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# Similarity Submodules and Semigroups

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**Abstract.** The similarity submodules for various lattices and modules of interest in the crystal and quasicrystal theory are determined and the corresponding semigroups, along with their zeta functions, are investigated. One application is to the colouring of crystals and quasicrystals.

## 1 Introduction

The symmetries of crystals or, more generally, of geometric objects with translational degrees of freedom, have been studied for ages and are well understood [**29**]. Recently, much attention has been given to generalizations to non-periodic objects with long-range orientational order, such as quasicrystals or non-periodic tilings and Delone sets [**18, 28, 9, 25, 24, 22**]. Here, the situation is very different and the development of symmetry as a mathematical tool in the theory is only in its infancy. In this case, it is well known that the objects to be called symmetries should not be restricted to usual point symmetries (i.e., to sets of isometries that fix a special point, such as the icosahedral group, say). In fact, without giving up invertibility (i.e., without leaving the environment of groups), one often finds a generalized type of symmetry called inflation/deflation symmetry. This is usually described in terms of similarity transformations that map a given discrete structure or Delone set to itself or to a locally equivalent one, see [**18, 9, 28**] for details.

At this point, a closer examination shows that the group structure actually results in a frame that is too rigid to access the full symmetry, or the full algebraic structure, of the objects under investigation. In fact, it is very natural to relax invertibility and extend the setting to that of semigroups where one asks for a certain class of operations that map the given structure into, but not necessarily

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onto, itself – with the "classical" symmetries forming a subgroup. Although this sounds rather straightforward, it has not been systematically attempted to our knowledge, and certainly not in the context of non-periodic crystals where it turns out to be rather important. Semigroups seem to be particularly useful for a better understanding of the inflation structure (e.g., in connection with the generation of such Delone sets and their Fourier transforms [**6**]), or for the classification of invariant sublattices and submodules together with their colour symmetries [**29, 4, 16, 17, 22, 5, 7**].

It is the aim of this illustrative contribution to begin to develop this structure, exemplifying the intricacies of the structures that can arise with a number of relevant examples in two and three dimensions. Further development of these ideas, together with a treatment of the 4D case, will appear in a forthcoming publication.

## 2 Definitions and Preliminaries

We are interested in $\mathbb{Z}$-modules $M$ of finite rank in Euclidean space $\mathbb{R}^n$ and the set of all affine similarities of $\mathbb{R}^n$ that map $M$ into itself. We do not assume that the $\mathbb{Z}$-rank $r$ of $M$ is the same as the dimension $n$ of the ambient space $\mathbb{R}^n$, but we will assume that $M$ spans $\mathbb{R}^n$, and hence $r \geq n$. If $M$ happens to be discrete (and hence $r = n$), we call it a lattice. An affine **self-similarity** is, by definition, a non-singular mapping

$$f \;:\; x \mapsto \alpha R x + a \tag{2.1}$$

of $M$ into itself where $R \in \mathrm{O}(n,\mathbb{R})$, $\alpha \in \mathbb{R}^{\bullet} = \mathbb{R}\backslash\{0\}$ (the inflation factor), and $a \in \mathbb{R}^n$ (the translational part). Since we want $f(M) \subset M$, we need $a = f(0) \in M$ and hence $\alpha R(M) = f(M) - a \subset M$. Thus we have

**Lemma 2.1** *The set of all self-similarities of a module $M$ forms a semigroup $\mathcal{S}(M)$ which decomposes into a linear part $\mathcal{L}(M)$ (those mappings fixing the origin) and a translational part isomorphic to $(M,+)$:*

$$\mathcal{S}(M) \;=\; (M,+) \times_s \mathcal{L}(M)\,. \tag{2.2}$$

As it contains 1, $\mathcal{S}(M)$ is actually a monoid, but we will usually speak of semigroups in any case. Note that the semi-direct product structure is directly inherited from the group of all affine transformations inside which all our semigroups live.

Obviously, the main interest is in the subsemigroup $\mathcal{L} = \mathcal{L}(M)$. $\mathcal{L}$ decomposes into $\mathcal{L}^+$ and $\mathcal{L}^-$, the orientation preserving and reversing elements, respectively. In what follows in this section, we deal with all of $\mathcal{L}$, but it is obvious that all the constructions work also for $\mathcal{L}^+$ as well. Later in the paper (starting with the section on two-dimensional examples) we will find it convenient to restrict our attention to $\mathcal{L}^+$. The extension to the full semigroup $\mathcal{L}$ is then an easy extra step.

If $f \in \mathcal{L}$, then $f(M)$ is a subgroup of $M$ and also a $\mathbb{Z}$-module of rank $n$. We call these subgroups the **similarity submodules** of $M$.

The index $[M : f(M)]$ of any similarity submodule of $M$ is finite. In fact, this index can be determined in the following way. First, choose a $\mathbb{Z}$-basis $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_r\}$ of $M$. Then, by representing each $f$ as a matrix of integers relative to this basis, we obtain a matrix representation

$$\nu \;:\; \mathcal{L} \;\longrightarrow\; \mathrm{Mat}(r,\mathbb{Z}) \tag{2.3}$$

of $\mathcal{L}$ and it is well known that

$$[M : f(M)] \;=\; |\det(\nu(f))|. \tag{2.4}$$

This leads at once to the semigroup homomorphisms

$$\mathrm{N} \;:\; \mathcal{L} \;\longrightarrow\; \mathbb{Z}^{\bullet} \tag{2.5}$$

(with $\mathbb{Z}^{\bullet} = \mathbb{Z}\backslash\{0\}$) and

$$\mathrm{ind} \;:\; \mathcal{L} \;\longrightarrow\; \mathbb{N} \tag{2.6}$$

given by $(\det\circ\nu)$ and $|\det\circ\nu|$, respectively, where $\circ$ denotes composition of mappings. We use the index map to produce the decomposition

$$\mathcal{L} \;=\; \bigcup_{k=1}^{\infty} \mathcal{L}_k, \tag{2.7}$$

where $\mathcal{L}_k$ denotes the set of self-similarities which map $M$ onto a submodule of index $k$. By the way, one should be aware that $\det(f) := \det(\alpha R)$ (see (2.1)) and $\det(\nu(f))$ (see (2.4)) are, in general, completely different (unless $r = n$).

We evidently have $\mathcal{L}_k\mathcal{L}_\ell \subset \mathcal{L}_{k\ell}$. Of course, $\mathcal{L}_1$ is the set of isometries of $M$, and hence is a group. We call this the **group of units** of $\mathcal{L}$. Obviously, it is the unique maximal subgroup of $\mathcal{L}$.

