal{A}_\nu(\omega_1))(\phi) - (\mathcal{A}_\nu(\omega_2))(\phi)\big| &= \left| \int_F \omega_1(\phi\circ f)\, \mathrm{d}\nu(f) - \int_F \omega_2(\phi\circ f)\, \mathrm{d}\nu(f) \right| \\ &= \left| \int_F \Big( \omega_1(\phi\circ f) - \omega_2(\phi \circ f) \Big)\, \mathrm{d}\nu(f) \right| \\ &\leq r \int_F \big| \omega_1(r^{-1}\phi\circ f) - \omega_2(r^{-1}\phi\circ f)\big|\, \mathrm{d}\nu(f) \\ &\leq r \int_F L(\omega_1,\omega_2)\, \mathrm{d}\nu(f) \;=\; r\, L(\omega_1,\omega_2)\; \nu(F) \\ &= r\, L(\omega_1,\omega_2)\,, \end{aligned}$$

since $r^{-1}\phi \circ f \in \mathrm{Lip}(\leq 1, W, \mathbb{R})$. This being true for all $\phi \in \mathrm{Lip}(1,W,\mathbb{R})$, we have

$$L(\mathcal{A}_\nu(\omega_1), \mathcal{A}_\nu(\omega_2)) \;\leq\; r\, L(\omega_1,\omega_2) \tag{2.5}$$

which is what we wanted. The existence of the unique measure $\rho^{(\nu)}$ follows now, once again, from the general contraction principle. $\square$

REMARK: More generally, we may replace $W$ in Proposition 2.4 by any other $W^+ \in \mathcal{K}X$ that satisfies $f(W^+) \subset W^+$ for all $f \in F$. Of course, $W \subset W^+$ and the invariant measure for $W^+$ is supported on $W$, hence is effectively the same as $\rho^{(\nu)}$.

## 3. Affine mappings in locally compact Abelian groups

In this Section, we treat the foregoing material in the setting of a locally compact Abelian group (LCAG). A convenient source for the results on LCAGs that we need is [**Ru**, Ch. 1].

**3.1. Affine mappings.** Let $H$ be an additive LCAG that is equipped with a translation invariant metric $d$ with respect to which $H$ is complete. For more information on metrizability, see [**HeRo**, Ch. 2, § 8]. We also assume that an automorphism $A$ is given (in particular, $A(0) = 0$) and that a Haar measure $\theta$ on $H$ has been fixed. It is unique up to normalization.

Since $A.\theta := \theta \circ A^{-1}$ is also $H$-invariant, it is another Haar measure, and we thus have $A.\theta = \alpha\theta$ for some $\alpha > 0$, the *modulus* of $A$. If $H = \mathbb{R}^n$, $\theta$ is Lebesgue measure, $A$ is simply an invertible linear map, and $\alpha = |\det(A^{-1})| = 1/|\det(A)|$.

LEMMA 3.1. *Let $H$ be as described with metric $d$. If $A$ is a contraction on $H$ relative to $d$, then $A$ has modulus $\alpha > 1$.*

PROOF: Since $H$ is locally compact, it contains a compact neighbourhood $U$ of 0. By assumption, the topology given on $H$ agrees with the metric topology. So, we can choose $U$ such that inside it we can find balls $B_r(0)$ and $B_s(0)$ with $r > s > 0$ and the property that $B_r(0) \setminus B_s(0)$ contains a non-empty open set which must then have positive measure.

On the other hand, $A$ is a contraction, $A(0) = 0$, and $d(0, A^n(U)) \to 0$ as $n \to \infty$ for any compact $U \subset H$. So, there must be some $m \in \mathbb{N}$ such that $A^m(B_r(0)) \subset B_s(0)$. Combined with the previous argument, this says $A^m(B_r(0))$ has smaller measure than $B_r(0)$. Consequently, the modulus of $A^m$ is $\alpha^m > 1$, and then also $\alpha > 1$. □

Clearly, the converse of Lemma 3.1 is not true. In view of the general context of this paper, we assume from now on that $A$ is a contraction on $H$ relative to $d$. A mapping $f: H \to H$ of the form

$$f: x \mapsto A(x) + v \tag{3.1}$$

with $v \in H$ is called an *affine map* with *automorphism* $A$ and *translation* $v$. We will sometimes denote this mapping by $A_v$.

Let $\mathcal{M}_{\mathbb{C}}(H)$ be the space of all bounded regular complex Borel measures $\lambda$ on $H$, i.e. measures with $\|\lambda\| = |\lambda|(H) < \infty$. We recall that the convolution of two measures $\lambda_1, \lambda_2 \in \mathcal{M}_{\mathbb{C}}(H)$ is defined by

$$(\lambda_1 * \lambda_2)(\phi) \;=\; \int_{H\times H} \phi(x+y)\, \mathrm{d}\lambda_1(x)\, \mathrm{d}\lambda_2(y)\,, \tag{3.2}$$

for $\phi \in C(H, \mathbb{C})$. If formulated for Borel sets $E$, the matching equation is

$$(\lambda_1 * \lambda_2)(E) \;=\; (\lambda_1 \otimes \lambda_2)(E^{(2)}) \tag{3.3}$$

where $E^{(2)} := \{(x, y) \in H \times H \mid x + y \in E\}$.

The Fourier-Stieltjes transform of $\lambda \in \mathcal{M}_{\mathbb{C}}(H)$ is the function $\widehat{\lambda}$ defined on the dual group $\widehat{H}$ of $H$ by

$$\widehat{\lambda}(k) \;=\; \int_H \overline{\langle k, x\rangle}\, \mathrm{d}\lambda(x) \tag{3.4}$$

where $x \mapsto \langle k, x\rangle$ is the continuous *character* on $H$ defined by $k \in \widehat{H}$.

The automorphism $A$ on $H$ induces an automorphism $A^T$ on $\widehat{H}$: $k \mapsto A^T k$ where $A^T k$, in turn, defines the character $x \mapsto \langle k, Ax\rangle$ on $H$.

We collect now some basic facts that we need. These are all elementary consequences of the definitions, whence we omit proofs. We write $A.h$ for the function defined by $x \mapsto h(A^{-1}(x))$ in analogy to $A.\mu = \mu \circ A^{-1}$ for measures, and $h\mu$, with

$h \in L^1(H)$, for the measure defined by $(h\mu)(\phi) = \mu(h\phi)$. Thus, $h\mu$ is absolutely continuous with respect to $\mu$, and $h$ is the corresponding Radon-Nikodym density (also called Radon-Nikodym derivative).

PROPOSITION 3.2. *Let $H, \theta, A, \alpha$ be as defined above. Let $\lambda_1, \lambda_2 \in \mathcal{M}_{\mathbb{C}}(H)$ and $h \in L^1(H)$. Then we have*

1. $\mathrm{d}\theta(A^{-1}x) \; = \; \alpha \, \mathrm{d}\theta(x)$
2. $A.(h\theta) \; = \; \alpha(A.h)\theta$
3. $A.(\lambda_1 * \lambda_2) \; = \; A.\lambda_1 * A.\lambda_2$
4. $\widehat{A.\lambda} \; = \; \widehat{\lambda} \circ A^T \; = \; (A^T)^{-1}.\widehat{\lambda}$ □

**3.2. Compact families of affine mappings.** Assume now that $F$ is a compact family of contractions on the LCAG $H$, each $f \in F$ being of the form

$$A_v : \; x \mapsto Ax + v$$

for some $v \in H$ (but all having the same $A$, namely our contractive automorphism fixed above). Evidently, $F$ is a compact admissible family of mappings from $H$ to $H$. Define

$$F_H \; = \; \{v \mid A_v \in F\} \; = \; \{f(0) \mid f \in F\} \; = \; F(0) \; \subset \; H \, . \tag{3.5}$$

In view of (1.4) and (1.5), the mapping $F \to F_H$ induced by $f \mapsto f(0)$ is continuous and hence $F_H$ is compact and homeomorphic to $F$. In particular, there is a natural isomorphism between $\mathcal{M}_{\mathbb{C}}(F)$ and the space of regular measures on $H$ that are supported on $F_H$.

Let $\nu_F \in \mathcal{P}(F)$ and $\nu \in \mathcal{P}(H)$ be such a corresponding pair of (probability) measures. We then have an averaging operator $\mathcal{A}_\nu : \, \mathcal{P}(W^+) \to \mathcal{P}(W^+)$ whenever $W^+ \subset H$ is any compact subset of $H$ for which $FW^+ \subset W^+$.

Let $\lambda \in \mathcal{P}(W^+)$. Then, for all Borel sets $E \subset H$,

$$\begin{aligned}
\mathcal{A}_\nu \lambda(E) \; &= \; \int_F f.\lambda(E) \, \mathrm{d}\nu_F(f) \; = \; \int_F \lambda(f^{-1}(E)) \, \mathrm{d}\nu_F(f) \\
&= \; \int_H \lambda(A^{-1}(E-v)) \, \mathrm{d}\nu(v) \; = \; \int_H (A.\lambda)(E-v) \, \mathrm{d}\nu(v) \\
&= \; \int_{H\times H} \mathbf{1}_E(u+v) \, \mathrm{d}(A.\lambda)(u) \, \mathrm{d}\nu(v) \\
&= \; (\nu * A.\lambda)(E) \qquad (3.6)
\end{aligned}$$

where $\mathbf{1}_E$ is the characteristic function of $E$. Since $\mathrm{supp}(\lambda) \subset W^+$, we have $f(\mathrm{supp}(\lambda)) \subset W^+$ for all $f \in F$. This implies $\mathrm{supp}(\mathcal{A}_\nu\lambda) \subset W^+$ and, more generally, $\mathrm{supp}(\mathcal{A}_\nu^\ell \lambda) \subset W^+$ for all $\ell \geq 0$. In particular, we can also infer

$$\mathrm{supp}(\mathcal{A}_\nu \lambda) \; = \; \mathrm{supp}(\nu * A.\lambda) \; \subset \; \mathrm{supp}(\nu) + A \, \mathrm{supp}(\lambda) \; \subset \; W^+ \, .$$

It is clear that we can now iterate (3.6) to get

$$\mathcal{A}_\nu^\ell \lambda \; = \; \nu * A.\nu * \ldots * A^{\ell-1}.\nu * A^\ell.\lambda$$

together with the inclusion relation

$$\mathrm{supp}(\mathcal{A}_\nu^\ell \lambda) \; \subset \; \mathrm{supp}(\nu) + A \, \mathrm{supp}(\nu) + \ldots + A^{\ell-1} \, \mathrm{supp}(\nu) + A^\ell \, \mathrm{supp}(\lambda) \; \subset \; W^+ \, .$$

Since $A$ is a contraction, $A^\ell \operatorname{supp}(\lambda) \longrightarrow \{0\}$ (in $\mathcal{K}H$) as $\ell \to \infty$ and we have

$$\sum_{\ell=0}^{\infty} A^\ell \operatorname{supp}(\nu) \subset W^+ .$$

In particular, since $W$ is $F$-invariant, we must also have

$$\sum_{\ell=0}^{\infty} A^\ell \operatorname{supp}(\nu) \subset W .$$

Define $W^+$ to be the smallest compact subset of $H$ which is $F$-invariant and contains $\sum_{j=0}^{\ell} A^j \operatorname{supp}(\nu)$ for all $\ell \geq 0$.

Define $\omega^{(\ell)} \in \mathcal{P}(W^+)$ by $\omega^{(0)} = \nu$, and (for $\ell \geq 0$)

$$\omega^{(\ell+1)} = \mathcal{A}_\nu \omega^{(\ell)} = \nu * A.\omega^{(\ell)} .$$

Next, let $\omega$ be the unique $\mathcal{A}_\nu$-invariant measure of $\mathcal{P}(W^+)$, see Proposition 2.4 and the Remark following it. We know, again by Proposition 2.4, that $\mathcal{A}_\nu$ is a contraction on $\mathcal{P}(W^+)$. Moreover, by (3.6), $\{\omega^{(\ell)}\}$ contracts to $\omega$ as $\ell \to \infty$. Thus

$$\lim_{\ell \to \infty} \omega^{(\ell)} = \omega$$

in $\mathcal{P}(W^+)$, with convergence in the vague topology.

Proposition 3.3. *Under the above assumptions, we have*

1. $\omega = \mathop{\ast}_{\ell=0}^{\infty} (A^\ell.\nu) \in \mathcal{P}(W)$, *which converges in the vague topology, is the unique self-similar probability measure for the compact admissible family of contractions* $F = \{A_v \mid v \in F_H\}$ *with respect to the measure* $\nu_F$ *on* $F$.
2. $\widehat{\omega} = \prod_{\ell=0}^{\infty} (A^T)^{-\ell}.\widehat{\nu}$, *convergence being uniform convergence on compact sets (compact convergence).*
3. *If the convolution product for* $\omega$ *converges also in the* $\|.\|$*-topology on* $\mathcal{P}(W)$, *the convergence of* $\widehat{\omega}$ *is (globally) uniform.*

Proof: Part 1 is clear from the discussion above. The support of $\omega$ is inside $W$ by Proposition 2.4. Part 2 follows from Proposition 3.2 and the continuity of the Fourier transform, sending measures $\mu$ to bounded and uniformly continuous functions $\widehat{\mu}$. The convergence statement is a direct consequence of Lévy's continuity theorem, see Theorem A.5 of the Appendix. Finally, the third assertion follows directly from $\|\widehat{\mu}\|_\infty \leq \|\mu\|$, see (3.4), without reference to Part 2. □

Remark: It is only a matter of convenience to start the above iteration with $\omega^{(0)} = \nu$. Any other choice $\lambda \in \mathcal{P}(W^+)$ is equally admissible and will lead to the same result, because $A^\ell.\lambda \to \delta_0$, as $\ell \to \infty$, and $\delta_0$, the unit point measure at 0, is the neutral element of convolution, i.e. $\mu * \delta_0 = \mu$ for all measures $\mu$.

**3.3. Self-similar functions.** If we assume that the measure $\nu$ on our compact family of affine contractions is absolutely continuous with respect to Haar measure, then Proposition 3.3 gets re-interpreted in terms of functions rather than measures.

We suppose the same notation as in Section 3.2 and assume in addition that the measure $\nu$ derived from $\nu_F$ on $F$ is absolutely continuous w.r.t. $\theta$, so $\nu$ is of the form $\nu = h\theta$, where $h \in L^1(H)$ and $\operatorname{supp}(h) \subset F_H$, with $F_H$ compact. For such measures, convolution matches the usual convolution of functions. Thus, using Proposition 3.2 and Part 1 of Proposition 3.3, we obtain

$$\begin{aligned} \omega^{(\ell)} &= \nu * A.\nu * \ldots * A^{\ell}.\nu \\ &= h\theta * \alpha(A.h)\theta * \ldots * \alpha^{\ell}(A^{\ell}.h)\theta \\ &= \Big( \mathop{\ast}\nolimits_{j=0}^{\ell} \alpha^{j}(A^{j}.h)\Big)\,\theta \end{aligned}$$

and $\omega = \big(\mathop{\ast}\nolimits_{j=0}^{\infty} \alpha^{j}(A^{j}.h)\big)\theta$, with convergence so far only in the vague topology. However, as the brackets already imply, convergence in the $\|.\|$-topology would be preferable. The situation is as follows. If we identify $L^1(H)$ with a subspace of $\mathcal{M}_{\mathbb{C}}(H)$ via $f \mapsto f\theta$, this is a *closed subspace* of $\mathcal{M}_{\mathbb{C}}(H)$ in the $\|.\|$-topology, see [**Ru**, § 1.3.5]. Consequently, the $\|.\|$-convergence of absolutely continuous measures is equivalent to the $L^1$-convergence of their Radon-Nikodym densities in $L^1(H)$.

To establish also the $\|.\|$-convergence in our case, recall first the following result [**Ru**, Thm. 1.1.8] on approximate units in the commutative convolution Banach algebra $L^1(H)$ (with norm $\|.\|_1$).

Lemma 3.4. *Given $f \in L^1(H)$ and $\varepsilon > 0$, there exists a neighbourhood $V$ of $0$ in $H$ with the following property: if $u$ is a non-negative Borel function which vanishes outside $V$, and if $\|u\|_1 = \int_H u(x)\,\mathrm{d}\theta(x) = 1$, then*

$$\| f - f * u \|_1 \; < \; \varepsilon \, . \qquad \square$$

We are now in the following situation. Our starting function is $h \in L^1(H)$, with $\mathrm{supp}(h)$ compact, $h \geq 0$ and $\int_H h \,\mathrm{d}\theta = \|h\|_1 = 1$. Let $f_\ell = \alpha^{\ell}(A^{\ell}.h)$ for $\ell \geq 0$, so that $f_\ell \geq 0$ and $\|f_\ell\|_1 = 1$. Also, $\mathrm{supp}(f_\ell) = A^{\ell}\,\mathrm{supp}(h)$, and we have the relation $\|f_\ell * f_{\ell+1} * \ldots * f_{\ell+k}\|_1 = 1$ for all $k \geq 0$.

Proposition 3.5. *Let $F$ be a compact family of affine mappings, with contractive automorphism $A$, modulus $\alpha$ and attractor $W \subset H$. Let $F_H$ be the corresponding set of translations and let $\nu = h\theta$ be an absolutely continuous probability measure on $F_H$, where $h \in L^1(H)$. Then, the infinite convolution product $\mathop{\ast}\nolimits_{\ell=0}^{\infty} f_\ell$ converges to an $L^1$-function, hence $\mathop{\ast}\nolimits_{\ell=0}^{\infty} f_\ell\theta$ converges also in the $\|.\|$-topology.*

Proof: Since $L^1(H)$ is complete, it suffices to show that $\big(\mathop{\ast}\nolimits_{\ell=0}^{n} f_\ell\big)_{n\geq 0}$ is a Cauchy sequence. Fix $\varepsilon > 0$ and let $V$ be the neighbourhood for the $L^1$-function $f = f_0 = h$ according to Lemma 3.4. Since $A$ is a contraction, there exists an integer $N$ so that $\sum_{\ell \geq N} \mathrm{supp}(f_\ell) \subset V$. In particular, any finite convolution of the form $\mathop{\ast}\nolimits_{\ell=N}^{N+k} f_\ell$, $k \geq 0$, is then an approximate unit for $h$ with bound $\varepsilon$.

