INVARIANT SUBMODULES AND SEMIGROUPS OF SELF-SIMILARITIES FOR FIBONACCI MODULES

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The problem of invariance and self-similarity in $\mathbb{Z}$-modules is investigated. For a selection of examples relevant to quasicrystals, especially Fibonacci modules, we determine the semigroup of self-similarities and encapsulate the number of similarity submodules in terms of Dirichlet series generating functions.

1 Introduction

In the theory of quasicrystals, one typically deals with coordinates (indices) that lie in some $\mathbb{Z}$-module of rank higher than the dimension of the ambient space, $\mathbb{R}^n$. Often, these $\mathbb{Z}$-modules are R-modules for some ring R and thereby admit sets of self-similarities (by scaling with elements of R), with a much richer structure than that available for lattices in $\mathbb{R}^n$. It is natural to study these and other self-similarities, partly because they are intrinsic to the mathematics of the models in question, and partly to compensate for the loss of translational symmetry in non-periodic structures.

In this contribution, we illustrate this concept with examples that show up in the context of quasicrystals and incommensurate structures. We shall focus here mainly on a class of modules with both crystallographic symmetries and scaling invariance, especially by $\tau = (1+\sqrt{5})/2$. We will describe both the semigroup of self-similarities and the set of invariant submodules, encapsulating the number of submodules of given index, $a(m)$, in terms of Dirichlet series generating functions. [1] They are more appropriate than power series because $a(m)$ will be a multiplicative function, i.e. $a(1) = 1$ and $a(mn) = a(m)a(n)$ for m, n coprime. Apart from providing a closed expression, such generating functions also allow for the determination of asymptotic properties; see Ref. 13 for some general background and Ref. 10 for some closely related examples.

[a] Heisenberg-fellow

2 Setup and general results

We are interested in $\mathbb{Z}$-modules M of finite rank r that span a Euclidean space $\mathbb{R}^n$, $r \geq n$, and the set of all affine *self-similarities*

$$f \; : \quad x \; \mapsto \; \alpha R x + v \tag{1}$$

that map M into itself, where $R \in \mathrm{O}(n,\mathbb{R})$, $\alpha \in \mathbb{R}^{\bullet} = \mathbb{R}\backslash\{0\}$ (the inflation factor)[b], and $v \in \mathbb{R}^n$ (the translational part). Since $f(M) \subset M$, $f(0) = v \in M$ and $\alpha R(M) = f(M) - v \subset M$. The set of all affine self-similarities of M thus forms a *semigroup*, $\mathcal{S}(M)$, which decomposes into a *linear* part, $\mathcal{L}(M)$ (those mappings fixing the origin), and a *translational* part isomorphic to $(M,+)$:

$$\mathcal{S}(M) \; = \; (M,+) \times_s \mathcal{L}(M) \, . \tag{2}$$

Obviously, the main interest is in the subsemigroup $\mathcal{L}(M)$ which we will thus concentrate on in the sequel. We call its (unique) maximal subgroup, denoted by $\mathcal{L}_1(M)$, the *group of units* of $\mathcal{L}(M)$, and the modules $f(M)$ with $f \in \mathcal{L}(M)$ the *similarity submodules* of M. They are *invariant* in the sense that they possess the same point symmetry as M (i.e. the point symmetry groups of M and $f(M)$ are conjugate in $\mathrm{O}(n,\mathbb{R})$) and the same scaling invariance. In our setting, all such submodules are of finite index, and one is then interested in the Dirichlet series generating function that counts them. Following the practice of analytic number theory, this is called the *zeta-function* of M:

$$\zeta_M(s) \; := \; \sum_{m=1}^{\infty} \frac{a(m)}{m^s} \; = \; \sum_{\tilde{M}} \frac{1}{[M : \tilde{M}]^s} \tag{3}$$

where $\tilde{M}$ runs through all similarity submodules of M. As mentioned above, $a(m)$ is an arithmetic function, and will be multiplicative in all our examples. Note that s is a complex number, and that such series have a nice analytic behaviour, e.g. they converge absolutely for $\mathrm{Re}(s) > c$ for some $c \in \mathbb{R}$, see Ref. 1 for a more general setting and further details.

In the happy instance that M may actually be viewed as the ring of integers of some number field K with class number 1 (i.e. unique factorization), the scaling invariance of M implies that the similarity submodules coincide with the non-zero *ideals* of M (i.e. subgroups $\mathfrak{a} \subset M$, $\mathfrak{a} \neq 0$, with $M\mathfrak{a} \subset \mathfrak{a}$), $\mathcal{L}_1(M)$ is the group of *units* of M (i.e. the group of invertible elements of M), and ζ_M is the *Dedekind zeta-function* of K (see Ref 3., Thm. 2). The simplest such case

[b] In general, if A is a ring, then $A^{\bullet}$ denotes its non-zero elements.

is $M = \mathbb{Z}$ where the non-zero ideals are $m\mathbb{Z}, m \in \mathbb{N}$, and the Dirichlet series for M then is the well-known Riemann zeta-function [1] itself, $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$.