If $f, g \in \mathcal{L}$, then one sees that $f(M) = g(M)$ if and only if $g^{-1}f \in \mathcal{L}_1$. Thus we have a one-to-one correspondence between the set of similarity submodules and the set $\overline{\mathcal{L}}$ of right cosets $\mathcal{L}/\mathcal{L}_1$. The index map factors through this to give us a mapping

$$\mathrm{ind} \;:\; \overline{\mathcal{L}} \;\longrightarrow\; \mathbb{N} \tag{2.8}$$

together with the decomposition

$$\overline{\mathcal{L}} \;=\; \bigcup_{k=1}^{\infty} \overline{\mathcal{L}}_k. \tag{2.9}$$

One of the very first things that one should be interested in is the number of distinct similarity submodules of each index, or, equivalently, the cardinality of the $\overline{\mathcal{L}}_k$, and this leads us to introduce a zeta function for $\mathcal{L}$:

$$\zeta_{\mathcal{L}}(s) \;:=\; \sum_{k=1}^{\infty} \frac{\mathrm{card}(\overline{\mathcal{L}}_k)}{k^s} \;=\; \sum_{a\in\overline{\mathcal{L}}} \frac{1}{\mathrm{ind}(a)^s}\,. \tag{2.10}$$

We will see that, in many cases, we can give an explicit description of the zeta functions of our semigroups.

## 3 Examples in One Dimension

The simplest possible example is the set of integers, $\mathbb{Z}$, seen as a lattice in 1D Euclidean space. The possible sublattices are $n\mathbb{Z}$ with $n \in \mathbb{N}$, and they are obviously similarity sublattices, all with the same point symmetry, $C_2$. Consequently, the semigroup of linear self-similarities is simply $(\mathbb{Z}\backslash\{0\}, \cdot)$. To simplify notation, we introduce

$$\mathbb{Z}^{\bullet} \;:=\; \mathbb{Z}\backslash\{0\} \tag{3.1}$$

and similarly for other sets.

Evidently, we get $\mathcal{L}_n = \{\pm n\}$, $\mathcal{L}_1 = \{\pm 1\} \simeq C_2$, and $\overline{\mathcal{L}}_n = n\mathcal{L}_1$. But more can be said since $\mathbb{Z}$ is a ring. There is precisely one sublattice of index $m$ for each $m \in \mathbb{N}$, and these sublattices are actually the nonzero *ideals* of the ring $\mathbb{Z}$, i.e., all the nonzero subgroups $\mathfrak{a} \subset \mathbb{Z}$ with $k\mathfrak{a} \subset \mathfrak{a}$ for all $k \in \mathbb{Z}$. The set of nonzero ideals itself forms a semigroup $\mathcal{I}$ under multiplication and $\mathcal{I}$ is isomorphic to $(\mathbb{N}, \cdot)$. Since $\mathcal{L}$ is commutative, the set of right cosets $\overline{\mathcal{L}}$ is also a semigroup in the obvious way, and in fact the one-to-one correspondence of $\overline{\mathcal{L}}$ and the set of similarity submodules is an isomorphism of $\overline{\mathcal{L}}$ and $\mathcal{I}$. We have

$$\mathcal{L}(\mathbb{Z}) \;=\; (\mathbb{Z}^{\bullet}, \cdot) \;\simeq\; C_2 \times \mathbb{N} \;\simeq\; \mathcal{L}_1 \times \overline{\mathcal{L}} \;\simeq\; \mathcal{L}_1 \times \mathcal{I}\,. \tag{3.2}$$

The zeta function of $\mathcal{L}(\mathbb{Z})$ is precisely the same as the zeta function of the rational field $\mathbb{Q}$ – that is, the ordinary Riemann zeta function [**1, 12**]:

$$\zeta_{\mathcal{L}(\mathbb{Z})}(s) \;=\; \zeta(s) \;=\; \sum_{m=1}^{\infty} \frac{1}{m^s} \;=\; \prod_p \frac{1}{1-p^{-s}}\,. \tag{3.3}$$

The product version (with $p$ running through all primes) is called its Euler product expansion [**1**].

The meaning of the Euler product expansion of $\zeta_{\mathcal{L}(\mathbb{Z})}$ is that the semigroup $\overline{\mathcal{L}}(\mathbb{Z})$ is freely generated (as an Abelian semigroup) by the 'primes' $\overline{\mathcal{L}}_p$ where $p$ runs over all the rational primes:

$$\overline{\mathcal{L}} \;=\; \prod_p \langle\, \overline{\mathcal{L}}_p \,\rangle \tag{3.4}$$

where $\langle\cdot\rangle$ denotes the cyclic monoid generated by set enclosed in the brackets.

Finally,

$$\mathcal{S}(\mathbb{Z}) \;=\; (\mathbb{Z}, +) \times_s (\mathbb{Z}^{\bullet}, \cdot) \tag{3.5}$$

is the full semigroup of self-similaritties of $\mathbb{Z}$ inside $\mathbb{R}$. The subsemigroup of orientation preserving self-similarities is

$$\mathcal{S}^{+}(\mathbb{Z}) \;=\; (\mathbb{Z}, +) \times_s (\mathbb{N}, \cdot)\,. \tag{3.6}$$

To summarize, we have

**Proposition 3.1** *The semigroup of injective affine transformations that map $\mathbb{Z}$ into itself is $\mathcal{S}(\mathbb{Z})$ of Equation (3.5). The submonoid of affine mappings that fix the origin is $\mathcal{L}(\mathbb{Z})$ of Equation (3.2). The similarity submodules of $\mathcal{L}(\mathbb{Z})$ are the nonzero ideals (or invariant sublattices) of $\mathbb{Z}$, and this leads to the decomposition (3.4). The zeta function of $\mathcal{L}(\mathbb{Z})$ is Riemann's zeta function (3.3).*

A slightly more interesting example is provided by the ring $\mathbb{Z}[\tau]$, with $\tau$ being the golden ratio, $\tau = (1+\sqrt{5})/2$. Although $\mathbb{Z}[\tau] \subset \mathbb{R}$, the rank of $\mathbb{Z}[\tau]$ as a $\mathbb{Z}$-module is 2. On the other hand, it is of rank 1 as a $\mathbb{Z}[\tau]$-module.

The affine self-similarities are mappings of the form $f:\; x \mapsto \alpha x + a$ where $\alpha$ and $a$ are real numbers. Applying $f$ to 0 shows that $a \in \mathbb{Z}[\tau]$ and then it follows that $\alpha \in \mathbb{Z}[\tau]^{\bullet}$. Thus we see

$$\mathcal{L} \;=\; \mathcal{L}(\mathbb{Z}[\tau]) \;\simeq\; (\mathbb{Z}[\tau]^{\bullet}, \cdot)\,. \tag{3.7}$$

It follows that the similarity submodules are the nonzero principal ideals of $\mathbb{Z}[\tau]$, and since $\mathbb{Z}[\tau]$ is a principal ideal domain, they are precisely all the nonzero ideals.