Let now $n, m \geq N$ and define $u = \mathop{\ast}\nolimits_{\ell=N}^{n} f_\ell$ and $v = \mathop{\ast}\nolimits_{\ell=N}^{m} f_\ell$. Then

$$\begin{aligned} \Big\| \mathop{\ast}_{\ell=0}^{n} f_\ell - \mathop{\ast}_{\ell=0}^{m} f_\ell \Big\|_1 &= \Big\| \Big( \mathop{\ast}_{\ell=0}^{N-1} f_\ell \Big) * (u - v) \Big\|_1 \\ &\leq \Big\| \mathop{\ast}_{\ell=1}^{N-1} f_\ell \Big\|_1 \cdot \big\| h * u - h * v \big\|_1 \\ &= \big\| (h * u - h) + (h - h * v) \big\|_1 \\ &\leq \big\| h - h * u \big\|_1 + \big\| h - h * v \big\|_1 \; < \; 2\varepsilon \end{aligned}$$

by application of Lemma 3.4. This gives part one of the claim, while the rest follows, once again, from the Radon-Nikodym theorem. $\square$

PROPOSITION 3.6. *Under the general assumptions of Proposition* 3.5, *we have:*

1. *There is a unique non-negative function* $g \in L^1(H)$ *which satisfies*[3]
$$g \;=\; \alpha \int_H g(A^{-1}(x-v))h(v)\,\mathrm{d}\theta(v)$$
*with normalization* $\int_H g\,\mathrm{d}\theta = 1$.
2. $g = \mathop{\ast}_{\ell=0}^{\infty} \alpha^\ell (A^\ell.h)$, *with convergence in the* $L^1$*-norm, and* $\mathrm{supp}(g) \subset W$.
3. *The Fourier transform of* $g$ *is the continuous function* $\widehat{g} = \prod_{\ell=0}^{\infty} \alpha^\ell\big(\widehat{h}.(A^T)^\ell\big)$, *with uniform convergence of the product.*
4. *If* $h \in L^1(H) \cap L^\infty(H)$, *then* $g$ *is continuous on* $H$.

PROOF: The convergence claimed in Part 2 follows from Proposition 3.5, so $g \in L^1(H)$ and $\widehat{g}$ is then continuous.

From $\omega = \nu * A.\omega$, we then have, by Proposition 3.2, $g = \alpha h * A.g$, which gives Part 1 by applying Proposition 3.3(1), and also the statement that $\mathrm{supp}(g) \subset W$.

The situation for Fourier transforms is even easier since $\widehat{g\theta} = \widehat{g}$ and we get the product formula in Part 3 from Proposition 3.3(2) with uniform convergence by means of Proposition 3.3(3).

Finally, suppose that $h \in L^1(H) \cap L^\infty(H)$. Since $h \in L^\infty(H)$ and $A.g \in L^1(H)$, we obtain ([**Ru**, Thm. 1.1.6]) the continuity of $h * A.g$, hence of $g$ itself. □

REMARK: If $H = \mathbb{R}^n$, we can actually iterate the last argument and arrive at the stronger statement that $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ implies that $g$ is a $C^\infty$-function with compact support contained in $W$. Furthermore, if $\nu$ is Lebesgue measure, then the self-similar function $g$ enjoys remarkable properties with respect to the averaging operator $\mathcal{A}_\nu$, namely its partial derivatives are eigenfunctions for the refinement operator with eigenvalues directly related to the spectrum of $A$. This is the situation in our previously studied examples [**BM1, BM2**] and these results may be found there.

## 4. Multi-component families of contractions

In this Section, we consider the generalization of the previous material to the case in which we have several compact metric spaces and sets of contractions between these spaces. This is the multi-component situation. The approach here is to consider the product of the various spaces in question. The basic theorem on the existence of attractors then reduces at once to the single-component situation already dealt with. The question of self-similar measures also fits naturally into the product formalism, though the situation now acquires some new features that did not appear before.

**4.1. Contractions and attractors.** Let $(X_1, d_1), \ldots, (X_n, d_n)$ be $n$ complete metric spaces and define $N := \{1, \ldots, n\}$. Also, let $d_{i,H}$ denote the corresponding Hausdorff metric for $\mathcal{K}X_i$, $i \in N$. We set

$$X_N \;:=\; X_1 \times \ldots \times X_n \tag{4.1}$$

and write $x = (x_1, \ldots, x_n)$ for the elements of $X_N$. We endow $X_N$ with the metric

$$d(x,y) \;:=\; \sup\{d_i(x_i, y_i) \mid i \in N\} \tag{4.2}$$

[3] In [**JLS**], a mapping on functions of this form is called a continuous refinement operator.

relative to which it is also complete, and denote by $d_H$ the attached Hausdorff metric on $\mathcal{K}X_N$.

For each pair $(i,j) \in N \times N$, let $F_{ij}$ be a compact admissible family of contractions $f = f_{ij} : X_j \longrightarrow X_i$. We extend this to allow the possibility that $F_{ij}$ is empty, though we require that for each $i$ there is at least one $j$ for which $F_{ij} \neq \emptyset$. We let $0 < r < 1$ be a uniform upper bound on the contractivity factors of all these mappings. For each pair $(i,j)$, we have from (1.11) the mapping ${}^{\cup}\mathcal{K}F_{ij} : \mathcal{K}X_j \longrightarrow \mathcal{K}X_i$. We define

$$ {}^{\cup}\mathcal{K}F : \quad \mathcal{K}X_1 \times \ldots \times \mathcal{K}X_n \longrightarrow \mathcal{K}X_1 \times \ldots \times \mathcal{K}X_n \tag{4.3} $$

where

$$ \begin{aligned} {}^{\cup}\mathcal{K}F(U_1,\ldots,U_n) \quad &:= \quad \Big(\bigcup\nolimits_j {}^{\cup}(\mathcal{K}F_{1j})\,(U_j),\ldots,\bigcup\nolimits_j {}^{\cup}(\mathcal{K}F_{nj})\,(U_j)\Big) \\ &= \quad \Big(\bigcup\nolimits_j \bigcup\nolimits_{f\in F_{1j}} f\,(U_j),\ldots,\bigcup\nolimits_j \bigcup\nolimits_{f\in F_{nj}} f\,(U_j)\Big)\,. \end{aligned} \tag{4.4} $$

Note that we write $(U_1,\ldots,U_n)$ rather than $U_1 \times \ldots \times U_n$ and that $\mathcal{K}X_1 \times \ldots \times \mathcal{K}X_n$ is a strict subset of $\mathcal{K}X_N$.

Proposition 4.1. *${}^{\cup}\mathcal{K}F : \mathcal{K}X_1 \times \ldots \times \mathcal{K}X_n \longrightarrow \mathcal{K}X_1 \times \ldots \times \mathcal{K}X_n$ is a contraction with Lipschitz constant at most $r$.*

Proof: For $U_i, V_i \in \mathcal{K}X_i$ , $i \in N$, we find

$$ \begin{aligned} & d_H\left({}^{\cup}\mathcal{K}F(U_1,\ldots,U_n), {}^{\cup}\mathcal{K}F(V_1,\ldots,V_n)\right) \\ &\quad = \; d_H\left((\ldots,\bigcup\nolimits_j {}^{\cup}(\mathcal{K}F_{ij})\,(U_j),\ldots)\,,\,(\ldots,\bigcup\nolimits_j {}^{\cup}(\mathcal{K}F_{ij})\,(V_j),\ldots)\right) \\ &\quad = \; \sup_i \left\{ d_{i,H}\big(\bigcup\nolimits_j {}^{\cup}(\mathcal{K}F_{ij})\,(U_j), \bigcup\nolimits_j {}^{\cup}(\mathcal{K}F_{ij})\,(V_j)\big)\right\} && \text{(by definition)} \\ &\quad \leq \; \sup_i \sup_j \left\{ d_{i,H}\big({}^{\cup}(\mathcal{K}F_{ij})\,(U_j), {}^{\cup}(\mathcal{K}F_{ij})\,(V_j)\big)\right\} && \text{(by (1.3))} \\ &\quad \leq \; \sup_i \sup_j \{ r_{\mathcal{K}F_{ij}}\, d_{j,H}(U_j,V_j)\} \;\leq\; r \sup_j \{ d_{j,H}(U_j,V_j)\} && \text{(by Prop. 1.2)} \\ &\quad = \; r\, d_H\big((U_1,\ldots,U_n),(V_1,\ldots,V_n)\big) \end{aligned} $$

which establishes our assertion. □

We conclude, using the usual contraction principle, that there is a unique attractor for ${}^{\cup}\mathcal{K}F$, in $\mathcal{K}X_1 \times \ldots \times \mathcal{K}X_n$, say $W_1 \times \ldots \times W_n$. The $W_i$ thus form the unique solution (in compact sets) to the system of equations:

$$ W_i \;=\; \bigcup_{j=1}^{n} \;\bigcup_{f\in F_{ij}} f(W_j)\;, \quad i \in N\,. \tag{4.5} $$

**4.2. Multi-component invariant measures.** The idea behind the invariant measures in the multi-component setting is straightforward in its conception but looks complicated in its details. We start with $n$ compact metric spaces $X_i$, $i \in N$, that are coupled by families $F_{ij}$ of contractions $f : X_j \longrightarrow X_i$. For the moment we can take each set of mappings $F_{ij}$ to be finite.

Each $f \in F_{ij}$ determines a transformation $\mu_j \mapsto f.\mu_j$ (see (2.2) for notation) of measure spaces $\mathcal{M}(X_j) \longrightarrow \mathcal{M}(X_i)$. Basically, we want to find a family of measures

$\{\mu_1, \ldots, \mu_n\}$ that is invariant under the average of these transformations:

$$\mu_i \;=\; \sum_{j=1}^{n} \frac{1}{\operatorname{card}(F_{ij})} \sum_{f \in F_{ij}} f.\mu_j \,. \tag{4.6}$$

There are some extensions and modifications that make this picture both more useful and easier to cope with mathematically:

1. We are at liberty to give each set of mappings $F_{ij}$ its own weighting.
2. We need not restrict ourselves to *finite* sets $F_{ij}$, nor need we assume that our averaging is uniform within each of these sets. In what follows, we only assume that the sets $F_{ij}$ are compact spaces of mappings. We then deal with these points simultaneously by assigning positive[4] measures $\sigma_{ij}$ to each of these spaces of mappings.
3. It is mathematically easiest to deal with all of the measures $\{\mu_1, \ldots, \mu_n\}$ as a single entity. Thus we prefer to deal with product measures $\mu_1 \otimes \ldots \otimes \mu_n$ on the space $X_1 \times \ldots \times X_n$. This means that we will be deriving a product form of the invariance equation (4.6).

After these considerations, the mathematics unfolds in much the same way as before, with one exception. Invariant measures $\mu_1 \otimes \ldots \otimes \mu_n$ can exist only if a certain eigenvector condition involving the total measures of the $\mu_i$ and the $\sigma_{ij}$ is met (see Eq. (4.8) below).

Let $N$ and $(X_i, d_i)$, $i \in N$, be as above. For each $J = (j_1, \ldots, j_n) \in N^n$, we define $X_J := X_{j_1} \times \ldots \times X_{j_n}$ and adopt standard multi-index notation, e.g. $x_J = (x_{j_1}, \ldots, x_{j_n})$. In particular, $X_N = X_1 \times \ldots \times X_n$ in agreement with our previous definition. We then define the metric $d_J$ on $X_J$ by $d_J(x_J, y_J) = \sup_{k=1}^n d_{j_k}(x_{j_k}, y_{j_k})$. For measures $\mu_i \in \mathcal{M}(X_i)$, $i \in N$, we write $\mu_J = \mu_{j_1} \otimes \ldots \otimes \mu_{j_n} \in \mathcal{M}(X_J)$ and $\mathrm{d}\mu_J = \mathrm{d}\mu_{j_1} \ldots \mathrm{d}\mu_{j_n}$.

For each $(i,j) \in N \times N$, let $F_{ij}$ be a compact admissible family of contractions $f : X_j \longrightarrow X_i$ (allowing, as above, the possibility that $F_{ij}$ is empty). We let $0 < r < 1$ be a uniform upper bound on the contractivity factors of all these mappings.

We let $F = \bigtimes_{i,j} F_{ij}$ be the product of all these spaces of maps, a typical element being a matrix of maps $\boldsymbol{f} = (f_{ij})$. For each such $\boldsymbol{f}$, and for all $J, K \in N^n$, let $f_{KJ} : X_J \longrightarrow X_K$ be given by

$$f_{KJ}(x_{j_1}, \ldots, x_{j_n}) \;=\; (f_{k_1 j_1}(x_{j_1}), \ldots, f_{k_n j_n}(x_{j_n}))\,. \tag{4.7}$$

We write $f_J$ for the special case $f_{NJ} : X_J \longrightarrow X_N$ and $F_J := \{f_J \mid \boldsymbol{f} \in F\}$. Note that now $f_{KJ}.\mu_J = (f_{k_1 j_1}.\mu_{j_1}) \otimes \ldots \otimes (f_{k_n j_n}.\mu_{j_n})$. Consequently, $f_{KJ}.\mu_J \in \mathcal{M}(X_K)$ and $f_J.\mu_J \in \mathcal{M}(X_N)$.

We assume that each space $F_{ij}$ is equipped with a positive Borel measure $\sigma_{ij}$ and define $s_{ij} := \sigma_{ij}(F_{ij})$, or $s_{ij} = 0$ if $F_{ij}$ is empty. For each $J, K \in N^n$, we define the measure $\sigma_J := \sigma_{NJ} = \sigma_{1j_1} \otimes \ldots \otimes \sigma_{nj_n}$ and $s_{KJ} := s_{k_1 j_1} \cdot \ldots \cdot s_{k_n j_n}$.

The matrix $\boldsymbol{s} := (s_{ij})$ is a non-negative matrix. We now make the following compatibility assumption:

[4]Strictly speaking, we should say non-negative measures, but we will always explicitly mention when the 0-measure occurs.

**CA** The total measures $m_i = \mu_i(X_i)$ of the $\mu_i$ are all (strictly) positive, and $\boldsymbol{m} := (m_1, \dots, m_n)^T$ is an eigenvector of $\boldsymbol{s}$ with eigenvalue 1:

$$\boldsymbol{s}\boldsymbol{m} \; = \; \boldsymbol{m} \, . \tag{4.8}$$

REMARK: If $\boldsymbol{s}$ is non-negative, but $\boldsymbol{s}\boldsymbol{m} = \boldsymbol{m}$ for a vector $\boldsymbol{m}$ with all $m_i > 0$ as we assume in **CA**, the eigenvalue 1 is also the spectral radius of $\boldsymbol{s}$ (see Appendix 2 of [**KT**], and Corollary 2.2 of it in particular) and thus its Perron-Frobenius (PF) eigenvalue. Under the additional assumption of irreducibility of $\boldsymbol{s}$ (which we do not make!), $\boldsymbol{m}$ would be the unique PF eigenvector, and primitivity of $\boldsymbol{s}$ would further imply that all other eigenvalues of $\boldsymbol{s}$ were less than 1 in absolute value.

Let us also mention that there is no need to choose any particular normalization here, but a convenient one would be $m_N := m_1 \cdot \ldots \cdot m_n = 1$.