An interesting class to study is that of the *Fibonacci modules*, i.e. of modules M such that $x \cdot y \subset \mathbb{Z}[\tau]$ for all $x, y \in M$. Since some of them have already been discussed elsewhere [3,10,2,5], we focus here on (hyper-)cubic $\mathbb{Z}[\tau]$-modules of the form $\mathbb{Z}[\tau]\boldsymbol{e}_1 \oplus \ldots \oplus \mathbb{Z}[\tau]\boldsymbol{e}_n$, with $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ the standard Euclidean basis of $\mathbb{R}^n$. Note that they possess (hyper-)cubic point symmetry because $\mathrm{O}(n, \mathbb{Z}[\tau]) = \mathrm{O}(n, \mathbb{Z}) \simeq (C_2)^n \times_s S_n$, which is the hyperoctahedral group (of order $2^n n!$). We also deal with $M = \mathbb{Z}[\sqrt{2}]\boldsymbol{e}_1 \oplus \mathbb{Z}[\sqrt{2}]\boldsymbol{e}_2$.

3 A one-dimensional example

The simplest Fibonacci module is the ring $\mathbb{Z}[\tau]$ itself. It is, at the same time, a $\mathbb{Z}$-module of rank 2 and a $\mathbb{Z}[\tau]$-module of rank 1. Seen as the former, it contains $\sigma_1(m)$ $\mathbb{Z}$-submodules of index m, where $\sigma_1(m) = \sum_{d|m} d$ is a divisor function, see Appendix A of Ref. 2 for details. Though all these submodules are invariant under inversion, most of them are *not* invariant under multiplication by τ. Since we also demand that, we have to search for all $\mathbb{Z}[\tau]$-submodules of $\mathbb{Z}[\tau]$, hence for all *ideals* $\mathfrak{a} \subset \mathbb{Z}[\tau]$, where we exclude the trivial case $\mathfrak{a} = 0$.

Since $\mathbb{Z}[\tau]$ is the ring of integers in the quadratic field $\mathbb{Q}(\tau)$, we are in the favorable situation mentioned above. Obviously, $\mathcal{L}(\mathbb{Z}[\tau]) = (\mathbb{Z}[\tau]^{\bullet}, \cdot)$, and the group of units is $\mathcal{L}_1(\mathbb{Z}[\tau]) = \{\pm\tau^m \mid m \in \mathbb{Z}\} \simeq C_2 \times C_\infty$. The Dedekind zeta-function of this field is the generating function we want. It reads

$$\begin{aligned} \zeta_{\mathbb{Q}(\tau)}(s) &= \frac{1}{1-5^{-s}} \cdot \prod_{p \equiv \pm 1\ (5)} \frac{1}{(1-p^{-s})^2} \cdot \prod_{p \equiv \pm 2\ (5)} \frac{1}{1-p^{-2s}} \qquad (4) \\ &= 1 + \tfrac{1}{4^s} + \tfrac{1}{5^s} + \tfrac{1}{9^s} + \tfrac{2}{11^s} + \tfrac{1}{16^s} + \tfrac{2}{19^s} + \tfrac{1}{20^s} + \tfrac{1}{25^s} + \tfrac{2}{29^s} + \tfrac{2}{31^s} + \tfrac{1}{36^s} + \tfrac{2}{41^s} + \cdots \end{aligned}$$

Here, we made use of another property of Dirichlet series of multiplicative arithmetic functions, namely the existence of an Euler product representation [1] where the product runs over all rational primes with the restrictions indicated. Note that this generating function, as in the examples below, allows for the determination of asymptotic properties, see Refs. 2, 3 and 10 for examples.

4 Planar examples

The most important planar examples are probably the modules with n-fold symmetry, such as the rings of cyclotomic integers, which are discussed in detail in Refs. 3, 5 and 10. To complement this, let us consider the Fibonacci module $\mathcal{M} = \mathbb{Z}[\tau]\boldsymbol{e}_1 \oplus \mathbb{Z}[\tau]\boldsymbol{e}_2$ which has only the point symmetry of the square lattice, and would appear in a τ-modulated version of it.

In this section, we describe the orientation preserving transformations only (indicating this by a superscript $+$). To obtain all transformations, we form the semi-direct product with another C_2 corresponding to complex conjugation.