The group $\mathcal{L}_1$ is the set of $a \in \mathbb{Z}[\tau]$ for which $a\mathbb{Z}[\tau] = \mathbb{Z}[\tau]$, which is nothing but the group of units of $\mathbb{Z}[\tau]$. $\mathcal{L}_1$ contains not only the group of point symmetries, $C_2 = \{\pm 1\}$, but also a group of inflation symmetries, $C_\infty$, consisting of multiplications by powers of $\tau$. So we have

$$\mathcal{L}_1 \;=\; \{\pm\tau^k \mid k \in \mathbb{Z}\} \;=\; \mathbb{Z}[\tau]^{\times} \;\simeq\; C_2 \times C_\infty\,. \tag{3.8}$$

Once again, $\overline{\mathcal{L}}$ is isomorphic with the semigroup $\mathcal{I}$ of nonzero ideals of $\mathbb{Z}[\tau]$. Thus the zeta function of $\mathcal{L}(\mathbb{Z}[\tau])$ is

$$\zeta_{\mathcal{L}(\mathbb{Z}[\tau])}(s) \;=\; \sum_{\mathfrak{a}\in\mathcal{I}} \frac{1}{[\mathbb{Z}[\tau]:\mathfrak{a}]^s}\,. \tag{3.9}$$

This coincides with the Dedekind zeta function of the quadratic number field $\mathbb{Q}(\tau) = \mathbb{Q}(\sqrt{5})$ which reads

$$\begin{aligned}\zeta_{\mathbb{Q}(\tau)}(s) &= \frac{1}{1-5^{-s}}\cdot \prod_{p\equiv\pm1\ (5)} \frac{1}{(1-p^{-s})^2}\cdot \prod_{p\equiv\pm2\ (5)} \frac{1}{1-p^{-2s}} \\ &= 1+\tfrac{1}{4^s}+\tfrac{1}{5^s}+\tfrac{1}{9^s}+\tfrac{2}{11^s}+\tfrac{1}{16^s}+\tfrac{2}{19^s}+\tfrac{1}{20^s}+\tfrac{1}{25^s}+\tfrac{2}{29^s}+\tfrac{2}{31^s}+\tfrac{1}{36^s}+\tfrac{2}{41^s}+\cdots\end{aligned} \tag{3.10}$$

Again, the Euler product expansion indicates that $\mathcal{I}$, and hence $\overline{\mathcal{L}}$, is freely generated as an Abelian semigroup by the 'primes' – this time the primes of the ring $\mathbb{Z}[\tau]$. We have

$$\mathcal{L}(\mathbb{Z}[\tau]) \;=\; (\mathbb{Z}[\tau]^{\bullet},\cdot) \;\simeq\; C_2\times C_\infty\times\overline{\mathcal{L}}\,. \tag{3.11}$$

Finally,

$$\mathcal{S}(\mathbb{Z}[\tau]) \;=\; (\mathbb{Z}[\tau],+)\times_s \mathcal{L}(\mathbb{Z}[\tau])\,. \tag{3.12}$$

To summarize:

**Proposition 3.2** *The semigroup of injective affine transformations that map $\mathbb{Z}[\tau]$ into itself is $\mathcal{S}(\mathbb{Z}[\tau])$ of Equation (3.12). The submonoid of affine mappings that fix the origin is $\mathcal{L}(\mathbb{Z}[\tau])$ of Equation (3.11). The similarity submodules are the nonzero ideals of $\mathbb{Z}[\tau]$, and the zeta function of $\mathcal{L}(\mathbb{Z}[\tau])$ is the Dedekind zeta function of the quadratic field $\mathbb{Q}(\tau)$, given in (3.10).*

Other modules of interest, such as $\mathbb{Z}[\sqrt{2}]$ etc., can be treated in the same fashion. With these simple examples in mind, we can now move on to the investigation of more interesting and relevant cases in two dimensions.

## 4 Examples in Two Dimensions

The simplest planar case is that of the square lattice, $\mathbb{Z}^2$, which may be identified with the ring of Gaussian integers [**12**] in the complex plane,

$$\mathbb{Z}^2 \;=\; \mathbb{Z}[i] \;=\; \{m+ni \mid m,n\in\mathbb{Z}\}\,. \tag{4.1}$$

In the complex plane, we can still write the orientation preserving similarity transformations as $f(z)=az+b$, now with $a,b,z\in\mathbb{C}\simeq\mathbb{R}^2$. The orientation reversing similarity transformations are all of the form $f\,:\,z\mapsto a\overline{z}+b$. In this and the subsequent section, we will consider only orientation preserving transformations.

Consider first the case $b=0$. Then, $f(\mathbb{Z}[i])\subset\mathbb{Z}[i]$ with $f$ regular implies $a\in\mathbb{Z}[i]^{\bullet}$ (since $a=f(1)$) and vice versa. So

$$\mathcal{L}^+(\mathbb{Z}[i]) \;=\; (\mathbb{Z}[i]^{\bullet},\cdot)\,. \tag{4.2}$$

The units of $\mathbb{Z}[i]$ are $\mathbb{Z}[i]^\times=\{1,i,-1,-i\}\simeq C_4$ and form the maximal subgroup $\mathcal{L}_1^+$ of $\mathcal{L}^+(\mathbb{Z}[i])$. $\overline{\mathcal{L}^+} := \mathcal{L}^+/\mathcal{L}_1^+$ is again a commutative semigroup isomorphic to the semigroup of nonzero ideals of $\mathbb{Z}[i]$. It is freely generated by the self-similarity classes corresponding to the prime ideals of $\mathbb{Z}[i]$.