Define $\mathcal{P}^{\boldsymbol{m}}(X_N)$ to be the space of all *product* measures $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ where $\mu_i \in \mathcal{M}_+^{m_i}(X_i)$, i.e. $\mu_i$ is a positive measure of total variation $\|\mu_i\| = \mu_i(X_i) = m_i$. For each $\boldsymbol{f} = (f_{ij}) \in F$, we define the operator

$$\begin{array}{rcl} \mathcal{A}_{\boldsymbol{f}} : \; \mathcal{P}^{\boldsymbol{m}}(X_N) & \longrightarrow & \mathcal{M}(X_N) \\ \mu & \mapsto & \mathcal{A}_{\boldsymbol{f}}(\mu) \; := \; \sum_{J \in N^n} (f_J . \mu_J) \, . \end{array} \tag{4.9}$$

For any $\phi \in C(X_N, \mathbb{R})$, we have

$$\mathcal{A}_{\boldsymbol{f}}(\mu)(\phi) \; = \; \sum_{J \in N^n} \mu_J(\phi \circ f_J) \, . \tag{4.10}$$

In particular, if $\phi(x_1, \dots, x_n) = \phi_1(x_1) \cdot \ldots \cdot \phi_n(x_n)$ for some $\phi_i \in C(X_i, \mathbb{R})$, then this can be rewritten as

$$\begin{aligned} \mathcal{A}_{\boldsymbol{f}}(\mu)(\phi) \; &= \; \sum_{j_1, \dots, j_n} \mu_{j_1}(\phi_1 \circ f_{1j_1}) \cdot \ldots \cdot \mu_{j_n}(\phi_n \circ f_{nj_n}) \\ &= \; \Big( \sum_j (f_{1j} . \mu_j)(\phi_1) \Big) \cdot \ldots \cdot \Big( \sum_j (f_{nj} . \mu_j)(\phi_n) \Big) \\ &= \; \sum_{J \in N^n} \big( f_{1j_1} . \mu_{j_1} \otimes \ldots \otimes f_{nj_n} . \mu_{j_n} \big)(\phi) \, , \end{aligned}$$

which, since the linear span of the product functions $\phi = (\phi_1, \dots, \phi_n)$ is dense in $C(X_N, \mathbb{R})$, shows that

$$\mathcal{A}_{\boldsymbol{f}}(\mu) \; = \; \sum_{J \in N^n} f_{1j_1} . \mu_{j_1} \otimes \ldots \otimes f_{nj_n} . \mu_{j_n} \; = \; \sum_{J \in N^n} f_J . \mu_J \, . \tag{4.11}$$

We define the *averaging* operator $\mathcal{A}$ on $\mathcal{P}^{\boldsymbol{m}}(X_N)$ by

$$\begin{aligned} \mathcal{A}(\mu) \; &:= \; \int_F \mathcal{A}_{\boldsymbol{f}}(\mu) \, \mathrm{d}\boldsymbol{\sigma}(\boldsymbol{f}) \\ &= \; \int_F \sum_{J \in N^n} \Big( f_{1j_1} . \mu_{j_1} \otimes \ldots \otimes f_{nj_n} . \mu_{j_n} \Big) \, \mathrm{d}\sigma_{1j_1}(f_{1j_1}) \cdot \ldots \cdot \mathrm{d}\sigma_{nj_n}(f_{nj_n}) \\ &= \; \Big( \sum_j \int_{F_{1j}} (f_{1j} . \mu_j) \, \mathrm{d}\sigma_{1j}(f_{1j}) \Big) \cdot \ldots \cdot \Big( \sum_j \int_{F_{nj}} (f_{nj} . \mu_j) \, \mathrm{d}\sigma_{nj}(f_{nj}) \Big) \\ &= \; \sum_{J \in N^n} \int_{F_J} (f_J . \mu_J) \, \mathrm{d}\sigma_J(f_J) \, , \end{aligned} \tag{4.12}$$

for $\mu = \mu_1 \otimes \ldots \otimes \mu_n$. For any $\phi \in C(X_N, \mathbb{R})$, this reads:

$$\int_F \mathcal{A}_{\boldsymbol{f}}(\mu)(\phi)\,\mathrm{d}\boldsymbol{\sigma}(\boldsymbol{f}) \;=\; \sum_{J\in N^n} \int_{F_J} \mu_J(\phi\circ f_J)\,\mathrm{d}\sigma_J(f_J) \;\in\; \mathcal{M}(X_N)\,, \tag{4.13}$$

and, if $\phi(x_1,\ldots,x_n) = \phi_1(x_1)\cdot\ldots\cdot\phi_n(x_n)$ for some $\phi_i \in C(X_i,\mathbb{R})$, this becomes

$$\begin{aligned}
&\int_F \mathcal{A}_{\boldsymbol{f}}(\mu)(\phi)\,\mathrm{d}\boldsymbol{\sigma}(\boldsymbol{f})\\
&\quad= \Big(\sum_j \int_{F_{1j}} (f.\mu_j)\,\mathrm{d}\sigma_{1j}(f)(\phi_1)\Big)\cdot\ldots\cdot\Big(\sum_j \int_{F_{nj}} (f.\mu_j)\,\mathrm{d}\sigma_{nj}(f)(\phi_n)\Big).
\end{aligned}$$

This shows that the averaging process is of the sort envisaged in (4.6) and

$$\mathcal{A}(\mu) \;=\; \Big(\sum_j \int_{F_{1j}} (f.\mu_j)\,\mathrm{d}\sigma_{1j}(f)\Big)\otimes\ldots\otimes\Big(\sum_j \int_{F_{nj}} (f.\mu_j)\,\mathrm{d}\sigma_{nj}(f)\Big). \tag{4.14}$$

Furthermore, consider

$$\sum_j \int_{F_{ij}} (f.\mu_j)\,\mathrm{d}\sigma_{ij}(f) \;\in\; \mathcal{M}(X_i). \tag{4.15}$$

Since $f\in F_{ij}$ implies $\mathbf{1}_{X_i}\circ f = \mathbf{1}_{X_j}$, (4.15) satisfies

$$\begin{aligned}
\sum_j \int_{F_{ij}} (f.\mu_j)\,\mathrm{d}\sigma_{ij}(f)\,(\mathbf{1}_{X_i}) \;&=\; \sum_j \int_{F_{ij}} \mu_j(\mathbf{1}_{X_i}\circ f)\,\mathrm{d}\sigma_{ij}(f)\\
= \sum_j m_j \int_{F_{ij}} \mathrm{d}\sigma_{ij}(f) \;&=\; \sum_j s_{ij} m_j \;=\; m_i\,.
\end{aligned}$$

This shows that our averaging operator stabilizes the space of product measures that we are considering:

Proposition 4.2. *The averaging operator* $\mathcal{A} := \int_F \mathcal{A}_{\boldsymbol{f}}\,\mathrm{d}\boldsymbol{\sigma}(\boldsymbol{f})$ *of Eq.* (4.12) *maps the space* $\mathcal{P}^{\boldsymbol{m}}(X_N)$ *of product measures with mass vector* $\boldsymbol{m}$ *into itself.* □

If $\phi\in C(X_N,\mathbb{R})$ is a contraction, then so is $\phi\circ f_J : X_J \longrightarrow \mathbb{R}$ for every $\boldsymbol{f}\in F$:

$$\begin{aligned}
\big|\phi\circ f_J(x) - \phi\circ f_J(x')\big| \;&\le\; r_\phi\, d_N(f_J(x), f_J(x'))\\
&=\; r_\phi\, \sup\nolimits_{i\in N} d_i\big(f_{ij_i}(x_{j_i}), f_{ij_i}(x'_{j_i})\big)\\
&\le\; r_\phi\, r\, \sup\nolimits_{i\in N} d_{j_i}(x_{j_i}, x'_{j_i})\\
&=\; r_\phi\, r\, d_J(x,x')
\end{aligned}$$

for all $x, x'\in X_J$.

Define a metric $L_J$ on $\mathcal{M}_+^{m_J}(X_J)$, the space of positive measures of total measure $m_J := \prod_{i=1}^n m_{j_i} > 0$, by

$$L_J(\mu,\nu) \;=\; \frac{1}{m_J}\,\sup\big\{\,|\mu(\psi)-\nu(\psi)| \mid \psi\in \mathrm{Lip}(\le 1, X_J,\mathbb{R})\big\}\,. \tag{4.16}$$

This makes $\mathcal{M}_+^{m_J}(X_J)$ into a complete metric space by Proposition 2.2.

Define $L$ on $\mathcal{P}^{\boldsymbol{m}}(X_N)$ by

$$L(\mu,\nu) \;=\; \sup\,\{L_J(\mu_J,\nu_J) \mid J\in N^n\}\,. \tag{4.17}$$

Proposition 4.3. *The operator* $\mathcal{A} : \mathcal{P}^{\boldsymbol{m}}(X_N) \longrightarrow \mathcal{P}^{\boldsymbol{m}}(X_N)$ *is a contraction with respect to the metric* $L$, *with contractivity factor at most* $r$.

PROOF: Let $\mu, \nu \in \mathcal{P}^{\boldsymbol{m}}(X_N)$. In order to determine the $L_K(\mathcal{A}(\mu), \mathcal{A}(\nu))$, we have to determine $\mathcal{A}(\mu)_K$ for any $K \in N^n$. Now, $\mathcal{A}\mu = \mathcal{A}(\mu)$ is a product measure, and from (4.14) we find

$$\begin{aligned}
(\mathcal{A}\mu)_K &= \Big(\sum_{j_1} \int_{F_{k_1 j_1}} (f.\mu_{j_1})\,\mathrm{d}\sigma_{k_1 j_1}(f)\Big) \otimes \ldots \otimes \Big(\sum_{j_n} \int_{F_{k_n j_n}} (f.\mu_{j_n})\,\mathrm{d}\sigma_{k_n j_n}(f)\Big) \\
&= \sum_J \int_{F_{KJ}} (f.\mu_J)\,\mathrm{d}\sigma_{KJ}(f)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
L_K\big((\mathcal{A}\mu)_K, (\mathcal{A}\nu)_K\big) &= L_K\Big(\sum_J \int_{F_{KJ}} (f.\mu_J)\,\mathrm{d}\sigma_{KJ}(f)\,, \sum_J \int_{F_{KJ}} (f.\nu_J)\,\mathrm{d}\sigma_{KJ}(f)\Big) \\
&= \frac{1}{m_K} \sup_\psi \left| \sum_J \int_{F_{KJ}} \big((f.\mu_J) - (f.\nu_J)\big)(\psi)\,\mathrm{d}\sigma_{KJ}(f) \right| \\
&\le \frac{1}{m_K} \sum_J \sup_\psi \int_{F_{KJ}} \big| (\mu_J - \nu_J)(\psi \circ f)\big|\,\mathrm{d}\sigma_{KJ}(f) \\
&\le \frac{r}{m_K} \sum_J \int_{F_{KJ}} \sup_\psi \big| (\mu_J - \nu_J)(r^{-1}\psi \circ f)\big|\,\mathrm{d}\sigma_{KJ}(f) \\
&\le \frac{r}{m_K} \sum_J \int_{F_{KJ}} m_J L_J(\mu_J, \nu_J)\,\mathrm{d}\sigma_{KJ}(f) \\
&\le \frac{rL(\mu,\nu)}{m_K} \sum_J m_J \int_{F_{KJ}} \mathrm{d}\sigma_{KJ}(f) \;=\; \frac{rL(\mu,\nu)}{m_K} \sum_J s_{KJ} m_J\,,
\end{aligned}$$

where $\psi$ runs through $\mathrm{Lip}(\le 1, X_K, \mathbb{R})$. Finally,

$$\begin{aligned}
L(\mathcal{A}\mu, \mathcal{A}\nu) &= \sup_K L_K\big((\mathcal{A}\mu)_K, (\mathcal{A}\nu)_K\big) \;\le\; rL(\mu,\nu) \sup_K \frac{1}{m_K} \sum_J s_{KJ} m_J \\
&= rL(\mu,\nu) \sup_K \frac{1}{m_K} \prod_{i=1}^n \sum_{j_i=1}^n s_{k_i j_i} m_{j_i} \\
&= rL(\mu,\nu) \sup_K \frac{1}{m_K} \prod_{i=1}^n m_{k_i} \;=\; rL(\mu,\nu)\,,
\end{aligned}$$

where we have used (4.8) in the last line. We have thus established the following result.

THEOREM 4.4. *Let $X_1, \ldots, X_n$ be compact metric spaces and, for each pair $(i,j) \in N \times N$, let $F_{ij}$ be a compact admissible family of contractions (possibly empty) from $X_j$ to $X_i$. Assume that each $F_{ij}$ is equipped with a positive Borel measure $\sigma_{ij}$ and define $s_{ij} = \sigma_{ij}(F_{ij})$, with $s_{ij} := 0$ if $F_{ij}$ is empty. Assume that $\boldsymbol{s} := (s_{ij})$ has a positive 1-eigenvector $\boldsymbol{m} = (m_1, \ldots, m_n)^T$.*

*Then there exists a unique family of measures $\omega_i \in \mathcal{M}(X_i)_+^{m_i}$ which satisfy*

$$\omega_i \;=\; \sum_{j=1}^n \int_{F_{ij}} (f.\omega_j)\,\mathrm{d}\sigma_{ij}(f)\,, \quad i \in N\,. \tag{4.18}$$

*The support of $\omega$ is contained in $W$, the attractor of (4.5).* □

We call $\omega = \omega_1 \otimes \ldots \otimes \omega_n$ the $(F, \boldsymbol{\sigma}, \boldsymbol{m})$-invariant measure on $X_N$, or simply $(F, \boldsymbol{\sigma})$-invariant measure, if $\boldsymbol{m}$ is understood from the context.

## 5. Multi-component families of affine mappings

In this Section, $H$ is an LCAG and $A$ is an automorphism of $H$ with modulus $\alpha > 1$. The Haar measure on $H$ is denoted by $\theta$. $H$ is assumed to be complete with respect to a metric $d$, relative to which $A$ is a contraction.

We assume that we are given $n$ copies of $H$, which we call $H_1, \ldots, H_n$, and compact families $F_{ij}$, $1 \leq i, j \leq n$, of affine mappings $f_{ij} : H_j \to H_i$, all of the form $f_{ij}(x) = Ax + u_{ij}$ with $u_{ij} \in H_i$. Our objective is to understand the multi-component system formed by

$$H^n \;=\; H_1 \times \ldots \times H_n \;=\; H \times \ldots \times H$$

and the admissible family of contractions $F = \bigtimes F_{ij}$ in a way that parallels our previous analysis in Section 3.

By Proposition 4.1, $F$ has a unique attractor $W = W_1 \times \ldots \times W_n \in (\mathcal{K}H^n)$. We wish to describe the unique $F$-self-similar measure $\omega = \omega_1 \otimes \ldots \otimes \omega_n$ on $H^n$ with respect to a system $\boldsymbol{\sigma} = (\sigma_{ij})$ of measures on $F$.

The mapping $F_{ij} \to H_i$ defined by $f_{ij} \mapsto f_{ij}(0) = u_{ij}$ is continuous and produces a homeomorphism between $F_{ij}$ and a compact subset $F'_{ij} := F_{ij}(0)$ of $H_i$.

We assume that each compact space $F_{ij}$ is supplied with a positive regular Borel measure $\sigma_{ij}$, supported on $F_{ij}$ with $s_{ij} := \sigma_{ij}(F_{ij})$. We may identify $\sigma_{ij}$ with a regular Borel measure on $H_i$ supported on $F'_{ij}$. It is understood that $F_{ij}$ may be empty, in which case $s_{ij} := 0$. Furthermore, we assume the existence of a mass vector $\boldsymbol{m} = (m_1, \ldots, m_n)^T > 0$ satisfying the compatibility assumption **CA**, i.e. $\boldsymbol{s}\boldsymbol{m} = \boldsymbol{m}$.

We define $X_i \in \mathcal{K}H_i$, $i = 1, \ldots, n$, to be compact subspaces with the following properties:

1. $X_1 \times \ldots \times X_n$ is invariant under the family of mappings $F$;
2. $\sum_{k=0}^{\ell} A^k \operatorname{supp}(\boldsymbol{\sigma}) \subset X_1 \times \ldots \times X_n$, for all $\ell \geq 0$.

It is easy to see that such sets exist because the mappings of $F$ and the automorphism $A$ are all contractive. The $F$-invariance already forces $W_i \subset X_i$.

Let notation be as in Section 4, so $X_N = X_1 \times \ldots \times X_n$ and $\mathcal{P}^{\boldsymbol{m}}(X_N)$ is the space of all product measures $\mu_1 \otimes \ldots \otimes \mu_n$ on $X_N$ for which $\mu_i \in \mathcal{M}_+^{m_i}(X_i)$. We know that the averaging operator $\mathcal{A}$ of (4.12) is a contraction on $\mathcal{P}^{\boldsymbol{m}}(X_N)$.

Let $E_i \subset X_i$, $i \in N$, be measurable sets. Then, from (4.14), we obtain

$$\begin{aligned}
\mathcal{A}\mu(E_1 \times \ldots \times E_n) &= \prod_{i=1}^{n} \sum_j \int_{F_{ij}} (f.\mu_j)(E_i)\, \mathrm{d}\sigma_{ij}(f) \\
&= \prod_{i=1}^{n} \sum_j \int_{F_{ij}} \mu_j(f^{-1}(E_i))\, \mathrm{d}\sigma_{ij}(f) \\
&= \prod_{i=1}^{n} \sum_j \int_{F'_{ij}} (A.\mu_j)(E_i - u)\, \mathrm{d}\sigma_{ij}(u) \\
&= \prod_{i=1}^{n} \sum_j (\sigma_{ij} * A.\mu_j)(E_i)\,. \qquad (5.1)
\end{aligned}$$

Thus

$$\mathcal{A}\mu \;=\; \Big(\sum_j \sigma_{1j} * A.\mu_j\Big) \otimes \ldots \otimes \Big(\sum_j \sigma_{nj} * A.\mu_j\Big).$$

Adopting matrix notation, with $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ written as $(\mu_1, \ldots, \mu_n)^T$ and $\boldsymbol{\sigma} = (\sigma_{ij})$, this reads

$$\mathcal{A}\mu \;=\; \boldsymbol{\sigma} * A.\mu \tag{5.2}$$

where $A.\mu := (A.\mu_1, \ldots, A.\mu_m)^T$. If we now define $A.\boldsymbol{\sigma} := (A.\sigma_{ij})$, we can iterate (5.2). Observing Proposition 3.2(3), we obtain

$$\mathcal{A}^{\ell}\mu \;=\; \boldsymbol{\sigma} * A.\boldsymbol{\sigma} * \ldots * A^{\ell-1}.\boldsymbol{\sigma} * A^{\ell}.\mu\,. \tag{5.3}$$

We now proceed as in Section 3 to define a suitable sequence of measures $\big(\omega^{(\ell)} \in \mathcal{P}^{\boldsymbol{m}}(X)\big)_{\ell\geq 0}$. First, let

$$\omega^{(0)} \;:=\; \delta^{\boldsymbol{m}} \;=\; (m_1\delta_0, \ldots, m_n\delta_0)^T \tag{5.4}$$

where $\delta_0$ is the unit point measure supported at $\{0\}$. Clearly, $\delta^{\boldsymbol{m}} \in \mathcal{P}^{\boldsymbol{m}}(X)$, but since $A$ is a contraction, we also have $A.\delta^{\boldsymbol{m}} = \delta^{\boldsymbol{m}}$, and $A^{\ell}.\mu \to \delta^{\boldsymbol{m}}$ as $\ell \to \infty$, for any $\mu \in \mathcal{P}^{\boldsymbol{m}}(X)$. Define iteratively, as before, $\omega^{(\ell+1)} = \mathcal{A}\omega^{(\ell)}$ for $\ell \geq 0$. Then

$$\omega^{(\ell+1)} \;=\; \boldsymbol{\sigma} * A.\boldsymbol{\sigma} * \ldots * A^{\ell}.\boldsymbol{\sigma} * \delta^{\boldsymbol{m}}\,.$$

We have $\operatorname{supp}(\omega^{(\ell)}) \subset \sum_{k=0}^{\ell} A^k \operatorname{supp}(\boldsymbol{\sigma}*\delta^{\boldsymbol{m}})$ and $\mathcal{A}\mathcal{P}^{\boldsymbol{m}}(X) \subset \mathcal{P}^{\boldsymbol{m}}(X)$, so we know, since $\delta^{\boldsymbol{m}} \in \mathcal{P}^{\boldsymbol{m}}(X)$, that $\omega^{(\ell)} \in \mathcal{P}^{\boldsymbol{m}}(X)$. Consequently, $\omega^{(\ell)}$ vaguely converges, as $\ell \to \infty$, to the unique $(F, \boldsymbol{\sigma}, \boldsymbol{m})$-self-similar measure $\omega = \omega_1 \otimes \ldots \otimes \omega_n \in \mathcal{P}^{\boldsymbol{m}}(X)$. Since $\operatorname{supp}(\omega) \subset W$ by Theorem 4.4, we find that $\omega \in \mathcal{P}^{\boldsymbol{m}}(W)$. To summarize:

Proposition 5.1. *Let $H$ be an LCAG which is a complete metric space. Let $A$ be a contractive automorphism on $H$ and let $F_{ij}$, $1 \leq i,j \leq n$, with attractor $W$, be compact admissible families of affine maps on $H$, all of the form $x \mapsto Ax + v$ with $v \in H$. Let $\sigma_{ij}$ be a positive regular Borel measure on $F_{ij}$, identified with a regular Borel measure on $H$ supported on $F_{ij}(0)$ (with $\sigma_{ij} := 0$ if $F_{ij} = \emptyset$). Let $\boldsymbol{s} = (s_{ij}) = (\sigma_{ij}(H))$ and, finally, let $\boldsymbol{m} = (m_1, \ldots, m_n)^T > 0$ satisfy $\boldsymbol{s}\boldsymbol{m} = \boldsymbol{m}$. Then*

1. *$\omega \;=\; \big(\ast_{\ell=0}^{\infty} A^{\ell}.\boldsymbol{\sigma}\big) * \delta^{\boldsymbol{m}}$ is the unique $(F, \boldsymbol{\sigma}, \boldsymbol{m})$-self-similar measure of (4.18), with $\omega \in \mathcal{P}^{\boldsymbol{m}}(H^n)$, $\operatorname{supp}(\omega) \subset W$, and $\delta^{\boldsymbol{m}}$ as in (5.4).*
2. *$\widehat{\omega} \;=\; \big(\prod_{\ell=0}^{\infty} (A^T)^{-\ell}.\widehat{\boldsymbol{\sigma}}\big)\; \boldsymbol{1}^{\boldsymbol{m}}$, where $\boldsymbol{1}^{\boldsymbol{m}} = (m_1\boldsymbol{1}_H, \ldots, m_n\boldsymbol{1}_H)^T$, the convergence of the product being uniform on compact sets.* □

If we assume that the measures $\sigma_{ij}$ are absolutely continuous with respect to Haar measure $\theta$ on $H$, then $\sigma_{ij} = h_{ij}\theta$ where $h_{ij} \in L^1(H)$ due to the Radon-Nikodym theorem. In particular, we have $\operatorname{supp}(h_{ij}) \subset F_{ij}(0) \subset H$, $h_{ij} \geq 0$ and $\|h_{ij}\|_1 = \int_H h_{ij}\,\mathrm{d}\theta = s_{ij}$, for all $1 \leq i,j \leq n$. Then

$$\begin{aligned}
\omega^{(\ell+1)} \;&=\; \boldsymbol{\sigma} * A.\boldsymbol{\sigma} * \ldots * A^{\ell}.\boldsymbol{\sigma} * \delta^{\boldsymbol{m}} \\
&=\; \boldsymbol{h}\Theta * A.(\boldsymbol{h}\Theta) * \ldots * A^{\ell}.(\boldsymbol{h}\Theta) \,*\, \delta^{\boldsymbol{m}} \\
&=\; \boldsymbol{h}\Theta * \alpha(A.\boldsymbol{h})\Theta * \ldots * \alpha^{\ell}(A^{\ell}.\boldsymbol{h})\Theta \,*\, \delta^{\boldsymbol{m}} \\
&=\; \boldsymbol{h} * \alpha(A.\boldsymbol{h}) * \ldots * \alpha^{\ell}(A^{\ell}.\boldsymbol{h})\,(\Theta * \delta^{\boldsymbol{m}})
\end{aligned}$$

where $\boldsymbol{h} = (h_{ij})$ and $\Theta = \mathrm{diag}(\theta, \dots, \theta)$ is a diagonal matrix. Thus we have $\Theta * \delta^{\boldsymbol{m}} = (m_1\theta, \dots, m_n\theta)^T$ and

$$\omega^{(\ell+1)} \; = \; \left( \mathop{\ast}_{k=0}^{\ell} \alpha^k(A^k.\boldsymbol{h}) \right) (m_1\theta, \dots, m_n\theta)^T \;\in\; \mathcal{P}^{\boldsymbol{m}}(H^n)\,. \tag{5.5}$$

Vague convergence of this sequence is clear, but the results of Section 4 suggest that we can expect more. However, $\|.\|$-convergence is technically more involved here. Let us thus first postpone this question and state first the result on the self-similar functions.

PROPOSITION 5.2. *Let notation and assumptions be as in Proposition* 5.1*, and suppose that the measures $\sigma_{ij} = h_{ij}\theta$ are absolutely continuous with respect to Haar measure $\theta$. Assume that the convolution in* (5.5)*, as $\ell \to \infty$, converges also in the $\|.\|$-topology. Then there is a unique vector $\boldsymbol{g} = (g_1, \dots, g_n)^T$ of non-negative functions in $L^1(H)$ that satisfies*

1. $\boldsymbol{g} = \left(\mathop{\ast}_{\ell=0}^{\infty} \alpha^\ell(A^\ell.\boldsymbol{h})\right)\boldsymbol{m}$
2. $g_i(x) = \sum_{j=1}^n \int_H h_{ij}(x-v)\, g_j(A^{-1}v)\,\mathrm{d}\theta(v)\,,\; i = 1, \dots, n,$ *with normalization* $\int_H g_i \,\mathrm{d}\theta = m_i$ *and* $\mathrm{supp}(g_i) \subset W_i$.

*Furthermore, if the $h_{ij}$ are functions in $L^1(H) \cap L^\infty(H)$, then the functions $g_i$ are continuous on $H$.*

PROOF: Part 1 is a direct reformulation of Eq. (5.5), and Part 2 is a componentwise recounting of Part 1. The continuity follows from the properties of the convolution product, as in Proposition 3.6(4). □

Infinite convolution products like that of Proposition 5.2(1) also appear in the context of matrix continuous refinement operators. These are introduced in [**JL**] (with $H$ being $\mathbb{R}^n$). In our paper [**BM2**], we relied on the results of [**JL**] for the existence of our self-similar densities. However, the methods of [**JL**] are from functional analysis and do not lend themselves to the general measure theoretic situation that we are trying to address here.

Let us come back to the convergence issue in Eq. (5.5). Unlike the situation in Section 3.3, with Lemma 3.4 and Proposition 3.5, the $\|.\|$-convergence in (5.5) is not entirely automatic. Note that the vector notation for the measures is handy for the formulation of the iteration, but it still represents a product measure. We are interested in the $\|.\|$-convergence of the sequence of product measures (5.5). For this, it is sufficient, but not necessary, that the sequence $K_\ell = \mathop{\ast}_{k=0}^{\ell} \alpha^k(A^k.\boldsymbol{h})$, seen as a sequence of linear operators, converges in the operator norm. However, for fixed $i, j$, $\|(K_\ell)_{ij}\|_1 = (\boldsymbol{s}^{\ell+1})_{ij}$, and convergence of this, for $\ell \to \infty$, does not follow from our general assumptions on the matrix $\boldsymbol{s}$, see the Remark after Eq. (4.8), because we did not assume primitivity of $\boldsymbol{s}$.

Nevertheless, there is an analogue of Proposition 3.5 which we will now derive. To this end, define $\boldsymbol{h}^{(k)} = \alpha^k(A^k.\boldsymbol{h})$ for $k \geq 0$. In particular, $\boldsymbol{h}^{(0)} = \boldsymbol{h} = (h_{ij})$, which is a matrix of functions in $L^1(H)$, and also each $(\boldsymbol{h}^{(k)})_{ij}$ is a non-negative $L^1$-function of norm $s_{ij}$. Recall that (5.5) means $\omega^{(\ell)} = f_1^{(\ell)}\theta \otimes \ldots \otimes f_n^{(\ell)}\theta$, with $L^1$-functions $f_i^{(\ell)} = \sum_j (\mathop{\ast}_{k=0}^{\ell-1} \boldsymbol{h}^{(k)})_{ij}\, m_j$ of norm $\|f_i^{(\ell)}\|_1 = \sum_j (\boldsymbol{s}^\ell)_{ij}\, m_j = m_i$. Consequently, showing that (5.5) converges also in the $\|.\|$-topology means showing that $f_i^{(\ell)}$ converges in $L^1(H)$ for each $i$ as $\ell \to \infty$.

Fix $\varepsilon > 0$, and let $V_{ij}$ be the corresponding neighbourhood for the function $h_{ij}$ according to Lemma 3.4. Let $V = \bigcap_{i,j} V_{i,j}$ and choose an integer $M$ such that, for all $k \geq 0$ and all $i, j$, the non-negative $L^1$-function $(\ast_{\ell=M}^{M+k} \boldsymbol{h}^{(\ell)})_{ij}$, of norm $(\boldsymbol{s}^{k+1})_{ij}$, has support inside $V$. Such an $M$ clearly exists. With Lemma 3.4, we then find, for all $i', j'$ simultaneously, the approximation formula

$$\big\| (\ast_{\ell=M}^{M+k} \boldsymbol{h}^{(\ell)})_{ij} * h_{i'j'} - (\boldsymbol{s}^{k+1})_{ij}\, h_{i'j'} \big\|_1 \;\leq\; (\boldsymbol{s}^{k+1})_{ij}\, \varepsilon\,.$$

Note that this formulation remains valid even in the limiting case that $(\boldsymbol{s}^{k+1})_{ij}$ happens to vanish.

Let now $n, m \geq M$ and define $u_{ij} = (\ast_{\ell=M}^{n} \boldsymbol{h}^{(\ell)})_{ij}$ and $v_{ij} = (\ast_{\ell=M}^{m} \boldsymbol{h}^{(\ell)})_{ij}$. Then, we can calculate as follows

$$\begin{aligned}
\big\| f_i^{(n)} - f_i^{(m)} \big\|_1 \;&=\; \big\| \textstyle\sum_{k,\ell,j} (\boldsymbol{h}^{(1)} * \ldots * \boldsymbol{h}^{(M-1)})_{k\ell} * h_{ik} * (u_{\ell j} - v_{\ell j})\, m_j \big\|_1 \\
&\leq\; \textstyle\sum_{k,\ell} \Big( \big\| (\boldsymbol{h}^{(1)} * \ldots * \boldsymbol{h}^{(M-1)})_{k\ell} \big\|_1 \cdot \big\| \sum_j h_{ik} * (u_{\ell j} - v_{\ell j})\, m_j \big\|_1 \Big) \\
&\leq\; \textstyle\sum_{k,\ell} (\boldsymbol{s}^{M-1})_{k\ell}\, \big\| \sum_j h_{ik} * (u_{\ell j} - v_{\ell j}) m_j \big\|_1
\end{aligned}$$

where we have used that convolution on the level of functions is commutative. Observe next that

$$\begin{aligned}
\big\| \textstyle\sum_j h_{ik} * (u_{\ell j} - v_{\ell j})\, m_j \big\|_1 \;&\leq\; \big\| h_{ik} \textstyle\sum_j \big( (\boldsymbol{s}^{n-M+1})_{\ell j} - (\boldsymbol{s}^{m-M+1})_{\ell j} \big) m_j \big\|_1 \\
&\quad + \textstyle\sum_j \Big( \big\| h_{ik} * u_{\ell j} - h_{ik}\, (\boldsymbol{s}^{n-M+1})_{\ell j} \big\|_1 + \big\| h_{ik} * v_{\ell j} - h_{ik}\, (\boldsymbol{s}^{m-M+1})_{\ell j} \big\|_1 \Big)\, m_j \\
&\leq\; \varepsilon \big( \textstyle\sum_j (\boldsymbol{s}^{n-M+1})_{\ell j}\, m_j + \sum_j (\boldsymbol{s}^{m-M+1})_{\ell j}\, m_j \big) \;=\; 2 m_\ell\, \varepsilon
\end{aligned}$$

where we have used the above approximation formula and the equation $\boldsymbol{s}\boldsymbol{m} = \boldsymbol{m}$. This finally gives

$$\big\| f_i^{(n)} - f_i^{(m)} \big\|_1 \;\leq\; 2\, \varepsilon \textstyle\sum_{k,\ell} (\boldsymbol{s}^{M-1})_{k\ell}\, m_\ell \;=\; 2\, (m_1 + \ldots + m_n)\, \varepsilon\,,$$

independently of $i$. This shows that all sequences $(f_i^{(n)})_{n \geq 0}$ are Cauchy, and we have thus established the expected analogue of Proposition 3.5 and strengthening of Proposition 5.2:

PROPOSITION 5.3. *Let notation and assumptions be as in Proposition* 5.1*, and suppose that the measures* $\sigma_{ij} = h_{ij}\theta$ *are absolutely continuous with respect to Haar measure* $\theta$*. Then, the sequence of product measures in* (5.5)*, as* $\ell \to \infty$*, converges not only vaguely, but also in the* $\|.\|$*-topology of* $\mathcal{P}^{\boldsymbol{m}}(H \times \ldots \times H)$*.* □

## 6. Model sets and Weyl's Theorem

To link our previous analysis to quasicrystals, let us now summarize some of the key ingredients to their mathematical description. A *cut and project scheme* consists of the following set of data:

- a real space $\mathbb{R}^m$
- a locally compact Abelian group $H$
- a lattice $\tilde{L} \subset \mathbb{R}^m \times H$

which satisfies the following properties. If $\pi$ and $\pi_H$ are the natural projections of $\mathbb{R}^m \times H$ onto $\mathbb{R}^m$ and $H$, respectively, then

- $\pi|_{\tilde{L}}$ is one-to-one.
- $\pi_H(\tilde{L})$ is dense in $H$.

This is summarized in the following diagram.

$$
\begin{array}{ccccc}
\mathbb{R}^m & \xleftarrow{\pi} & \mathbb{R}^m \times H & \xrightarrow{\pi_H} & H \\
 & {\scriptstyle 1\text{-}1}\ \nwarrow & \cup & \nearrow\ {\scriptstyle \text{dense}} & \\
 & & \tilde{L} & &
\end{array}
\tag{6.1}
$$

To say that $\tilde{L}$ is a *lattice* in $\mathbb{R}^m \times H$ means that $\tilde{L}$ is a discrete subgroup of $\mathbb{R}^m \times H$ such that $(\mathbb{R}^m \times H)/\tilde{L}$ is compact.

We set $L = \pi(\tilde{L})$, a subgroup of $\mathbb{R}^m$, and define the *star map* $(.)^* : L \to H$ by $x^* = \pi_H \circ \left(\pi|_{\tilde{L}}\right)^{-1}(x)$. Although $(.)^*$ is a group homomorphism, it has, in general, no natural extension to $\mathbb{R}^m$ and, indeed, it is typically totally discontinuous in the topology on $L$ induced by $\mathbb{R}^m$. In fact, it is this property that makes it useful!

Given any subset $U \subset H$, we define

$$
\Lambda(U) \ := \ \{x \in L \mid x^* \in U\} \ = \ \{\pi(\tilde{x}) \mid \tilde{x} \in \tilde{L}\,,\ \pi_H(\tilde{x}) \in U\} \ \subset \ L \ \subset \ \mathbb{R}^m\,. \tag{6.2}
$$

A set $\Lambda \subset \mathbb{R}^m$ is a *model set* relative to (6.1), if $\Lambda = \Lambda(W)$ for some $W \subset H$ that is compact and equals the closure of its non-empty interior[5].

Model sets have remarkable properties that make them important objects of study in the theory of mathematical quasicrystals. We refer the reader to [**B, M, P**] and references therein for more details, but we mention here a few of the key points:

1. If $\Lambda \subset \mathbb{R}^m$ is a model set, then it is a *Delone set* in $\mathbb{R}^m$, i.e. $\Lambda$ is both uniformly discrete and relatively dense.
2. Generically, model sets have no translational symmetries, although they certainly still have a high degree of long-range order.
3. If $W$ is *Riemann measurable*, i.e. if $\partial W$ has vanishing Haar measure in $H$, then $\Lambda$ has a well-defined density (see Proposition 6.1 below).
4. If $W$ is *Riemann measurable*, then $\Lambda$ is pure point diffractive [**Hof1, Sch2**].

Model sets appeared very early in the theory of quasicrystals (under the name of cut and project sets, see [**B**] and references given there), but originally only with the internal group $H$ being another real space. However, model sets had been defined much earlier and in full generality in a totally different context by Y. Meyer [**Mey**]. Recent papers [**M, BMS, Sch2**] show that the more general setting is completely relevant in the mathematical theory of quasicrystals, aperiodic tilings and substitution systems, both geometric and algebraic.

A key feature of a model set is that it puts together a discrete geometric object, $\Lambda(W)$, with a relatively compact set $W \subset H$ on which we can use the powerful array of tools from analysis on locally compact Abelian groups. The essential mathematical link is Weyl's Theorem on uniform distribution [**We**] which connects densities on $\Lambda(W)$ to measures on $W$. We refer to this theory in the more general setting of LCAG's, see [**KN**, Ch. 4.4] for background material.