The Dirichlet series generating function of the number of different ideals of index $m$ is the Dedekind zeta function of the quadratic field $\mathbb{Q}(i)$ and also the zeta function of $\mathcal{L}^+(\mathbb{Z}[i])$, i.e., $\zeta_{\mathcal{L}^+(\mathbb{Z}[i])}(s) = \zeta_{\mathbb{Q}(i)}(s)$, where [**12, 30**]

$$\begin{aligned}\zeta_{\mathbb{Q}(i)}(s) = \sum_{m=1}^{\infty} \frac{a_4(m)}{m^s} &= \frac{1}{1-2^{-s}} \cdot \prod_{p\equiv 1\ (4)} \frac{1}{(1-p^{-s})^2} \cdot \prod_{p\equiv 3\ (4)} \frac{1}{1-p^{-2s}} \\ &= 1 + \tfrac{1}{2^s} + \tfrac{1}{4^s} + \tfrac{2}{5^s} + \tfrac{1}{8^s} + \tfrac{1}{9^s} + \tfrac{2}{10^s} + \tfrac{2}{13^s} + \tfrac{1}{16^s} + \tfrac{2}{17^s} + \tfrac{1}{18^s} + \tfrac{2}{20^s} + \tfrac{3}{25^s} + \cdots \, .\end{aligned} \tag{4.3}$$

Finally, the semigroup $\mathcal{S}^+(\mathbb{Z}[i])$, by analogous arguments as in the 1D case, is given by

$$\mathcal{S}^+(\mathbb{Z}[i]) \;=\; (\mathbb{Z}[i],+) \times_s (\mathbb{Z}[i]^\bullet, \cdot) \tag{4.4}$$

and we have

**Proposition 4.1** *The semigroup of injective orientation preserving affine transformations that map $\mathbb{Z}[i]$ into itself is $\mathcal{S}^+(\mathbb{Z}[i])$ of Equation (4.4). The submonoid of affine mappings that fix the origin is $\mathcal{L}^+(\mathbb{Z}[i])$ of Equation (4.2). The images of elements of $\mathcal{L}^+(\mathbb{Z}[i])$ are the finite-index ideals (or invariant sublattices) of $\mathbb{Z}[i]$, and the Dirichlet series generating function for the number of ideals of index $m$ is the Dedekind zeta function (4.3).*

A similar argument applies to the triangular lattice which may be identified with the ring of Eisenstein integers $\mathbb{Z}[(1+i\sqrt{3})/2]$. More generally, one can consider the standard planar module [**20**]

$$\mathcal{M}_n \;:=\; \mathbb{Z} \oplus \mathbb{Z}\xi \oplus \mathbb{Z}\xi^2 \oplus \ldots \oplus \mathbb{Z}\xi^{n-1} \tag{4.5}$$

with $\xi = e^{2\pi i/n}$. $\mathcal{M}_n$ is a $\mathbb{Z}$-module of rank $\phi(n)$ (Euler's totient function) in the plane, and shows $N$-fold rotational symmetry, where

$$N \;=\; \begin{cases} n, & \text{if n even,} \\ 2n, & \text{if n odd.} \end{cases} \tag{4.6}$$

This module is nothing but the ring of cyclotomic integers, $\mathbb{Z}[\xi]$, the maximal order of the cyclotomic field $\mathbb{Q}(\xi)$, compare [**30**] for details.

Clearly, in obvious generalization of our previous examples,

$$\mathcal{L}^+(\mathcal{M}_n) \;=\; (\mathbb{Z}[\xi]^\bullet, \cdot) \tag{4.7}$$

and

$$\mathcal{S}^+(\mathcal{M}_n) \;=\; (\mathbb{Z}[\xi],+) \times_s \mathcal{L}^+(\mathcal{M}_n)\,. \tag{4.8}$$

If we want to relate the elements of $\mathcal{L}^+(\mathcal{M}_n)$ with the ideals of $\mathbb{Z}[\xi]$, we encounter a complication. Clearly, $f(\mathbb{Z}[\xi])$ is an ideal of $\mathbb{Z}[\xi]$, but, in general, not all ideals are of this form. The semigroup $\mathcal{L}^+$ (and also the set of similarity submodules) corresponds to the subsemigroup of nonzero ideals of $\mathbb{Z}[\xi]$ that are principal ideals. In fact, for almost all $n$, the ideal class number is larger than 1 and new types of ideals emerge. For precisely 29 planar modules, namely those with

$$\begin{aligned} n = {} & 3,4,5,7,8,9,11,12,13,15,16,17,19,20,21, \\ & 24,25,27,28,32,33,35,36,40,44,45,48,60,84 \end{aligned}$$

(where $n \not\equiv 2 \bmod 4$ to avoid double counting, see Equation (4.6)), the class number is 1 and all ideals of $\mathbb{Z}[\xi]$ are similarity images of $\mathbb{Z}[\xi]$ itself, see [**19**] or Theorem 11.1 of [**30**]. For some examples, with applications to quasicrystals, we refer to [**20, 26, 4**]. Let us summarize the situation as follows.

**Theorem 4.2** *Let $\xi = e^{2\pi i/n}$ and $\mathcal{M}_n = \mathbb{Z}[\xi]$. Then $\mathcal{L}^+(\mathcal{M}_n)$ and $\mathcal{S}^+(\mathcal{M}_n)$ are given by Equations (4.7) and (4.8), respectively. If $n$ is one of the 29 numbers given above, all invariant submodules of $\mathcal{M}_n$ are obtained as images under elements of $\mathcal{L}^+(\mathcal{M}_n)$, and the number of such ideals of index $m$ is given by the corresponding coefficient of the Dedekind zeta function $\zeta_{\mathbb{Q}(\xi)}(s)$.*

Note that the zeta function always counts the ideals, but that this does not coincide with counting the similarity sublattices resp. submodules, unless $n$ is one of the 29 numbers discussed above.

Let us give an explicit example here. Of particular interest in the quasicrystalline context is $n = 5$. The module $\mathcal{M}_5 = \mathbb{Z}[\xi]$ with $\xi = e^{2\pi i/5}$ is intimately related to objects such as the Penrose tiling or the Tübingen triangle tiling, compare [**9**]. Here, the zeta function reads

$$\begin{aligned}\zeta_{\mathbb{Q}(\xi)}(s) &= \sum_{m=1}^{\infty} \frac{a_{10}(m)}{m^s} \qquad (4.9)\\ &= \frac{1}{1-5^{-s}} \cdot \prod_{p\equiv 1\ (5)} \frac{1}{(1-p^{-s})^4} \cdot \prod_{p\equiv -1\ (5)} \frac{1}{(1-p^{-2s})^2} \cdot \prod_{p\equiv \pm 2\ (5)} \frac{1}{1-p^{-4s}}\\ &= 1 + \tfrac{1}{5^s} + \tfrac{4}{11^s} + \tfrac{1}{16^s} + \tfrac{1}{25^s} + \tfrac{4}{31^s} + \tfrac{4}{41^s} + \tfrac{4}{55^s} + \tfrac{4}{61^s} + \tfrac{4}{71^s} + \tfrac{1}{80^s} + \tfrac{1}{81^s} + \cdots .\end{aligned}$$

Apart from being the generating function for the number of invariant submodules of index $m$, it also gives the number of possible colourings of one-component Delone sets that are compatible with five-fold symmetry [**4**] and possess $\mathbb{Z}[\xi]$ as its limit translation module [**9**]. Here, the term one-component Delone set means that all points of such a set (which is the object to be coloured) are in one translation class w.r.t. the limit translation module of the set.