In its usual form, Weyl's Theorem is stated for real spaces, but it works at the level of locally compact Abelian groups, too. Here we establish the theorem in this more general setting. The basis of the theorem in the context of model sets is the phenomenon of 'uniformity of projection', which we state here in the generality

[5]There are variations on the exact conditions imposed on $W$ depending on the delicateness of the results required. Our assumptions imply the Delone property of $\Lambda$, and are rather convenient for many other purposes. There is still something unsatisfying about our present understanding of model sets. To say that $\Lambda \subset \mathbb{R}^m$ is a model set is to say that it arises from some cut and project scheme. But we still lack a useful direct characterization of such sets, compare [**Sch1**].

that we will need. In fact, our proof demonstrates that Weyl's Theorem, in this context, is actually equivalent to the uniformity of projection. For an approach to uniformity of projection via ergodic theory, see [**Hof2, Sch2**].

In the following two Propositions, it is understood that a cut and project scheme according to (6.1) has been given. In addition, let $\tilde{\theta}$ denote the product measure on $\mathbb{R}^m \times H$, formed from Lebesgue measure on $\mathbb{R}^m$ and our fixed Haar measure $\theta$ on $H$. Let now $T$ be any measurable fundamental domain of $\mathbb{R}^m \times H$ with respect to the action of its discrete subgroup $\tilde{L}$, and define $|\tilde{L}| := \tilde{\theta}(T)$ as its volume. Note that the value of $|\tilde{L}|$ does not depend on the actual choice of $T$. Its meaning really is the averaged number of lattice points per unit volume (in Haar measure).

PROPOSITION 6.1. (Schlottmann [**Sch1**]) *Let a cut and project setup according to diagram* (6.1) *be given, with* $|\tilde{L}|$ *as described above. Let* $U \subset H$ *be totally bounded and Riemann measurable* (*i.e.* $U$ *measurable with* $\theta(\partial U) = 0$). *Then*

$$\lim_{r\to\infty} \frac{1}{\operatorname{vol} B_r(0)} \left( \sum_{x\in\Lambda(U)\cap B_r(a)} 1 \right) \;=\; \frac{\theta(U)}{|\tilde{L}|}\,.$$

*Furthermore, the limit is uniform in* $a$. □

This limit is called the *density*, $\operatorname{den}(\Lambda(U))$, of $\Lambda(U)$. Note that the totally bounded set $U$ in this Proposition need not be closed. It is only demanded that its boundary has vanishing Haar measure. If $U$ itself is of measure 0, the density of $\Lambda(U)$ vanishes.

REMARK: There is another way to explain the meaning of $|\tilde{L}|$ which is perhaps more natural from the group theoretic point of view. Consider the factor group $T' := (\mathbb{R}^m \times H)/\tilde{L}$ (which is compact) and let $\mu$ be its normalized Haar measure. If now $f$ is a continuous function on $\mathbb{R}^m \times H$ with compact support, define a new function by $F(x) = \sum_{u\in\tilde{L}} f(x+u)$. So, $F$ results from $f$ by averaging over the canonical Haar measure of the lattice $\tilde{L}$, which is counting measure. The function $F$ can then also be viewed as a function on $T'$, and we can determine its integral, $\mu(F)$. If we then define a new measure on $\mathbb{R}^m \times H$ by $\tilde{\theta}'(f) := \mu(F)$, it is another Haar measure on $\mathbb{R}^m \times H$, and we must have $\tilde{\theta}' = c\,\tilde{\theta}$. The constant $c$ is nothing but $|\tilde{L}|$, see [**D**, Ch. XIV.4] for background material.

THEOREM 6.2. (Weyl's Theorem for general model sets) *Let* $\Lambda = \Lambda(W)$ *be a model set in the above sense, with compact, Riemann measurable* $W \subset H$. *Let* $f : H \to \mathbb{R}$ *be continuous with* $\operatorname{supp}(f) \subset W$. *Let* $p : L \to \mathbb{R}$ *be defined by* $p(x) = f(x^*)$. *Then*

$$\lim_{r\to\infty} \frac{1}{\operatorname{vol} B_r(0)} \sum_{x\in\Lambda\cap B_r(a)} p(x) \;=\; \frac{1}{|\tilde{L}|} \int_H f(y)\,\mathrm{d}\theta(y)$$

*uniformly in* $a$.

PROOF: The strategy will be to derive this more general result from Proposition 6.1. To this end, we approximate $f$ by a step function $\psi$ on a (Riemann) admissible partition $\{U_1, \ldots, U_n\}$ of $W$, i.e. $W = \bigcup_{i=1}^n U_i$ with pairwise disjoint sets $U_i \subset W$ that are all Riemann measurable.

Fix $\varepsilon > 0$. By Lemma A.6, there is a step function $\psi$ on such an admissible partition (with suitable $n = n(\varepsilon)$) of $W$, $\psi = \sum_{i=1}^n c_i\,\mathbf{1}_{U_i}$, with $\|f-\psi\|_\infty < \varepsilon$.

Choose a radius $R$ big enough so that we have, for all $r > R$,

$$\left| \frac{1}{\operatorname{vol} B_r(0)} \left( \sum_{x \in \Lambda \cap B_r(a)} 1 \right) - \frac{\theta(W)}{|\tilde{L}|} \right| < \varepsilon$$

and also

$$\left| \frac{1}{\operatorname{vol} B_r(0)} \left( \sum_{x \in \Lambda(U_i) \cap B_r(a)} 1 \right) - \frac{\theta(U_i)}{|\tilde{L}|} \right| < \frac{\varepsilon}{n}$$

for all $1 \le i \le n$, uniformly in $a$. Such a radius clearly exists. Then we have, since $p(x) = f(x^*)$, for all $r > R$ the following $3\varepsilon$-type argument,

$$\begin{aligned}
&\left| \frac{1}{\operatorname{vol} B_r(0)} \left( \sum_{x \in \Lambda \cap B_r(a)} p(x) \right) - \frac{1}{|\tilde{L}|} \int_W f(y) \, \mathrm{d}\theta(y) \right| \\
&\le \frac{1}{\operatorname{vol} B_r(0)} \left( \sum_{x \in \Lambda \cap B_r(a)} \big| f(x^*) - \psi(x^*) \big| \right) + \frac{1}{|\tilde{L}|} \left| \int_W \big( f(y) - \psi(y) \big) \, \mathrm{d}\theta(y) \right| + \\
&\quad \sum_{i=1}^{n} \left| \frac{c_i}{\operatorname{vol} B_r(0)} \left( \sum_{x \in \Lambda(U_i) \cap B_r(a)} 1 \right) - \frac{1}{|\tilde{L}|} \int_{U_i} \psi(y) \, \mathrm{d}\theta(y) \right| \\
&< \frac{\varepsilon}{\operatorname{vol} B_r(0)} \left( \sum_{x \in \Lambda \cap B_r(a)} 1 \right) + \frac{\theta(W)}{|\tilde{L}|} \varepsilon + \\
&\quad \sum_{i=1}^{n} |c_i| \left| \frac{1}{\operatorname{vol} B_r(0)} \left( \sum_{x \in \Lambda(U_i) \cap B_r(a)} 1 \right) - \frac{\theta(U_i)}{|\tilde{L}|} \right| \\
&< \Big( 2 \frac{\theta(W)}{|\tilde{L}|} + 1 \Big) \varepsilon + \sum_{i=1}^{n} \|\psi\|_\infty \frac{\varepsilon}{n} < \Big( 2 \frac{\theta(W)}{|\tilde{L}|} + \|f\|_\infty + 2 \Big) \varepsilon ,
\end{aligned}$$

from which the Theorem follows. □

REMARK: Weyl's Theorem also extends to all functions that are only continuous on the compact set $W$, see the Remark following the proof of Lemma A.6.

## 7. Self-similar densities on model sets

Finally, we have collected all results that we need to construct self-similar measures for model sets on their "internal" side, and then, under certain circumstances, also invariant densities on the model sets themselves.

**7.1. Self-similar systems.** An affine self-similar system of model sets consists of the the following data (**SS1**–**SS4**):

- **SS1** a cut and project scheme (6.1) whose internal space $H$, in addition to being an LCAG, is a complete metric space with translation invariant metric $d$.
- **SS2** a family of regular model sets $\Lambda_i = \Lambda(W_i)$, $i = 1, \ldots, n$, for this cut and project scheme, with each $W_i$ compact.
- **SS3** an invertible linear mapping $Q: \mathbb{R}^m \to \mathbb{R}^m$ which satisfies $Q(L) \subset L$, where $L$ is the projection $\pi(\tilde{L})$ of the lattice $\tilde{L}$ in (6.1).

**SS4** sets $F_{ij}$, $1 \le i, j \le n$, some of which may be empty, of affine mappings

$$C \;=\; C_a \;:\quad x \mapsto Qx + a \quad (a \in L)$$

which map $\Lambda_j$ to $\Lambda_i$ and satisfy

$$\Lambda_i \;=\; \bigcup_{j=1}^{n} \; \bigcup_{C \in F_{ij}} C(\Lambda_j)\,, \quad 1 \le i \le n\,. \tag{7.1}$$

The sets $F_{ij}$ may (and usually will) be infinite. Because all the affine mappings involved have the same linear part, $Q$, each $F_{ij}$ is parameterized by the translational parts $a \in L$. In the sequel, we will thus mostly view $F_{ij}$ as a subset of $L$. In this case, we will, from now on, use the notation $F'_{ij}$, i.e. $F'_{ij} := F_{ij}(0)$.

Such systems of model sets can arise quite naturally in the study of self-similar tilings. Each proto-tile is marked in some suitable way with a finite set of points, call them proto-points or *control points*. This provides a marking of the tiling by points, and the sets $\Lambda_i$ are then taken to be the set of points that correspond to each class of control points. In this case, the union in (7.1) would typically be disjoint, but in our study we definitely wish to include non-disjoint unions as well.

The idea of this Section is to pass the self-similar system to the internal side $H$ of the cut and project scheme, to apply our theory of self-similar measures there, and finally to pull back the results to the physical side, namely to the model sets $\Lambda_i$ themselves. We will see that pulling back is not automatically possible, and we need to make various types of assumptions to guarantee it. Still, these assumptions are not unnatural, and they are actually met in many interesting cases.

The situation for simple model sets $\Lambda = \Lambda(W)$ is just the special case of the general situation here, where $n = 1$. In this case, the matrix $\boldsymbol{s}$ that appears below is simply the unit matrix $(1)$.

We can directly lift $Q : \; L \to L$ to a group homomorphism $\tilde{Q} : \; \tilde{L} \to \tilde{L}$, and then to a group homomorphism $Q^* : \; L^* \to L^*$. We assume

**SS5** $Q^*$ is contractive with respect to the metric $d$.

In this case, since $L^*$ is dense in $H$, $Q^*$ extends to a continuous contractive automorphism

$$A : \quad H \longrightarrow H$$

with $A|_{L^*} = Q^*$. Due to Lemma 3.1, the modulus $\alpha$ of $A$ with respect to the Haar measure $\theta$ on $H$ satisfies $\alpha > 1$, see Section 3 for details.

For each affine map $C_a : \; x \mapsto Qx + a$, $a \in L$, we define the affine mapping $C^*_a$ on $H$ by $y \mapsto Ay + a^*$. In this way, we arrive at admissible families of contractions $F^*_{ij}$ on $H$ (see Section 1). We let $\mathcal{F}_{ij}$ be the closure of $F^*_{ij}$ in the space $C(H,H)$ of continuous mappings on $H$. If we identify the mappings in $F_{ij}$ with their translational parts in $L$, then $F'^*_{ij}$ is viewed as a subset of $L^*$ and $\mathcal{F}'_{ij}$ is the closure of this in $H$ (see Section 3.2 where we did a similar thing). Let us summarize our notation in the following diagram, where $*$ stands for the $*$-map and $'$ for the mapping that links affine transformations with their translational parts.

$$\begin{array}{ccccccc}
F_{ij} & \stackrel{*}{\longleftrightarrow} & F^*_{ij} & \subset & \mathcal{F}_{ij} & \subset & C(H,H) \\
{}'\updownarrow & & {}'\updownarrow & & {}'\updownarrow & & \\
F'_{ij} & \stackrel{*}{\longleftrightarrow} & F'^*_{ij} & \subset & \mathcal{F}'_{ij} & \subset & H
\end{array}$$

From (7.1), we have, for all $1 \le i \le n$,

$$\Lambda_i^* \;=\; \bigcup_{j=1}^{n} \; \bigcup_{C \in F_{ij}} C^*(\Lambda_j^*) \tag{7.2}$$

and taking closures gives us

$$W_i \;\supset\; \bigcup_{j=1}^{n} \; \bigcup_{C^* \in F_{ij}^*} C^*(W_j)\,. \tag{7.3}$$

Since the $W_i$ are compact and $C^*(W_j) = AW_j + a^* \subset W_i$ for $C = C_a$, we see that the translational parts $F'_{ij}$ of the affine mappings are bounded (with respect to $d$). Thus we have that $\mathcal{F}'_{ij}$ is compact in $H$ and $\mathcal{F}_{ij}$ is compact in $C(H, H)$.

Proposition 7.1. *Under the above conditions, we have*

$$W_i \;=\; \bigcup_{j=1}^{n} \; \bigcup_{D \in \mathcal{F}_{ij}} D(W_j)\,, \quad i = 1, \ldots, n\,. \tag{7.4}$$

*and* $W_1 \times \ldots \times W_n$ *is the attractor of* $\mathcal{F}$.

To prove this result, we first establish

Lemma 7.2. *Let $F$ be a relatively compact set of continuous mappings from $H$ to $H$. Suppose that $U, V$ are compact subsets of $H$ such that $C(U) \subset V$ for all $C \in F$. Then, $D(U) \subset V$ for all $D \in \overline{F}$.*

Proof: Let $D \in \overline{F}$. Fix any $\varepsilon > 0$ and let $K = K(U, B_\varepsilon(0))$ be the set of all continuous mappings of $H$ to itself that map $U$ inside $B_\varepsilon(0)$. This is an open neighbourhood of 0 in $C(H, H)$, so $D - C \in K$ for some $C \in F$. Thus

$$D(U) \;\subset\; C(U) + B_\varepsilon(0) \;\subset\; V + B_\varepsilon(0) \;\subset\; [V]_\varepsilon\,.$$

This being true for all $\varepsilon > 0$, we have $D(U) \subset V$. □

Proof of Proposition 7.1: Consider (7.3). Using Lemma 7.2, we get

$$W_i \;\supset\; \bigcup_{j=1}^{n} \; \bigcup_{D \in \mathcal{F}_{ij}} D(W_j)\,.$$

The right hand side is compact by (1.10) and contains $\Lambda_i^*$ by (7.2), hence also $\overline{\Lambda_i^*} = W_i$. □

Remark: A solution to (7.4) is guaranteed by the general theory of contractions of Section 4. However, in the present situation, we know more: the $W_i$ have non-empty interiors, since the $\Lambda_i$ are model sets. In general, it is hard to know when such a self-similar system of mappings leads to an attractor with non-empty interior.

**7.2. Self-similar measures.** We now assume that each $\mathcal{F}_{ij}$ is equipped with a positive regular Borel measure $\sigma_{ij}$, with $\sigma_{ij} := 0$ if $\mathcal{F}_{ij} = \emptyset$ as before. As in Proposition 5.1, we usually identify $\sigma_{ij}$ with a positive Borel measure on $H$ that is supported on $\mathcal{F}'_{ij}$. We set $s_{ij} = \sigma_{ij}(\mathcal{F}'_{ij}) = \sigma_{ij}(H)$ and restate the compatibility assumption **CA** for the matrix $\boldsymbol{s} = (s_{ij})$, namely that it has a positive 1-eigenvector:

**SS6** There is a positive vector $\boldsymbol{m} = (m_1, \ldots, m_n)^T$ which satisfies $\boldsymbol{s}\boldsymbol{m} = \boldsymbol{m}$.

By Proposition 5.1, we have the existence of a positive measure $\omega_1 \otimes \ldots \otimes \omega_n$, supported on $W_1 \times \ldots \times W_n$, which satisfies

$$\omega_i \;=\; \sum_{j=1}^{n} \int_{\mathcal{F}_{ij}} (D.\omega_j)\, \mathrm{d}\sigma_{ij}(D)\,, \tag{7.5}$$

with $\omega_i(H) = m_i$, for all $i = 1, \ldots, n$, and which is explicitly given by the infinite product formula in Proposition 5.1. The next task is to convert (7.5) into a statement about densities on the model sets $\Lambda_1, \ldots, \Lambda_n$. Our basic assumption is

**SS7** Each $\omega_i$ is *continuously representable* in the sense that, for $i = 1, \ldots, n$,

$$\omega_i \;=\; g_i \theta \tag{7.6}$$

where $g_i \geq 0$ is a function on $H$ which is supported on $W_i$ and has the property that its restriction to $W_i$ is continuous on $W_i$.

Given (7.6), we define the corresponding weights or *densities*

$$p_i : \, L \to \mathbb{R}\,, \quad p_i(x) \;=\; g_i(x^*)\,, \quad i = 1, \ldots, n\,.$$

Since $g_i$ is supported on $W_i$, $p_i$ is supported on $\Lambda_i = \{x \in L \mid x^* \in W_i\}$. Our assumption that each $\Lambda_i$ is regular, i.e. that each $W_i$ is Riemann measurable, allows us to apply Weyl's Theorem (Theorem 6.2) to prove the existence of the average density for each $p_i$:

$$\lim_{r\to\infty} \frac{1}{\mathrm{vol}(B_r(a))} \sum_{x \in L \cap B_r(a)} p_i(x) \;=\; \frac{1}{|\tilde{L}|} \int_H g_i \, \mathrm{d}\theta \;=\; \frac{m_i}{|\tilde{L}|} \tag{7.7}$$

where the convergence of the limit is uniform in $a$.

For any affine mapping $D : \, x \mapsto Ax + a$ and any $h \in L^1(H)$, we have

$$D.(h\theta) \;=\; \alpha (D.h) \theta$$

as easily follows by applying both sides to a test function and then using the formula $\mathrm{d}\theta(A^{-1}y) = \alpha\, \mathrm{d}\theta(y)$, see Proposition 3.2(1). Plugging this into (7.5) gives

$$g_i \;=\; \alpha \sum_{j=1}^{n} \int_{\mathcal{F}_{ij}} (D.g_j)\, \mathrm{d}\sigma_{ij}(D)$$

and then, for $i = 1, \ldots, n$,

$$g_i(x) \;=\; \alpha \sum_{j=1}^{n} \int_{\mathcal{F}'_{ij}} g_j\big(A^{-1}(x-u)\big)\, \mathrm{d}\sigma_{ij}(u)\,. \tag{7.8}$$

To pass this to the physical side, we need to be able to deal with the integral. There are two situations in which we know how to do this, namely when the $F_{ij}$ are finite and we basically use counting measures on the $\mathcal{F}'_{ij}$, and when the sets $\mathcal{F}'_{ij}$ are Riemann measurable subsets of $H$ and the measures $\sigma_{ij}$ are basically restrictions of the Haar measure on $H$. Let us now discuss these cases.

7.2.1. *$F_{ij}$ finite, $\sigma_{ij}$ counting measure.* If each set $F_{ij}$ is finite, then so is $F^*_{ij}$ and $\mathcal{F}_{ij} = \overline{F^*_{ij}} = F^*_{ij}$. The model sets $\Lambda_i$ are linked by a finite collection of finite unions as follows,

$$\Lambda_i \;=\; \bigcup_{j=1}^{n} \bigcup_{k=1}^{N_{ij}} \big(Q\Lambda_j + a_{ijk}\big)\,. \tag{7.9}$$

We suppose that $\sigma_{ij}$ is counting measure normalized to total measure $s_{ij}$ satisfying **SS2**. Then

$$g_i = \alpha \sum_{j=1}^{n} \frac{s_{ij}}{\operatorname{card}(F_{ij})} \sum_{a \in F'_{ij}} C_a^* . g_j \tag{7.10}$$

$$p_i(x) = \alpha \sum_{j=1}^{n} \frac{s_{ij}}{\operatorname{card}(F_{ij})} \sum_{a \in F'_{ij}} p_j\big(Q^{-1}(x-a)\big) . \tag{7.11}$$

We do not know many conditions that guarantee the existence of the representing functions $g_i$.

However, suppose that the unions in (7.2) are *non-overlapping*; more precisely, we assume that

**NO1** $F_{ij}$ is finite
**NO2** $m_i = \theta(W_i)$
**NO3** $\sigma_{ij}$ is counting measure scaled by $\alpha^{-1}$, i.e. $s_{ij} = \sigma_{ij}(F_{ij}) = \alpha^{-1}\operatorname{card}(F_{ij})$.
**NO4** for each $i$, the sets $D(W_j)$ entering into the union in (7.4) intersect at most on sets of measure 0, i.e. they are *just touching*.

Taking measures in (7.4), we see how the compatibility condition **SS3** fits in:

$$m_i = \sum_{j=1}^{n} \sum_{D \in \mathcal{F}_{ij}} \alpha^{-1} m_j = \alpha^{-1} \sum_{j=1}^{n} \operatorname{card}(F_{ij})\, m_j = \sum_{j=1}^{n} s_{ij} m_j .$$

The self-similar measures $\omega_i$ are easy to find, $\omega_i = \mathbf{1}_{W_i}\theta$. In fact, the equation

$$\mathbf{1}_{W_i} = \sum_{j=1}^{n} \sum_{D \in \mathcal{F}_{ij}} \mathbf{1}_{D(W_j)} \qquad (a.e.)$$

(which is what non-overlapping means in (7.4)) is equivalent to

$$\mathbf{1}_{W_i}\theta = \alpha^{-1} \sum_{j=1}^{n} \sum_{D \in \mathcal{F}_{ij}} D.(\mathbf{1}_{W_j}\theta) = \sum_{j=1}^{n} \int_{\mathcal{F}_{ij}} D.(\mathbf{1}_{W_j}\theta)\, \mathrm{d}\sigma_{ij}(D) .$$

Consequently, the self-similar densities $p_i$ are simply

$$p_i(x) = \begin{cases} 1 & \text{if } x \in \Lambda_i \\ 0 & \text{otherwise} \end{cases}$$

and (7.11), for $i = 1, \ldots, n$, reads as

$$p_i(x) = \sum_{j=1}^{n} \sum_{a \in F'_{ij}} p_j\big(Q^{-1}(x-a)\big) \qquad (a.e.) \tag{7.12}$$

where $(a.e.)$ means that equality holds for $x \in L$, possibly up to a set of density 0.

Equation (7.12) is the point set analogue of the situation in an inflation tiling where the measure of the inflated tile is the sum of the measures of the tiles into which it decomposes. Here, density replaces measure and the “tiling” condition is effectively put on the attractor in our assumptions in equations (7.4).

Let us briefly mention that Lagarias and Wang [**LW**] have recently begun an investigation of multi-component point sets with self-similarities. Their paper is taken from the discrete and combinatorial point of view, but does address the issue of counting multiplicities due to overlapping in the substitution process, and hence implicitly the question of self-similar measures.

7.2.2. *$\mathcal{F}'_{ij}$ Riemann measurable, $\sigma_{ij}$ Haar measure.* By definition,

$$F'_{ij} \;\subset\; \{a \in L \mid Q\Lambda_j + a \subset \Lambda_i\}\,,$$

and so $F'^{*}_{ij}$ is a subset of

$$\mathcal{G}'_{ij} \;:=\; \{b \in H \mid AW_j + b \subset W_i\}\,. \tag{7.13}$$

Since $\mathcal{G}'_{ij} = \bigcap_{w\in W_j}(W_i - Aw)$ is closed (and compact), we have $\mathcal{F}'_{ij} \subset \mathcal{G}'_{ij}$ for all $i,j$. Thus the $\mathcal{G}'_{ij}$ give us an upper bound on the $\mathcal{F}'_{ij}$ and, in any case,

$$F'_{ij} \;\subset\; \{a \in L \mid a^* \in \mathcal{G}'_{ij}\}\,.$$

The right hand side has the interesting property that, provided that $\mathcal{G}'_{ij}$ is equal to the closure of its interior, it constitutes a model set of the cut and project scheme (6.1). If, furthermore, the $\mathcal{G}'_{ij}$ are Riemann measurable, then we have access to Weyl's Theorem again. This suggests that we may use *all possible* self-similarities of a given collection of model sets, replacing the $F'_{ij}$ by the sets

$$G'_{ij} \;=\; \{a \in L \mid a^* \in \mathcal{G}'_{ij}\}\,. \tag{7.14}$$

Thus given regular model sets $\Lambda_i = \Lambda(W_i)$, $i = 1,\dots,n$, and an inflation $Q$ satisfying **SS1**–**SS4** above, we can define $\mathcal{G}'_{ij}$ by (7.13) and replace the given system $\mathcal{F}$ of affine mappings by the new set $\mathcal{G}$ with

$$G_{ij} \;:=\; \{C : x \mapsto Qx + a \mid a \in L, a^* \in \mathcal{G}'_{ij}\}\,. \tag{7.15}$$

With this motivation in mind, we go back to our original setup with **SS1**–**SS5**. We now assume in addition that

**SS8** each $\mathcal{F}'_{ij}$ is Riemann measurable and, if $\theta(\mathcal{F}'_{ij}) > 0$, we have

$$\sigma_{ij} \;=\; \frac{s_{ij}}{\theta(\mathcal{F}'_{ij})}\,\mathbf{1}_{\mathcal{F}'_{ij}}\,\theta\,,$$

i.e. $\sigma_{ij}$ is Haar measure restricted to $\mathcal{F}'_{ij}$ and normalized to total measure $s_{ij}$ where $\boldsymbol{s} = (s_{ij})$ is an arbitrary non-negative matrix satisfying **SS2**. If $\theta(\mathcal{F}'_{ij}) = 0$, $\sigma_{ij}$ is defined as the 0-measure.

We now apply the results of Section 5. The $\sigma_{ij}$ are absolutely continuous with respect to Haar measure $\theta$ since $\sigma_{ij} = h_{ij}\theta$ where $h_{ij} = \frac{s_{ij}}{\theta(\mathcal{F}'_{ij})}\mathbf{1}_{\mathcal{F}'_{ij}}$ and the density is $h_{ij} \in L^1(H) \cap L^\infty(H)$.

Assuming that the convolution (5.5) converges and using Proposition 5.2, we obtain a family of *continuous* non-negative functions $g_1,\dots,g_n$ which satisfy the self-similarity equations

$$g_i(x) \;=\; \alpha \sum_{j=1}^{n} w_{ij} \int_{\mathcal{F}_{ij}} g_j\big(A^{-1}(x-u)\big)\,\mathrm{d}\theta(u) \tag{7.16}$$

where $w_{ij} := s_{ij}/\theta(\mathcal{F}'_{ij})$. Applying Theorem 6.2 we obtain

THEOREM 7.3. *Let $\Lambda_1,\dots,\Lambda_n$ be a self-similar system of model sets which satisfy the assumptions* **SS1**–**SS6** *and* **SS8**. *Then there exist non-negative functions*

*$p_i : L \to \mathbb{R}$, supported on $\Lambda_i$, $i = 1, \ldots, n$, with the following properties*

$$m_i = \lim_{r\to\infty} \frac{|\tilde{L}|}{\mathrm{vol}(B_r(a))} \sum_{x\in L\cap B_r(a)} p_i(x)\,, \tag{7.17}$$

$$p_i(x) = \lim_{r\to\infty} \frac{\alpha\,|\tilde{L}|}{\mathrm{vol}(B_r(a))} \sum_{j=1}^{n} w_{ij} \sum_{x\in L\cap B_r(a)} p_j\big(Q^{-1}(x-u)\big)\,, \tag{7.18}$$

*for all $x \in L$ and for all $i = 1, \ldots, n$, where the limits are uniform in $a \in \mathbb{R}^m$.* □

This may be compared with the similar formula that we derived in [**BM2**]. There, $\tilde{Q}$ was assumed to be an automorphism of $\tilde{L}$, and the set of $Q$-affine mappings was the set of all possible mappings. There, however, the scaling constants $\nu^{ij}$ (which are our $s_{ij}$ here) had no general interpretation.

REMARK: For simplicity, the assumptions made in **SS8** are actually a little stronger than necessary – there is no general need to exclude the case that some $\mathcal{F}'_{ij}$ are singleton sets, but still carry a positive (point) measure. Though this is then not absolutely continuous, it is 'harmless' in the convolution process because the convolution of a function with a unit point mass only results in a shift of the function. We will meet this case in Section 8.1.4 below.

## 8. Concrete examples

A number of explicit examples have been presented earlier, in [**BM1**] and [**BM2**]. The case of the planar Penrose pattern in its rhombic version is an example of a multi-component model set, because the vertex points fall into 4 classes. This was described in detail in [**BM2**] and will not be repeated here. Instead, let us look into a few other examples. First, we describe the silver mean chain in one dimension and look at it in different ways, both as a single and as a multi-component model set. Next, we briefly describe the Ammann-Beenker pattern [**AGS**] in the plane, an eightfold symmetric relative of the Penrose pattern. This example appears also in other contributions to this volume. Finally, we look at a more unusual example that involves the 3-adic numbers.

**8.1. The silver mean chain.** The silver mean chain is a 2-sided sequence on the alphabet $\{a, b\}$. It can generated by iterating the substitution

$$a \mapsto aba \quad , \quad b \mapsto a$$

which, when starting from $a|a$, leads to the palindromic fixed point

$$\ldots abaaabaabaabaaaba|abaaabaabaabaaaba \ldots \tag{8.1}$$

where $|$ simply marks the centre. With $a$ and $b$ interpreted as intervals (=tiles) of length $\alpha := 1 + \sqrt{2}$ and 1 respectively, this gives rise to a tiling of the line, see [**HRB**] for details of its structure. The number $\alpha$ is called the *silver mean*.

Let $\Lambda_1$ (resp. $\Lambda_2$) denote the coordinates of the left end points of the tiles of type $a$ (resp. $b$), assuming that the initial block $a|a$ was centred at 0, i.e. its left end points are located at $-\alpha$ and 0. Then $\Lambda_1, \Lambda_2$, and $\Lambda := \Lambda_1 \cup \Lambda_2$ are model sets for the following cut and project scheme:

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\;\pi\;} & \mathbb{R}\times\mathbb{R} & \xrightarrow{\;\pi_{\mathbb{R}}\;} & \mathbb{R} \\ \cup & & \cup & & \cup \\ \mathbb{Z}[\sqrt{2}] & \longleftarrow & \tilde{L} & \longrightarrow & \mathbb{Z}[\sqrt{2}] \end{array} \tag{8.2}$$

where $\mathbb{Z}[\sqrt{2}\,] = \mathbb{Z} \oplus \mathbb{Z}\sqrt{2}$ is the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{2}\,)$, the $^*$-map from $\mathbb{Z}[\sqrt{2}]$ to $\mathbb{Z}[\sqrt{2}]$ is the algebraic conjugation defined by $\sqrt{2} \mapsto -\sqrt{2}$, and the lattice is $\tilde{L} := \{(x, x^*) \,|\, x \in \mathbb{Z}[\sqrt{2}]\}$. Explicitly, we obtain

$$\begin{aligned} \Lambda_1 &= \{x \in \mathbb{Z}[\sqrt{2}] \mid x^* \in W_1\} \\ \Lambda_2 &= \{x \in \mathbb{Z}[\sqrt{2}] \mid x^* \in W_2\} \\ \Lambda &= \{x \in \mathbb{Z}[\sqrt{2}] \mid x^* \in W\,\} \end{aligned} \tag{8.3}$$

where the corresponding windows are intervals, namely

$$W_1 := \Big[\frac{1}{\sqrt{2}} - 1, \frac{1}{\sqrt{2}}\Big]\,,\; W_2 := \Big[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} - 1\Big]\,,\; W := \Big[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big]\,, \tag{8.4}$$

where $W = W_1 \cup W_2$ and $W_1 \cap W_2 = \{\alpha^*/\sqrt{2}\,\}$.

This can be verified in the following way. Let $Q$ be the linear mapping which is scalar multiplication by $\alpha = 1 + \sqrt{2}$ and let $A$ be its conjugate map, which is scalar multiplication by $\alpha^* = -\alpha^{-1} = 1 - \sqrt{2}$. With the explicit coordinatization given above, the substitution rules say that

$$\begin{aligned} \Lambda_1 &= Q\Lambda_1 \cup Q\Lambda_2 \cup (Q\Lambda_1 + \alpha + 1) \\ \Lambda_2 &= Q\Lambda_1 + \alpha \end{aligned} \tag{8.5}$$

so that we now have a system of $Q$-inflations which, in the notation of **SS4**, is defined by

$$F' = (F'_{ij}) = \begin{pmatrix} \{0, \alpha+1\} & \{0\} \\ \{\alpha\} & \emptyset \end{pmatrix}. \tag{8.6}$$

The corresponding contractions satisfy:

$$\begin{aligned} W_1 &= AW_1 \cup AW_2 \cup (AW_1 + \alpha^* + 1) \\ W_2 &= AW_1 + \alpha^* \end{aligned} \tag{8.7}$$

which shows that $W_1 \times W_2$ is the attractor for the system of $A$-affine contractions given by $F'^*$.

Since the generators of $\Lambda$ (which correspond to the $a$-tiles with coordinates 0 and $-\alpha$) are mapped into $W_1$ by $^*$, all subsequent points generated from them are $^*$-mapped into $W$. Thus the model set $\Lambda(W)$ assuredly contains our set $\Lambda$. On the other hand, it is easy to see that the minimum separation between the points of $\Lambda(W)$ is 1, and no point can be added to $\Lambda$ without violating this. In short, $\Lambda = \Lambda(W), \Lambda_1 = \Lambda(W_1), \Lambda_2 = \Lambda(W_2)$.

We now examine this situation in four different ways, namely as single component and multi-component case, and each then with minimal and maximal families of affine contractions. By this we mean either the case when the window system is minimally generated by affine contractions or when we use the entire set of affine contractions available.

We content ourselves with a few remarks about each case and one figure illustrating the continuous case. We indicate the appropriate Sections of the paper as we go along and freely use the notation from these Sections. Note that $\alpha = 1 + \sqrt{2}$ as we have defined it in this Section is the modulus of the contraction $A$, in keeping with previous notation.

8.1.1. *The single model set* $\Lambda = \Lambda(W)$ *with* $F$ *minimal.* (see Section 3.2) The contractivity factor is $\alpha^*$. Since $|\alpha^*| < 1/2$, we need at least three affine mappings to get the full window as the attractor – we would end up with a Cantor subset of $W$ if we would start with only two. One possible choice is

$$W \;=\; (AW + \alpha^*) \cup AW \cup (AW - \alpha^*)$$

and our family of mappings is then $F_H = F_{\mathbb{R}} = \{\alpha^*, 0, -\alpha^*\}$. We take the simplest of all probability measures on $F_{\mathbb{R}}$, i.e. counting measure,

$$\nu \;=\; \tfrac{1}{3}(\delta_{\alpha^*} + \delta_0 + \delta_{-\alpha^*})\,,$$

so that this is an example of a finite IFS. The corresponding invariant measure on $W$ and its Fourier transform are given by

$$\begin{aligned} \omega \;&=\; \mathop{\Large *}_{\ell=1}^{\infty} \tfrac{1}{3}\big(\delta_{(\alpha^*)^\ell} + \delta_0 + \delta_{-(\alpha^*)^\ell}\big) \\ \hat{\omega} \;&=\; \prod_{\ell=1}^{\infty} \tfrac{1}{3}\big(1 + 2\cos(2\pi i(\alpha^*)^\ell k)\big)\,. \end{aligned} \tag{8.8}$$

This measure is similar to those studied in the context of the binary addressing problem, see [**So**] and references therein. Although most of them are absolutely continuous if the IFS covers the full interval (which is does here), exceptions emerge, see [**BoGi**, Thm. 4], if the scaling factor of the IFS is the inverse of a Pisot-Vijayaraghavan number (which is the case here, too). These exceptional self-similar measures will then be purely singular continuous. Note that this might be very difficult to detect if one is not aware of it – the fractal dimension of such a measure can be tantalizingly close to 1, see [**Lal**, Sec. 8]. In any case, we cannot pull back such a measure to the physical side, and thus do not gain much insight into the structure of our model set from it.