Motivated by their appearance in the description of planar quasicrystals with $N$-fold symmetry, we have focused on the cyclotomic integers so far. It is clear, however, that one can also treat other submodules of the plane if they have a ring structure. We have

**Theorem 4.3** *Let $\mathcal{R} \subset \mathbb{R}^2 \simeq \mathbb{C}$ be a planar $\mathbb{Z}$-module that is the ring of integers of a complex number field $K$. Then one has*

(1) $\mathcal{L}_1^+ \simeq \mathcal{R}^\times = \{$*the group of units of* $\mathcal{R}\}$;
(2) $\overline{\mathcal{L}^+}$ *is isomorphic to the semigroup of nonzero principal ideals of* $\mathcal{R}$;
(3) $\mathcal{L}^+ \simeq \mathcal{L}_1^+ \times \overline{\mathcal{L}^+}$.

The proof is an obvious application of the ideas used for the above examples and need not be repeated here. Among the examples we have in mind here are direct products such as $\mathbb{Z}[\tau]^2 \simeq \mathbb{Z}^2 \oplus \tau\mathbb{Z}^2$ or $\mathbb{Z}[\sqrt{2}]^2$, the latter actually being rather tricky. Both modules have the point symmetry of the square lattice and appear in the context of modulated structures. Here also, a Dirichlet series generating function can be calculated, see [**7**] for details.

Let us instead turn to three dimensions where things become more complicated because the rotation group SO(3) is not Abelian.

## 5 Examples in Three Dimensions

In 3-space, the (orientation preserving) similarity transformations have the form

$$\boldsymbol{x} \mapsto f(\boldsymbol{x}) = \alpha R \boldsymbol{x} + \boldsymbol{a} \qquad (5.1)$$

where $\boldsymbol{a} \in \mathbb{R}^3$, $R \in \mathrm{SO}(3,\mathbb{R})$ and $\alpha \in \mathbb{R}^+$. We again discuss only orientation preserving transformations here. If necessary, orientation reversing transformations can easily be included because $\mathrm{O}(3,\mathbb{R}) = \mathrm{SO}(3,\mathbb{R}) \times \{\pm\mathbb{1}\}$.

Let us first explain the situation with the simplest example, that of the primitive cubic lattice, $\mathbb{Z}^3$. As usual, we have the decomposition

$$\mathcal{S}^+(\mathbb{Z}^3) \;=\; (\mathbb{Z}^3,+) \times_s \mathcal{L}^+(\mathbb{Z}^3) \tag{5.2}$$

and we can focus our attention to linear similarity transformations, i.e., those fixing the origin.

Let us use the explicit realization

$$\mathbb{Z}^3 \;=\; \mathbb{Z}\boldsymbol{e}_1 \oplus \mathbb{Z}\boldsymbol{e}_2 \oplus \mathbb{Z}\boldsymbol{e}_3 \tag{5.3}$$

with the standard Euclidean basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. But then, $\alpha R \mathbb{Z}^3 \subset \mathbb{Z}^3$ implies $\alpha R$ integral because each $\boldsymbol{e}_i$ has to be mapped onto an integer linear combination of the basis vectors again. Now, $R$ is orthogonal, so $R^t = R^{-1}$. With $\alpha R$, also its transpose must be integral, and $(\alpha R)(\alpha R)^t = \alpha^2 \mathbb{1}$ then shows $\alpha^2 \in \mathbb{Z}$. But $\det(\alpha R) = \alpha^3 \det(R) = \alpha^3$ is also an integer, hence $\alpha = \alpha^3/\alpha^2 \in \mathbb{Q}$. So we have $\alpha \in \mathbb{Q}$ with $\alpha^2 \in \mathbb{Z}$ which is only possible for $\alpha \in \mathbb{Z}$. Note that in this example the norm map N coincides with the usual determinant map for $3 \times 3$ matrices, and so $\mathrm{ind}(\alpha R) = |\alpha|^3$.

Now, with $\alpha \in \mathbb{Z}$, $\alpha R$ integral means $R \in \mathrm{SO}(3,\mathbb{Q})$, and any such matrix is possible as similarity transformation that maps $\mathbb{Z}^3$ into itself, if $\alpha$ is a multiple of the denominator of $R$,

$$\mathrm{den}(R) \;:=\; \gcd\{m \in \mathbb{N} \mid mR \;\text{ integral}\}\,. \tag{5.4}$$

It is interesting to note that $\mathrm{den}(R)$ precisely takes all odd integers [**3**] while $R$ runs through the group $\mathrm{SO}(3,\mathbb{Q})$. If we define the set

$$\mathcal{R} \;:=\; \{\mathrm{den}(R) \cdot R \mid R \in \mathrm{SO}(3,\mathbb{Q})\}, \tag{5.5}$$

we see that each element of $\mathcal{L}^+(\mathbb{Z}^3)$ factors uniquely as $kS$, where $k \in \mathbb{N}$ and $S \in \mathcal{R}$. We have thus shown that the semigroup $\mathcal{L}^+(\mathbb{Z}^3)$ is

$$\mathcal{L}^+(\mathbb{Z}^3) \;=\; (\{\mathbb{N} \times \mathcal{R}\}, \cdot)\,. \tag{5.6}$$

Note that, even though the set $\mathcal{R}$ does *not* form a semigroup, the set $\{\mathbb{N} \times \mathcal{R}\}$, together with usual multiplication, does.

At this point, let us consider the statistics of similarity submodules in some more detail. It is clear that sets of 24 different elements of $\mathcal{R}$ will result in the same sublattice, because $\mathbb{Z}^3$ admits 24 rotational symmetries, the units of $\mathcal{L}^+(\mathbb{Z}^3)$, namely the octahedral group [**11, 2**] $\mathrm{SO}(3,\mathbb{Z})$:

$$\mathcal{L}^+_1 \;\simeq\; \mathrm{SO}(3,\mathbb{Z}) \;\simeq\; \mathfrak{S}_4\,. \tag{5.7}$$

On the other hand, given an $f \in \mathcal{L}^+(\mathbb{Z}^3)$, $f(\boldsymbol{x}) = k \cdot \mathrm{den}(R) \cdot R\boldsymbol{x}$, the image $f(\mathbb{Z}^3)$ is a cubic sublattice of $\mathbb{Z}^3$ of index

$$\mathrm{ind}(f) \;=\; [\mathbb{Z}^3 : f(\mathbb{Z}^3)] \;=\; k^3 \cdot \mathrm{den}(R)^3\,. \tag{5.8}$$

This shows that the corresponding generating function is a product of two contributions, one from the possibilities of the form $\mathrm{den}(R) \cdot R$ and the other from the freedom to go from any sublattice to another (sub-)sublattice by scaling with an integer.

Consider the contribution from the rational matrices first. The Dirichlet series generating function for the number $a_m$ of $\mathrm{SO}(3,\mathbb{Q})$-matrices of denominator $m$, divided by 24, is known from the closely related coincidence site lattice problem, compare [**8, 3**], and reads

$$\Phi_{cub}(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} = \prod_{p\neq 2} \frac{1+p^{-s}}{1-p^{1-s}} = \frac{1-2^{1-s}}{1+2^{-s}} \cdot \frac{\zeta(s)\zeta(s-1)}{\zeta(2s)} \tag{5.9}$$
$$= 1 + \tfrac{4}{3^s} + \tfrac{6}{5^s} + \tfrac{8}{7^s} + \tfrac{12}{9^s} + \tfrac{12}{11^s} + \tfrac{14}{13^s} + \tfrac{24}{15^s} + \tfrac{18}{17^s} + \tfrac{20}{19^s} + \tfrac{32}{21^s} + \tfrac{24}{23^s} + \tfrac{30}{25^s} + \cdots .$$

Together with the proper account of the possibilities coded by the factor $k^3$ in (5.8), we thus obtain the zeta function for $\mathcal{L}^+(\mathbb{Z}^3)$:

$$\zeta_{\mathcal{L}^+(\mathbb{Z}^3)}(s) = \zeta(3s)\Phi_{cub}(3s) \tag{5.10}$$
$$= 1 + \tfrac{1}{8^s} + \tfrac{5}{27^s} + \tfrac{1}{64^s} + \tfrac{7}{125^s} + \tfrac{5}{216^s} + \tfrac{9}{343^s} + \tfrac{1}{512^s} + \tfrac{17}{729^s} + \tfrac{7}{1000^s} + \cdots .$$

This can also be read as the generating function for the number of possibilities to colour the points of $\mathbb{Z}^3$ with $m$ different colours such that one colour occupies a similarity sublattice of index $m$ while the others code the cosets. Note that this arrangement possesses a colour symmetry in the sense that any rotation of $\mathrm{SO}(3,\mathbb{Z})$ or its conjugate under the similarity transformation, followed by a suitable permutation of the colours, maps the coloured lattice onto itself.

Let us summarize our findings as follows.

**Theorem 5.1** *The semigroup of orientation preserving self-similarities of $\mathbb{Z}^3$ is $\mathcal{S}^+(\mathbb{Z}^3)$ of Equation (5.2). $\mathcal{L}^+(\mathbb{Z}^3)$ is described by Equations (5.5, 5.6) and its group of units by Equation (5.7). For $f \in \mathcal{L}^+(\mathbb{Z}^3)$, $f = \alpha R$, the image $f(\mathbb{Z}^3)$ is a primitive cubic sublattice of $\mathbb{Z}^3$, invariant under the action of $R \cdot \mathrm{SO}(3,\mathbb{Z}) \cdot R^{-1}$. Finally, the zeta function of $\mathcal{L}^+(\mathbb{Z}^3)$ is given by Equation (5.10).*

Let us add, without going into any detail, that the same result also applies to the face-centred and the body-centred cubic lattice in 3-space, see [**3**] for additional material and the necessary arguments to prove this.

We now move into the interesting problem of icosahedral symmetry. As an intermediate step, we consider the module

$$\mathcal{M}_C = \mathbb{Z}[\tau]\boldsymbol{e}_1 \oplus \mathbb{Z}[\tau]\boldsymbol{e}_2 \oplus \mathbb{Z}[\tau]\boldsymbol{e}_3 = \mathbb{Z}^3 \oplus \tau\mathbb{Z}^3 \tag{5.11}$$

which, by definition, is both a $\mathbb{Z}[\tau]$-module of rank 3 and a $\mathbb{Z}$-module of rank 6. Although this object has only cubic symmetry, it plays an important role in the icosahedral context.

Going back to the arguments given above for the derivation of $\mathcal{S}^+(\mathbb{Z}^3)$ and $\mathcal{L}^+(\mathbb{Z}^3)$, one quickly realizes that the same reasoning applies here, with $\mathbb{Z}$ replaced by $\mathbb{Z}[\tau]$ and $\mathbb{Q}$ by $\mathbb{Q}(\tau)$. This is so because $\mathbb{Q}(\tau)$ is a field and $\mathbb{Z}[\tau]$ is a Euclidean domain. In particular, for $R \in \mathrm{SO}(3,\mathbb{Q}(\tau))$, we can still define its denominator,

$$\mathrm{den}(R) := \gcd\{\gamma \in \mathbb{Z}[\tau] \mid \gamma R \text{ integral}\} \tag{5.12}$$

where integral now means integral w.r.t. $\mathbb{Z}[\tau]$ and the greatest common divisor is only defined up to units of $\mathbb{Z}[\tau]$, i.e., up to $\pm\tau^m$, $m \in \mathbb{Z}$. Let us assume $\mathrm{den}(R)$ to be positive to preserve orientation. All this does not disturb the argument though because $\mathcal{M}_C$ is a $\mathbb{Z}[\tau]$-module and thus invariant under multiplication by any of these units. Now, with $\mathbb{Z}[\tau]^+ = \mathbb{Z}[\tau] \cap \mathbb{R}^+$ and the set

$$\mathcal{R}_C := \{\mathrm{den}(R) \cdot R \mid R \in \mathrm{SO}(3,\mathbb{Q}(\tau))\}, \tag{5.13}$$

we can formulate

**Proposition 5.2** *The semigroups $\mathcal{L}^+$ and $\mathcal{S}^+$ of similarity transformations that map $\mathcal{M}_C$ into itself are given by*

$$\mathcal{L}^+(\mathcal{M}_C) \;=\; (\{\mathbb{Z}[\tau]^+ \times \mathcal{R}_C\}, \cdot)$$

*and*

$$\mathcal{S}^+(\mathcal{M}_C) \;=\; (\mathcal{M}_C, +) \times_s \mathcal{L}^+(\mathcal{M}_C)\,. \tag{5.14}$$

*Furthermore, the group of units is*

$$\mathcal{L}^+_1(\mathcal{M}_C) \;\simeq\; C_2 \times C_\infty \times \mathfrak{S}_4\,. \tag{5.15}$$

Also, the Dirichlet series generating function for the number of similarity submodules of $\mathcal{M}_C$ of index $m$ can be calculated explicitly. It turns out to be

$$F_{\mathcal{M}_C}(s) \;=\; \zeta_{\mathbb{Q}(\tau)}(3s) \cdot \Phi_{\mathcal{M}_C}(3s) \tag{5.16}$$

with the series $\Phi_{\mathcal{M}_C}(s)$ taken from Equation (5.43) of [**3**], see [**7**] for details.