8.1.2. *The single model set* $\Lambda = \Lambda(W)$ *with* $F$ *maximal.* (see Section 3.3) Here, we start with the observation

$$W \;=\; \bigcup_{u\in[\alpha^*,-\alpha^*]} AW + u\,.$$

We now use the complete set of available self-similarities $\mathcal{F}' = [\alpha^*, -\alpha^*]$. For our pre-assigned probability measure $\nu$, we choose

$$\nu \;=\; \frac{\mathbf{1}_{[\alpha^*,-\alpha^*]}}{2|\alpha^*|}\,\theta \;=\; \frac{\mathbf{1}_{\mathcal{F}'}}{2|\alpha^*|}\,\theta$$

where $\theta$ is Lebesgue measure on $\mathbb{R}$. Then we are the situation of Proposition 3.6:

$$\begin{aligned} g \;&=\; \mathop{\Large *}_{\ell=0}^{\infty} \left(\frac{\mathbf{1}_{|\alpha^*|^\ell \mathcal{F}'}}{2\,|\alpha^*|^{\ell+1}}\right) \\ \hat{g}(k) \;&=\; \prod_{\ell=1}^{\infty} \left(\frac{\sin(2\pi(\alpha^*)^\ell k)}{2\pi(\alpha^*)^\ell k}\right)\,. \end{aligned} \tag{8.9}$$

This is illustrated in Figure 1.

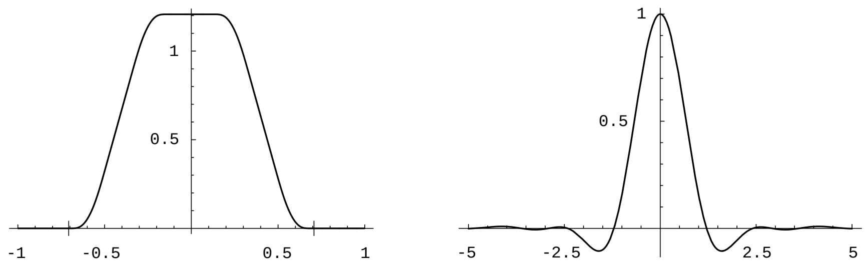


FIGURE 1. Invariant density (left) and its Fourier transform (right) for the silver mean chain. The support of the $C^\infty$-density $g$ of (8.9) is the window $W$ of (8.4), indicated by extra markers.

8.1.3. *The system of model sets* $\{\Lambda_1, \Lambda_2\}$ *with windows* $\{W_1, W_2\}$, *with* $F$ *minimal.* (see Section 7.2, non-overlapping case)

We consider $\Lambda = \Lambda_1 \cup \Lambda_2$ as a multi-component model set through (8.5) and (8.6) with the windows $W_1$ and $W_2$ given by (8.4) and related by (8.7). The non-overlapping conditions **NO1** and **NO4** hold. To obtain **NO2** and **NO3**, we define $\boldsymbol{m} = (1, |\alpha^*|)^T$ and (since $|\alpha^*| = \alpha - 2 = 1/\alpha$)

$$\boldsymbol{s} \; = \; \begin{pmatrix} 2/\alpha & 1/\alpha \\ 1/\alpha & 0 \end{pmatrix}, \tag{8.10}$$

which is clearly primitive. The self-similar measures are $\omega_1 = \mathbf{1}_{W_1}\theta$ and $\omega_2 = \mathbf{1}_{W_2}\theta$, and the corresponding densities on the physical side are $p_1$ and $p_2$, which are the characteristic functions (defined on $\mathbb{Z}[\sqrt{2}\,]$) of $\Lambda_1$ and $\Lambda_2$, respectively.

8.1.4. *The system of model sets* $\{\Lambda_1, \Lambda_2\}$ *with windows* $\{W_1, W_2\}$, *with* $F$ *maximal.* (see Section 7.2, Riemann measurable case)

We continue with the multi-component picture of Section 8.1.3, but now we use all available $\alpha$-affine self-similarities. These are easily computed from (8.4) and (8.7), resulting in

$$\mathcal{F}' \; = \; \begin{pmatrix} [0, 1+\alpha^*] & [\alpha^*, -\alpha^*] \\ \{\alpha^*\} & \emptyset \end{pmatrix}.$$

The sets appearing here are all measurable. Let us fix the same vector $\boldsymbol{m}$ and matrix $\boldsymbol{s}$ as in Section 8.1.3. If we use Lebesgue measure $\theta$ on $\mathbb{R}$, we obtain **SS7** in the form

$$\boldsymbol{\sigma} \; = \; \begin{pmatrix} (1-\alpha^*)\mathbf{1}_{[0,1+\alpha^*]}\,\theta & \frac{1}{2}\mathbf{1}_{[\alpha^*,-\alpha^*]}\,\theta \\ -\alpha^*\delta_{\alpha^*} & 0 \end{pmatrix}.$$

Using Proposition 5.1, we have the self-similar measure

$$\omega \; = \; \mathop{\ast}_{\ell=0}^{\infty} \alpha^\ell \begin{pmatrix} (1-\alpha^*)\mathbf{1}_{(\alpha^*)^\ell[0,1+\alpha^*]}\,\theta & \frac{1}{2}\mathbf{1}_{(\alpha^*)^\ell[\alpha^*,-\alpha^*]}\,\theta \\ -\alpha^*\delta_{(\alpha^*)^{\ell+1}} & 0 \end{pmatrix} \ast \begin{pmatrix} \delta_0 \\ |\alpha^*|\,\delta_0 \end{pmatrix}. \tag{8.11}$$

The solution here is mildly different from the one appearing in **SS8** because of the appearance of the point measure, compare the Remark following Theorem 7.3. However, the convolution of a delta and a function simply translates the function. Furthermore, the convolution of two functions $f$ and $g$ is point-wise bounded by $||f||_\infty ||g||_1$. Thus the partial convolution products in $\omega^{(n)}$ of $\omega$ in (8.11), for all

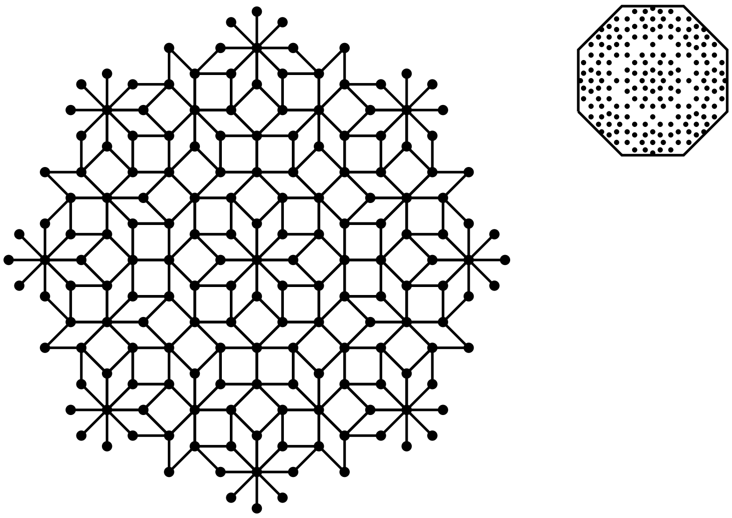

FIGURE 2. The Ammann-Beenker model set, seen as the vertex set of a tiling with squares and rhombi. The window in internal space is a regular octagon. It is shown to the right with the $*$-images of the vertex points from the patch to the left.

$n \geq 1$, are *functions*

$$\begin{pmatrix} f_{11}^{(n)} & f_{12}^{(n)} \\ f_{21}^{(n)} & f_{22}^{(n)} \end{pmatrix}$$

where the $f_{ij}^{(n)}$ are supported on $W_j$, are uniformly bounded, and are increasingly differentiable as $n \to \infty$. Alternatively, one could also work with the square of the averaging operator here, which would match **SS8** from the beginning.

The resulting self-similar measure is represented by two $C^{\infty}$ functions, $g_1$ and $g_2$. They are of the kind shown on the left of Figure 1, but now supported on $W_1$ and $W_2$, respectively, and with total mass 1 and $|\alpha^*|$, in accordance with $\boldsymbol{m}$.

**8.2. The Ammann-Beenker model set.** This relative of the rhombic Penrose tiling is usually described as a model set obtained from the primitive cubic lattice $\mathbb{Z}^4$ in 4-space, see [**BJ**] and references therein for details. Here, we prefer the number theoretic approach given in [**HRB**], which is more compatible with the above description of the silver mean chain. With $\xi := e^{2\pi i/8}$, we use the following cut and project scheme

$$\begin{array}{ccccc} \mathbb{R}^2 & \xleftarrow{\;\pi\;} & \mathbb{R}^2 \times \mathbb{R}^2 & \xrightarrow{\;\pi_{\mathbb{R}^2}\;} & \mathbb{R}^2 \\ \cup & & \cup & & \cup \\ \mathbb{Z}[\xi] & \longleftarrow & \tilde{L} & \longrightarrow & \mathbb{Z}[\xi] \end{array} \tag{8.12}$$

where $\mathbb{Z}[\xi]$ is the ring of cyclotomic integers generated by the primitive 8th roots of unity. It is the maximal order in the cyclotomic field $\mathbb{Q}(\xi)$. Then, the lattice is

$$\tilde{L} \;=\; \{(x, x^*) \mid x \in \mathbb{Z}[\xi]\}$$

where the $*$-map is given by algebraic conjugation $\xi \mapsto \xi^3$.

In this setting, the window $W$ of the Ammann-Beenker model set $\Lambda = \Lambda(W)$ is simply a regular octagon of edge length 1, centred at the origin. Its area is $2\alpha$, with $\alpha = 1+\sqrt{2}$ as above. With this choice, the model set $\Lambda$ is both *regular* ($W$ is a polytope, hence Riemann measurable) and *generic* ($L = \mathbb{Z}[\xi]$ does not intersect $\partial W$). A symmetric patch and its lift to internal space is shown in Figure 2.

Let us now consider $\Lambda = \Lambda(W)$ as a single model set and let us determine the invariant density on $W$ that results from the set of *all* self-similarities of the form $\Lambda \mapsto \alpha\Lambda + v \subset \Lambda$ (with $v \in \mathbb{Z}[\xi]$). So, $Q$ is again multiplication by $\alpha$, and $A$ multiplication by $\alpha^*$, a contraction. It then follows that

$$\mathcal{F}' \;=\; \{u \mid \alpha^* W + u \subset W\} \;=\; (2-\sqrt{2})W\,,$$

i.e. $\mathcal{F}' \subset \mathbb{R}^2$ is another octagon centred at the origin, but with reduced edge length $|\alpha^*|\sqrt{2} = 2-\sqrt{2}$. Consequently, $\mathcal{F}'$ has area $4|\alpha^*|$. For the a priori probability measure on $\mathcal{F}'$, we choose

$$\nu \;=\; \frac{\mathbf{1}_{\mathcal{F}'}}{4|\alpha^*|}\,\theta$$

where $\theta$ is now Lebesgue measure on $\mathbb{R}^2$. Proposition 3.6 then gives

$$g \;=\; \mathop{\ast}_{\ell=0}^{\infty} \left(\frac{\mathbf{1}_{|\alpha^*|^\ell \mathcal{F}'}}{4\,|\alpha^*|^{2\ell+1}}\right). \tag{8.13}$$

The invariant density is a $C^\infty$-function supported on the window $W$, see Figure 3. The deviations from circular symmetry are rather faint. However, in contrast to the Penrose case investigated in [**BM2**], it has a central plateau and then rolls off smoothly towards the boundary of $W$. The same phenomenon is actually visible for the invariant density of the silver mean chain in Figure 1, in contrast to the one for the Fibonacci chain in [**BM1**]. It is due to the larger absolute value of $\alpha$ in comparison to the golden ratio $\tau = (1+\sqrt{5})/2$ which appears there. This is interesting in relation to the rather widespread experimental finding that "real world" quasicrystals are to be described by window functions with a smooth roll-off. Although this is usually explained as a random tiling effect, our above examples show that other mechanisms are possible as well, and the attractive feature is then that they result from some residual (or statistical) inflation symmetry.

**8.3. A 3-adic example.** In this final example, which will be fully developed along with some other $p$-adic examples in [**HM**], we indicate what invariant measures can look like in a very different situation. The result is quite surprising.

This time, we begin with the 2-sided chain on the ternary alphabet $\{a,b,c\}$, generated by the substitution rule

$$a \mapsto ab \quad , \quad b \mapsto abc \quad , \quad c \mapsto abcc\,.$$

Starting from $c|a$, iteration leads to the 2-sided fixed point

$$\dots babcabccabccababcabccabcc|ababcababcabccababcababc\dots \tag{8.14}$$

With $a,b,c$ assigned intervals of length $1,2,3$ respectively, which is the natural geometric realization here, this again gives rise to a tiling of the line. This system was studied in [**BMS**] where it was shown that, when it is coordinatized, the resulting sets of points $\Lambda_1, \Lambda_2, \Lambda_3$ are model sets based on the 3-adic integers $\hat{\mathbb{Z}}_3$.

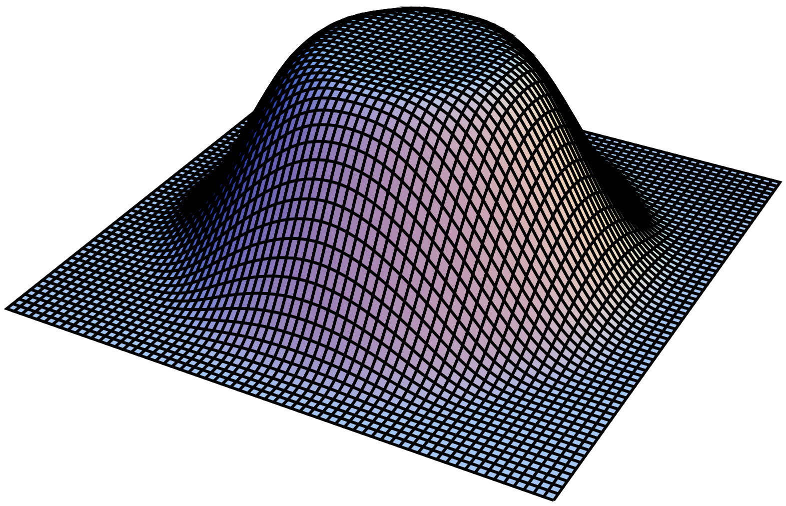

FIGURE 3. The self-similar density $g$ for the Ammann-Beenker model set, according to Eq. (8.13). The support of the $C^\infty$-function $g$ is a regular octagon, the window $W$ in the upper right corner of Figure 2.

The coordinatization starting at 0 and using *right* end points is

$$\begin{aligned} \Lambda_1 &= \bigcup_{k=2}^{\infty}\Big(\big(\textstyle\sum_{i=0}^{k-2} 3^i\big) + 3^k\mathbb{Z}\Big) \\ \Lambda_2 &= \bigcup_{k=2}^{\infty}\Big(\big(2+\textstyle\sum_{i=0}^{k-2} 3^i\big) + 3^k\mathbb{Z}\Big) \\ \Lambda_3 &= 3^2\mathbb{Z} \cup \Big(\bigcup_{k=3}^{\infty}\Big(\big(-\textstyle\sum_{i=1}^{k-2} 3^i\big) + 3^k\mathbb{Z}\Big)\Big)\,. \end{aligned} \tag{8.15}$$

The corresponding model sets are

$$\Lambda_i = \{x \in \mathbb{Z} \mid x \in W_i\}$$

where the windows are given by

$$\begin{aligned} W_1 &= \bigcup_{k=2}^{\infty}\Big(\big(\textstyle\sum_{i=0}^{k-2} 3^i\big) + 3^k\hat{\mathbb{Z}}_3\Big) \\ W_2 &= \bigcup_{k=2}^{\infty}\Big(\big(2+\textstyle\sum_{i=0}^{k-2} 3^i\big) + 3^k\hat{\mathbb{Z}}_3\Big) \\ W_3 &= 3^2\hat{\mathbb{Z}}_3 \cup \Big(\bigcup_{k=3}^{\infty}\Big(\big(-\textstyle\sum_{i=1}^{k-2} 3^i\big) + 3^k\hat{\mathbb{Z}}_3\Big)\Big)\,. \end{aligned} \tag{8.16}$$

We consider this as a multi-component model set and the basic inflationary maps as affine mappings of the form $x \mapsto 3x + u$, $u \in \mathbb{Z}$. These maps transfer over to the internal side without any symbolic change, but there they are contractions with respect to the standard 3-adic metric topology. The most interesting case seems to be when we allow all possible mappings of this type between the various windows (this corresponds to the case of the silver mean example in Section 8.1.4). These are quite straightforward to work out and give rise to a $3 \times 3$ matrix $\mathcal{F}$. The invariant measures on $\hat{\mathbb{Z}}_3$ are then given by the matrix equation

$$\omega \;=\; \Big( \mathop{\ast}_{\ell=0}^{\infty} Y^{(\ell)} \Big) * \delta^{\boldsymbol{m}} \tag{8.17}$$

where

$$Y^{(\ell)} \;:=\; \left( \frac{s_{ij}}{\theta(3^{\ell}\mathcal{F}_{ij})}\, \mathbf{1}_{3^{\ell}\mathcal{F}_{ij}} \right)_{1\le i,j\le 3}\,, \tag{8.18}$$

$\theta$ is the Haar measure on $\hat{\mathbb{Z}}_3$ normalized to total measure 1, and $\boldsymbol{s}\boldsymbol{m} = \boldsymbol{m}$.