In the context of quasicrystals, structures with icosahedral symmetry are among the most important ones. The corresponding algebraic structures are the 3 icosahedral $\mathbb{Z}$-modules of rank 6 in $\mathbb{R}^3$ [**27**], called $\mathcal{M}_B$, $\mathcal{M}_P$ and $\mathcal{M}_F$. With the definition of $\mathcal{M}_C$ as used above, they can be given explicitly as

$$\begin{aligned} \mathcal{M}_B &= \{\; \boldsymbol{x} \in \mathcal{M}_C \mid \tau^2\alpha_1 + \tau\alpha_2 + \alpha_3 \equiv 0 \;\; (2)\,\} \\ \mathcal{M}_P &= \{\; \boldsymbol{x} \in \mathcal{M}_B \mid \alpha_1 + \alpha_2 + \alpha_3 \equiv 0 \;\text{ or }\; \tau \;\; (2)\,\} \\ \mathcal{M}_F &= \{\; \boldsymbol{x} \in \mathcal{M}_B \mid \alpha_1 + \alpha_2 + \alpha_3 \equiv 0 \;\; (2)\,\}\,. \end{aligned} \tag{5.17}$$

The standard Euclidean basis used is understood to point along three mutually orthogonal twofold axes of an icosahedron. These modules can be obtained as projection of the three types of 6D hypercubic lattices into 3-space. In our setting, $\mathcal{M}_F$ is the $\mathbb{Z}$-span of the 30 vertices of an icosidodecahedron [**10**] (which is the root system for the non-crystallographic Coxeter group $H_3$ [**13, 23**]), and $\mathcal{M}_B$ is its dual module (up to a constant which can be absorbed into the definition of the corresponding quadratic form). $\mathcal{M}_F$ and $\mathcal{M}_B$ are invariant under multiplication by $\tau$ (they are actually $\mathbb{Z}[\tau]$-modules of rank 3), while $\mathcal{M}_P$ is only invariant under multiplication by $\tau^3$, see [**27, 3**] for details. The relation with the cubic module $\mathcal{M}_C$ is given by

$$2\mathcal{M}_C \overset{4}{\subset} \mathcal{M}_F \overset{2}{\subset} \mathcal{M}_P \overset{2}{\subset} \mathcal{M}_B \overset{4}{\subset} \mathcal{M}_C \tag{5.18}$$

where the integers on top of the inclusion symbols indicate the corresponding indices.

Let us take a closer look at the module $\mathcal{M}_F$ which can also be seen as a $\mathbb{Z}[\tau]$-module of rank 3. This is also evident from (5.17), as $\mathcal{M}_F$ is defined as a submodule of the $\mathbb{Z}[\tau]$-module $\mathcal{M}_C$ with constraints invariant under multiplication by $\tau$.

Each similarity submodule $\Lambda$ is evidently a $\mathbb{Z}[\tau]$-module. For each $\alpha \in \mathbb{Z}[\tau]$, $\alpha\Lambda$ has index

$$[\Lambda : \alpha\Lambda] \;=\; |\mathrm{N}_2(\alpha)|^3 \tag{5.19}$$

in $\Lambda$, where $\mathrm{N}_2(\gamma)$ denotes the (number theoretic) norm in the quadratic field $\mathbb{Q}(\tau)$. For $\gamma = p + q\tau$ it is

$$\mathrm{N}_2(\gamma) \;:=\; \gamma\gamma' \;=\; p^2 + pq - q^2 \tag{5.20}$$

where $\gamma' = p + q\tau'$ is the algebraic conjugate of $\gamma$ defined through $\tau' = -1/\tau = 1 - \tau$.

Let us assume that $\Lambda \subset \mathcal{M}_F$ is a maximal similarity submodule in the sense that $\alpha\Lambda \subset \mathcal{M}_F$ implies $\alpha \in \mathbb{Z}[\tau]$. (In particular, $\mathcal{M}_F$ itself is considered to be maximal.) We want to determine what these maximal similarity submodules are, resp. which similarity transformations lead to them. So, consider $\alpha R$ such that

$$\alpha R \mathcal{M}_F \;\subset\; \mathcal{M}_F\,. \tag{5.21}$$

From (5.18), we may now conclude that $\alpha R{\cdot}2\mathcal{M}_C \subset \mathcal{M}_F \subset \mathcal{M}_C$, so $2\alpha R$ must have $\mathbb{Z}[\tau]$ entries only. This is a necessary, but not a sufficient condition, while taking $\alpha R$ to be $\mathbb{Z}[\tau]$-integral is sufficient, but not necessary. In any case, we can apply Proposition 5.2 and conclude that $2\alpha \in \mathbb{Z}[\tau]$ and $R \in \mathrm{SO}(3,\mathbb{Q}(\tau))$ is a rational matrix.

From this we can deduce the structure of $\mathcal{L}_1^+(\mathcal{M}_F)$. If $\alpha R$ is a unit then so are all its powers, and $(\alpha R)^{-1} = \alpha^{-1} R^t$ in particular. Consequently, $2\alpha^k \in \mathbb{Z}[\tau]$ for all integers $k \in \mathbb{Z}$. This forces $\alpha$ to be a (positive) unit of $\mathbb{Z}[\tau]$, i.e., $\alpha \in (\mathbb{Z}[\tau]^+)^\times$. Then it follows that $R \in \mathrm{SO}(3,\mathbb{Z}[\tau])$, which is the icosahedral group [**11**] $\mathfrak{A}_5$ of order 60 (actually a maximal finite subgroup of $\mathrm{SO}(3,\mathbb{R})$). So we have

$$\mathcal{L}_1^+(\mathcal{M}_F) \;\simeq\; (\mathbb{Z}[\tau]^+)^\times \times \mathfrak{A}_5\,. \tag{5.22}$$

For each $R \in \mathrm{SO}(3,\mathbb{Q}(\tau))$, we thus obtain one maximal $\mathcal{M}_F$-submodule $\Lambda$, and precisely 60 different $R$'s will result in the same $\Lambda$. To actually reach $\Lambda$, we have to multiply $R$ by a suitable $\mathbb{Z}[\tau]$ number, $\sigma(R)$, and, in contrast to the cubic case above, this number is not always the denominator of $R$, but sometimes a true divisor of it (then coinciding with $\mathrm{den}(R)/2$ up to units in $\mathbb{Z}[\tau]$). Without loss of generality, we may assume $\sigma(R)$ to be totally positive, i.e., $\sigma(R) > 0$ (otherwise we multiply by $-1$) and $\mathrm{N}_2(\sigma(R)) > 0$ (otherwise we multiply $\sigma(R)$ by $\tau$). So, $\sigma(R)$ is a number in $\mathbb{Z}[\tau]$ whose norm is the so-called *coincidence index* $\Sigma(R)$, defined through [**8, 3**]

$$\Sigma(R) \;:=\; [\mathcal{M}_F : (\mathcal{M}_F \cap R\mathcal{M}_F)] \;=\; \mathrm{N}_2(\sigma(R))\,. \tag{5.23}$$

The corresponding similarity transformation maps $\mathcal{M}_F$ to a maximal submodule $\Lambda$ which is of index $\Sigma(R)^3$ in $\mathcal{M}_F$. Let us add that, while $R$ runs through $\mathrm{SO}(3,\mathbb{Q}(\tau))$, $\Sigma(R)$ takes *all* positive integers that can be represented by the integer quadratic form (5.20). These are the numbers all of whose prime factors congruent to 2 or 3 (mod 5) occur with even exponent only [**8, 3**].

At this point, we can describe $\mathcal{L}^+(\mathcal{M}_F)$:

$$\mathcal{L}^+(\mathcal{M}_F) \;=\; (\{\mathbb{Z}[\tau]^+ \times \mathcal{R}_F\}, \cdot) \tag{5.24}$$

where $\mathcal{R}_F$ is different from $\mathcal{R}_C$ precisely in the necessary prefactors, i.e.,

$$\mathcal{R}_F \;:=\; \{\sigma(R)\cdot R \mid R \in \mathrm{SO}(3,\mathbb{Q}(\tau))\}\,. \tag{5.25}$$

Note that the (totally positive) number $\sigma(R)$ is only determined up to multiplication by $\tau^{2m}$, $m \in \mathbb{Z}$, but the meaning of (5.24) is well defined and the semigroup unique. The actual calculation of $\Sigma(R)$ (and hence that of $\sigma(R)$) is a bit tricky and requires the knowledge of the ideal structure of the icosian ring [**21, 24**]. The latter enters through Cayley's parametrization of $\mathrm{SO}(3,\mathbb{Q}(\tau))$-matrices [**3**]. Details of this, and an extension of this analysis to four dimensions, will be given in a forthcoming publication.

Let us nevertheless present the statistics of the similarity submodules of index $m$ in form of the zeta function or Dirichlet series generating function,

$$\begin{aligned} \zeta_{\mathcal{L}^+(\mathcal{M}_F)}(s) \;&=\; \zeta_{\mathbb{Q}(\tau)}(3s)\Phi_{ico}(3s) \\ &=\; 1 + \tfrac{6}{64^s} + \tfrac{7}{125^s} + \tfrac{11}{729^s} + \tfrac{26}{1331^s} + \tfrac{26}{4096^s} + \tfrac{42}{6859^s} + \tfrac{42}{8000^s} + \tfrac{37}{15625^s} + \cdots\,. \end{aligned} \tag{5.26}$$

It is derived in the same spirit as the corresponding formula for the cubic lattices, but now with the zeta function of $\mathbb{Q}(\tau)$ (instead of Riemann's zeta function) and with the icosahedral generating function

$$\Phi_{ico}(s) = \frac{1+5^{-s}}{1-5^{1-s}} \prod_{p\equiv\pm 2\ (5)} \frac{1+p^{-2s}}{1-p^{2(1-s)}} \prod_{p\equiv\pm 1\ (5)} \left(\frac{1+p^{-s}}{1-p^{1-s}}\right)^2 \tag{5.27}$$
$$= 1 + \tfrac{5}{4^s} + \tfrac{6}{5^s} + \tfrac{10}{9^s} + \tfrac{24}{11^s} + \tfrac{20}{16^s} + \tfrac{40}{19^s} + \tfrac{30}{20^s} + \tfrac{30}{25^s} + \tfrac{60}{29^s} + \tfrac{64}{31^s} + \tfrac{50}{36^s} + \tfrac{84}{41^s} + \cdots$$

from [**8, 3**] (instead of its cubic counterpart met above).

To summarize this part:

**Theorem 5.3** *The semigroup of (orientation-preserving) self-similarities of the icosahedral module $\mathcal{M}_F$ is $\mathcal{S}^+(\mathcal{M}_F) = (\mathcal{M}_F, +) \times_s \mathcal{L}^+(\mathcal{M}_F)$. The group of units is given by Equation (5.22), $\mathcal{L}^+(\mathcal{M}_F)$ by Equations (5.24, 5.25) and its zeta function by Equation (5.26).*

The same reasoning, and hence the same result, applies to the semigroups and the Dirichlet series of the module $\mathcal{M}_B$. The situation of the intermediate module $\mathcal{M}_P$, however, is more complicated because it is not $\tau$-invariant. For one possibility to overcome this by considering triples of the form $(\mathcal{M}_P, \tau\mathcal{M}_P, \tau^2\mathcal{M}_P)$, we refer to [**4**].

In this 3D case, it is more difficult to specify relevant common properties that allow for a generalization similar to that of Theorem 4.3. Clearly, there is no obvious ring structure connected with $\mathbb{R}^3$, and it will be difficult in general to determine $\mathcal{L}$ or $\mathcal{L}^+$. On the other hand, for any given module $\mathcal{M} \subset \mathbb{R}^3$, $\mathcal{S}^+$ still has the form $\mathcal{S}^+(\mathcal{M}) = (\mathcal{M}, +) \times_s \mathcal{L}^+(\mathcal{M})$. If we relate the rotations of 3-space with the action of quaternions in 4-space (where the real part is left invariant, compare [**15**] for a full description), we see that the determination of $\mathcal{L}^+$ is intimately related with the structure of maximal orders inside the quaternions, i.e., Hurwitz' ring of integer quaternions [**14**] for the case of $\mathbb{Z}^3$ or the icosian ring [**21, 24**] for the icosahedral modules.

Both cases are even more important if one attempts to generalize the approach to four dimensions. There, with proper tools from algebra and number theory, one can determine $\mathcal{S}^+$ both for the 4D cubic lattices and for the icosian ring itself, seen as the module generated by the non-crystallographic root system of the Coxeter group $H_4$. This will be reported separately in a forthcoming publication.

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