Remarkably, this convolution can be explicitly computed and yields

$$\omega \;=\; 3^2 \begin{pmatrix} \mathbf{1}_{3^2\hat{\mathbb{Z}}_3+1}\, m_1\,\theta \\ \mathbf{1}_{3^2\hat{\mathbb{Z}}_3+3}\, m_2\,\theta \\ \mathbf{1}_{3^2\hat{\mathbb{Z}}_3}\, m_3\,\theta \end{pmatrix}. \tag{8.19}$$

In particular, the self-similar measures are absolutely continuous and give rise to the following self-similar densities on our three model sets $\Lambda_{\{1,2,3\}}$ in $\mathbb{Z}$:

$$p_{\{1,2,3\}}(\ell) \;=\; \begin{cases} 9\, m_{\{1,2,3\}} & \text{if } \ell \equiv \{1,3,0\} \bmod 9, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the self-similar densities are *periodic* although the point sets $\Lambda_i$ are aperiodic. In fact, the support of the densities consists of three different cosets of $9\mathbb{Z}$, each of which lies entirely inside one of the three point sets $\Lambda_i$, $i = 1,2,3$. We do not know many other mechanisms that result in periodic states, although it is not clear whether this inflation induced periodicity could be interpreted physically.

## Appendix

For the sake of completeness, we collect a number of results in this Appendix. Let us first show that real (complex) Lipschitz functions are dense in $C(X,\mathbb{R})$ (in $C(X,\mathbb{C})$) if $X$ a compact metric space. We first need the following result which is essentially stated in [**Bou**, Prop. IX.2.2.3].

Lemma A.1. *Let $X$ be an arbitrary metric space. For every non-empty $A \subset X$, the function $\phi\colon x \mapsto d(x,A)$ is in* $\mathrm{Lip}(\le 1, X, \mathbb{R})$*. The Lipschitz constant is $r_\phi = 0$ iff $\phi \equiv 0$ on $X$, and $r_\phi = 1$ otherwise.*

Proof: Recall that $d(x,A) = \inf\{d(x,z) \mid z \in A\}$. Let $x,y \in X$. Fix $\varepsilon > 0$ and choose $z \in A$ such that $d(y,z) \le d(y,A) + \varepsilon$. Then

$$\begin{aligned} d(x,A) - d(y,A) \;&\le\; d(x,A) - d(y,z) + \varepsilon \;\le\; d(x,z) - d(y,z) + \varepsilon \\ &\le\; d(x,y) + \varepsilon\,. \end{aligned}$$

Now, since $\varepsilon > 0$ was arbitrary and the argument is essentially symmetric in $x$ and $y$, we can conclude

$$|\phi(x) - \phi(y)| \;=\; |d(x,A) - d(y,A)| \;\le\; d(x,y)\,.$$

This means that $\phi : x \mapsto d(x,A)$ is Lipschitz with constant $r_\phi \le 1$.

If $\phi \equiv 0$, we trivially have $r_\phi = 0$. Otherwise, there exists an $x \in X \setminus A$ with $\phi(x) = d(x,A) > 0$. We know from above that $r_\phi \le 1$. Fix $\varepsilon > 0$ and $y \in A$ with $d(x,y) \le d(x,A) + \varepsilon$. Then, we have $d(y,A) = 0$ and get

$$0 \;<\; d(x,y) \;\le\; d(x,A) + \varepsilon \;=\; d(x,A) - d(y,A) + \varepsilon \;\le\; r_\phi\, d(x,y) + \varepsilon\,.$$

Since $\varepsilon > 0$ was arbitrary, this is only possible with $r_\phi \geq 1$, hence $r_\phi = 1$. $\square$

LEMMA A.2. *If $X$ is a compact metric space, the real (complex) Lipschitz functions are dense in $C(X,\mathbb{R})$ (in $C(X,\mathbb{C})$).*

PROOF: This is a straight-forward application of the Stone-Weierstraß theorems. The real Lipschitz functions form a subalgebra of $C(X,\mathbb{R})$ under pointwise addition and multiplication because $r_{\phi+\psi} \leq r_\phi + r_\psi$ and $r_{\phi\psi} \leq \|\phi\|_\infty r_\psi + \|\psi\|_\infty r_\phi$. Furthermore, the constant functions are Lipschitz, and the separation property follows from Lemma A.1. So, the real variant, see [**RS**, Thm. IV.9], gives the one claim, while the complex variant, [**RS**, Thm. IV.10], gives the other. In the latter case, the additional requirement that the algebra of complex Lipschitz functions is closed under complex conjugation is obvious. $\square$

We also need a number of convergence results, which we collect here. The following standard result is a special case of [**Q**, Thm. 14.22 and Cor. 14.23] or [**Kel**, Thms. 7.14 and 7.15], see also [**Q**, A 14.8].

LEMMA A.3. *Let $X$ be a locally compact space and $\{g_n\}$ an equi-continuous sequence of functions in $C(X,\mathbb{C})$ which converges pointwise. Then the limit is a continuous function, $g \in C(X,\mathbb{C})$, and the convergence is uniform on compact subsets of $X$ (compact convergence).* $\square$

LEMMA A.4. *Let $W$ be a compact subset of an LCAG $H$, with dual group $\widehat{H}$. Let $(\mu_n)_{n\in\mathbb{N}}$ be a vaguely convergent sequence of probability measures in $\mathcal{P}(W)$, with limit $\mu$. If $K \subset \widehat{H}$ is compact, the family $\{\widehat{\mu}_n\}$ of functions from $C(H,\mathbb{C})$ is equi-continuous (and even equi-uniformly continuous) on $K$.*

PROOF: Let $\varepsilon > 0$ and $V_\varepsilon := \{k \in \widehat{H} \mid \sup_{x\in W} |\langle k,x\rangle - 1| < \varepsilon\}$. Note that $0 \in V_\varepsilon$ (since the trivial character is $\langle 0,x\rangle \equiv 1$) and that $V_\varepsilon$ is a neighbourhood of 0 in $\widehat{H}$ because it is a typical open set in the compact-open topology of $\widehat{H} \subset C(H,\mathbb{C})$. Fix some $k_1 \in K$ and choose $k_2 \in K$ so that $k_1 - k_2 \in V_\varepsilon$. Then we have, for all $x \in W$,

$$\left| \overline{\langle k_1,x\rangle} - \overline{\langle k_2,x\rangle} \right| \;=\; \big|\langle k_1 - k_2, x\rangle - 1\big| \;<\varepsilon\,,$$

where we used $|\langle k_2,x\rangle| = 1$ and the multiplication rule for characters.

Let $\widehat{\mu}_n$ be an arbitrary element of our sequence. We then get

$$\begin{aligned}
\big|\widehat{\mu}_n(k_1) - \widehat{\mu}_n(k_2)\big| \;&=\; \left|\int_H \Big(\overline{\langle k_1,x\rangle} - \overline{\langle k_2,x\rangle}\Big)\, \mathrm{d}\mu_n(x)\right| \\
&\leq\; \int_W \Big|\overline{\langle k_1,x\rangle} - \overline{\langle k_2,x\rangle}\Big|\, \mathrm{d}\mu_n(x) \\
&<\; \varepsilon\, \|\mu_n\| \;=\; \varepsilon\,.
\end{aligned}$$

Since this is independent both of $n$ and of $k_1 \in K$, equi-uniform continuity of $\{\widehat{\mu}_n\}$ on $K$ follows. $\square$

REMARK: With little extra complication, the result can be extended to vaguely convergent sequences of measures $\mu_n \in \mathcal{M}_+^m(H)$, see [**Bau**, Lemma 23.7] for a proof that can easily be adapted to this case.

THEOREM A.5. (Continuity Theorem of P. Lévy) *Let $W, H, \widehat{H}$ be defined as in Lemma A.4. If $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of measures in $\mathcal{P}(W)$ that vaguely converges to $\mu \in \mathcal{P}(W)$, then the corresponding sequence of Fourier-Stieltjes transforms, $(\widehat{\mu}_n)_{n\in\mathbb{N}}$, converges compactly to $\widehat{\mu}$.*

Proof: Each of the functions $x \mapsto \langle k, x\rangle$, $k \in \widehat{H}$, lies in $C(H,\mathbb{C})$. Therefore, $\lim_{n\to\infty} \widehat{\mu}_n(k) = \widehat{\mu}(k)$ by the very definition of vague convergence. So, we have a sequence of pointwise converging functions in $C(H,\mathbb{C})$ that are equi-continuous on compact subsets $K \subset \widehat{H}$ due to Lemma A.4, hence the convergence of $(\widehat{\mu}_n)_{n\in\mathbb{N}}$ is uniform on $K$ by Lemma A.3 which proves the assertion. □

Remark: Once again, the restriction to $\mathcal{P}(W)$ is not essential, and the result holds for sequences in $\mathcal{M}^m_+(H)$ as well, compare [**Bau**, Thm. 23.8], but we do not need the stronger result here.

Finally, we formulate the approximation property that we need in Section 6 to prove Theorem 6.2.

Lemma A.6. *Let $H$ be an LCAG with Haar measure $\mu$, and let $W$ be a Riemann measurable compact subset of $H$. Then each $f \in C(H,\mathbb{C})$ with* $\mathrm{supp}(f) \subset W$ *can be uniformly approximated by a sequence of step functions $\psi^{(\ell)}$ of the form*

$$\psi^{(\ell)} \; = \; \sum_{i=1}^{n(\ell)} c_i^{(\ell)} \, \mathbf{1}_{U_i}$$

*where $c_i^{(\ell)} \in \mathbb{C}$ and $W = \bigcup_{i=1}^{n(\ell)} U_i$ is a partition into pairwise disjoint sets $U_i \subset W$ that are all Riemann measurable.*

Proof: Although this is a standard type of result, we give an explicit proof because the additional condition of Riemann measurability of the partition requires some attention. It is sufficient to prove the statement for real functions because any $f$ may be split into its real and imaginary parts, both of which are continuous and supported on $W$.

Assume $f$ is real. For all $s \in \mathbb{R}$, $f^{-1}(s)$ is a closed subset of $H$, hence measurable. For $0 \neq s \neq t \neq 0$, $f^{-1}(s)$ and $f^{-1}(t)$ are disjoint subsets of $W$, and we thus have $\sum_{s\in\mathbb{R}\setminus\{0\}} \mu(f^{-1}(s)) \leq \mu(W) < \infty$. But this means, as usual, that

$$P \; := \; \{ s \in \mathbb{R} \mid \mu(f^{-1}(s)) > 0 \}$$

is at most a *countable* set, called the set of “bad” points. Since $f$ is continuous, $f(W) \subset \mathbb{R}$ is compact, hence $f(W) \subset \; (a,b)$ for some $a, b \in \mathbb{R}$.

Fix some $\varepsilon > 0$ and choose an integer $n > (b-a)/\varepsilon$. We can then subdivide $[a,b)$ into non-empty intervals $I_1, \ldots, I_n$, each of the form $[\alpha_i, \beta_i)$ with $\alpha_i, \beta_i \notin P$ and $0 < \mathrm{length}(I_i) = \beta_i - \alpha_i < \varepsilon$. Define $U_i = f^{-1}(I_i) \cap W$. Then $\{U_1, \ldots, U_n\}$ clearly is a partition of $W$. In addition, we have, for all $1 \leq i \leq n$,

$$\left.\begin{array}{r} f^{-1}((\alpha_i,\beta_i)) \text{ is open} \\ f^{-1}([\alpha_i,\beta_i]) \text{ is closed} \\ \mu\{f^{-1}(\alpha_i) \cup f^{-1}(\beta_i)\} = 0 \end{array}\right\} \;\Longrightarrow\; \text{each } U_i \text{ is Riemann measurable.}$$

Now, $\overline{U}_i \subset W$ is closed and hence compact. Define $c_i := \inf\{ f(x) \mid x \in U_i\}$ and $\psi := \sum_{i=1}^{n} c_i \, \mathbf{1}_{U_i}$. Then $\psi$ is supported on $W$. If $x \in W$, then $x$ is in precisely one of the sets of the partition, $U_i$ say, and $|f(x) - \psi(x)| = |f(x) - c_i| \leq \mathrm{length}(I_i) < \varepsilon$, independently of $x$. Consequently, $\| f - \psi \|_\infty < \varepsilon$. Finally, we can construct a sequence $\psi^{(\ell)}$ this way (e.g. via $\varepsilon^{(\ell)} = 1/\ell$) which establishes our claim. □

Remark: The result of this Lemma extends to all continuous functions on $W$, even if they do not define continuous functions on $H$. This follows from Tietze's extension theorem [**RS**, Thm. IV.11].

## References


[A] E. Akin, *The General Topology of Dynamical Systems*, Amer. Math. Soc., Providence, RI (1993); corr. reprint (1996).

[AGS] R. Ammann, B. Grünbaum and G. C. Shephard, *Aperiodic Tiles*, Discr. Comput. Geom. **8** (1992), 1–25.

[B] M. Baake, *A guide to mathematical quasicrystals*, in: *Quasicrystals*, eds. J.-B. Suck, M. Schreiber and P. Häußler, Springer, Berlin (2001), in press; preprint math-ph/9901014.

[BJ] M. Baake and D. Joseph, *Ideal and defective vertex configurations in the planar octagonal quasilattice*, Phys. Rev. **B 42** (1990), 8091–8102.

[BM1] M. Baake and R. V. Moody, *Self-similarities and invariant densities for model sets*, in: *Algebraic Methods and Theoretical Physics*, ed. Y. Saint-Aubin, Springer, New York (2000), in press; preprint math-ph/9809006.

[BM2] M. Baake and R. V. Moody, *Multi-component model sets and invariant densities*, in: *Aperiodic '97*, eds. M. de Boissieu, J.-L. Verger-Gaugry and R. Currat, World Scientific, Singapore (1998), pp. 9–20; math-ph/9809005.

[BMS] M. Baake, R. V. Moody and M. Schlottmann, *Limit-(quasi)periodic point sets as model sets with p-adic internal spaces*, J. Phys. **A 31** (1998), 5755–5765; math-ph/9901008.

[Bau] H. Bauer, *Probability Theory*, de Gruyter, Berlin (1996).

[BoGi] J. M. Borwein and R. Girgensohn, *Functional equations and distribution functions*, Results in Math. **26** (1994), 229–237.

[Bou] N. Bourbaki, *General Topology*, vol. 1, Addison-Wesley, Reading, MA (1966).

[D] J. Dieudonné, *Treatise on Analysis*, vol. II, Academic Press, New York (1970).

[HRB] J. Hermisson, C. Richard and M. Baake, *A guide to the symmetry structure of quasiperiodic tiling classes*, J. Phys. I (France) **7** (1997), 1003–1018.

[HeRo] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, vol. 1, 2nd ed., Springer, New York (1979); corr. 3rd printing (1997).

[HM] M. Höffe and R. V. Moody, *Self-similar measures for p-adic model sets*, in preparation.

[Hof1] A. Hof, *On diffraction by aperiodic structures*, Commun. Math. Phys. **169** (1995), 25–43.

[Hof2] A. Hof, *Uniform distribution and the projection method*, in: *Quasicrystals and Discrete Geometry*, ed. J. Patera, Fields Institute Monographs, vol. 10, Amer. Math. Soc., Providence, RI (1998), pp. 201–206.

[Hut] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–743.

[JLS] R. Q. Jia, S. L. Lee and A. Sharma, *Spectral properties of continuous refinement operators*, Proc. Amer. Math. Soc. **126** (1998), 729–737.

[JL] Q. Jiang and S. L. Lee, *Spectral properties of matrix continuous refinement operators*, Adv. Comput. Math. **7** (1997), 361–382.

[KT] S. Karlin and H. M. Taylor, *A First Course in Stochastic Processes*, 2nd ed., Academic Press, Boston, MA (1975)

[Kel] J. L. Kelley, *General Topology*, Van Nostrand, Princeton, NJ (1955); reprint, Springer, New York (1975).

[KN] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, New York (1974).

[LW] J. C. Lagarias and Y. Wang, *Substitution Delone sets*, preprint (2000).

[Lal] S. P. Lalley, *Random Series in powers of algebraic integers: Hausdorff dimension of the limit distribution*, J. London Math. Soc. (2) **57** (1998), 629–654.

[LM] J.-Y. Lee and R. V. Moody, *Lattice substitution systems and model sets*, Discr. Comput. Geom., to appear; preprint math.MG/0002019.

[Mey] Y. Meyer, *Algebraic Numbers and Harmonic Analysis*, North-Holland, Amsterdam (1972).

[M] R. V. Moody, *Meyer sets and their duals*, in: *The Mathematics of Long-Range Aperiodic Order*, ed. R. V. Moody, NATO ASI Series C 489, Kluwer, Dordrecht (1997), pp. 403–441.

[P] P. A. B. Pleasants, *Designer quasicrystals: cut-and-project sets with pre-assigned properties*, in: *Directions in Mathematical Quasicrystals*, eds. M. Baake and R. V. Moody, CRM Monograph Series, Amer. Math. Soc., Providence, RI (2000); this volume.

[Q] B. von Querenburg, *Mengentheoretische Topologie*, 2nd ed., Springer, Berlin (1979).

[RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, 2nd ed., Academic Press, San Diego, CA (1980).

[Ru] W. Rudin, *Fourier Analysis on Groups*, Wiley, New York (1962); reprint (1990).

[Sch1] M. Schlottmann, *Cut-and-project sets in locally compact Abelian groups*, in: *Quasicrystals and Discrete Geometry*, ed. J. Patera, Fields Institute Monographs, vol. 10, Amer. Math. Soc., Providence, RI (1998), 247–264.

[Sch2] M. Schlottmann, *Generalized model sets and dynamical systems*, in: *Directions in Mathematical Quasicrystals*, eds. M. Baake and R. V. Moody, CRM Monograph series, Amer. Math. Soc., Providence, RI (2000); this volume.

[So] B. Solomyak, *On the random series* $\sum \pm\lambda^n$ *(an Erdős problem)*, Ann. Math. **142** (1995), 611–625.

[We] H. Weyl, *Über die Gleichungverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352.

[Wi] K. R. Wicks, *Fractals and Hyperspaces*, Lecture Notes in Mathematics, vol. 1492, Springer, Berlin (1991).

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