We first observe that we may view $\mathcal{M}$ as a ring, namely $\mathcal{M} = \mathbb{Z}[i\tau] = \mathbb{Z}[i,\tau]$. It is not hard to show that $\mathcal{M}$ is the ring of integers of the quartic field $K := \mathbb{Q}(i\tau)$, the splitting field of the polynomial $x^4 + 3x^2 + 1$. The Galois group of K is $C_2 \times C_2$ and K is a totally imaginary extension of its maximal real subfield, $\mathbb{Q}(\tau) = \mathbb{Q}(\sqrt{5}\,)$ (so, K has degree 2 over $\mathbb{Q}(\tau)$). The latter has class number $h(\mathbb{Q}(\tau)) = 1$, and hence unique factorization. From Ref. 6 (see pp. 11, 46, and the Table of relative class numbers), one can now calculate that the class number of $\mathbb{Q}(i\tau)$ is $h(\mathbb{Q}(i\tau)) = h^* \cdot h(\mathbb{Q}(\tau))$ where h^*, the relative class number of $\mathbb{Q}(i\tau)/\mathbb{Q}(\tau)$, is 1. Hence, $h(\mathbb{Q}(i\tau)) = 1$, and we have unique factorization again. This means that all ideals of $\mathbb{Z}[i\tau]$ are principal and thus coincide with the similarity submodules. So, $\mathcal{L}^+(\mathcal{M}) = (\mathcal{M}^\bullet, \cdot)$ and we now have to determine the group of units and the zeta-function of K. One can prove (though we have to omit that here) that the former is given by

$$\mathcal{L}_1^+(\mathcal{M}) \;=\; \{ i^k \tau^\ell \mid k = 0,1,2,3 \text{ and } \ell \in \mathbb{Z} \} \;\simeq\; C_4 \times C_\infty \,. \tag{5}$$

To calculate the Dirichlet series generating function of the number of similarity submodules (= ideals) of index m, one now realizes that $\mathbb{Q}(i\tau)$ is a degree 2 subfield of the 20th cyclotomic field, $K_{20} = \mathbb{Q}(i\xi)$, where $\xi = e^{2\pi i/5}$. Now, one can apply the technique of Dirichlet characters, see chapters 3 and 4 of Ref. 12, to determine the Dedekind zeta-function of $\mathbb{Q}(i\tau)$ as a product of the L-functions of four characters. The result reads

$$\begin{aligned} \zeta_{\mathcal{M}}(s) \;&=\; \frac{1}{1-4^{-s}} \cdot \frac{1}{(1-5^{-s})^2} \cdot \prod_p \frac{1}{(1-p^{-s})^4} \cdot \prod_{\tilde p} \frac{1}{(1-\tilde p^{-2s})^2} \qquad (6) \\ &=\; 1 + \tfrac{1}{4^s} + \tfrac{2}{5^s} + \tfrac{2}{9^s} + \tfrac{1}{16^s} + \tfrac{2}{20^s} + \tfrac{3}{25^s} + \tfrac{4}{29^s} + \tfrac{2}{36^s} + \tfrac{4}{41^s} + \tfrac{4}{45^s} + \tfrac{2}{49^s} + \tfrac{4}{61^s} + \cdots \end{aligned}$$

where p runs over primes $\equiv 1, 9$ (20) and $\tilde p$ over those $\equiv 3, 7, 11, 13, 17, 19$ (20).

Before we continue with the Fibonacci modules, let us briefly mention another example, $M = \mathbb{Z}[\sqrt{2}]\boldsymbol{e}_1 \oplus \mathbb{Z}[\sqrt{2}]\boldsymbol{e}_2 = \mathbb{Z}^2 \oplus \sqrt{2}\cdot\mathbb{Z}^2$. With $\xi = e^{2\pi i/8}$, one sees that M is the ring $\mathbb{Z}[i,\sqrt{2}\,] \subset \mathbb{Z}[\xi]$, and the field generated by M is $\mathbb{Q}(\xi)$. This time, although $\mathbb{Q}(\xi)$ has class number 1, M lies with index 2 in its ring of integers, $\mathbb{Z}[\xi]$. In fact, M is not integrally closed, nor is it a principal ideal domain. However, $\mathcal{L}^+(M) = (M^\bullet, \cdot)$, and the group of units, since $\xi \notin M$, turns out to be (with $\lambda = 1 + \sqrt{2}\,$)

$$\mathcal{L}_1^+(M) \;=\; \{ i^k \lambda^\ell \mid k = 0,1,2,3 \;\text{ and }\; \ell \in \mathbb{Z} \} \;\simeq\; C_4 \times C_\infty \,. \tag{7}$$

Our further analysis revolves around the prime 2, which splits (ignoring units) as $(1+\xi)^4$. However, the prime $(1+\xi)$ itself does not occur in M, and, in fact, $M \cap (1+\xi)\mathbb{Z}[\xi] = (1+\xi)^2\mathbb{Z}[\xi] = (1+i)\mathbb{Z}[\xi]$. Not all ideals of M are principal, but only these are similarity submodules. There is a one-to-one correspondence between odd principal ideals of M (i.e. those of odd index) and those of $\mathbb{Z}[\xi]$, given by $aM \leftrightarrow a\mathbb{Z}[\xi]$, $a \in M$. The even principal ideals of M require some care. There are none of index $2(2\ell+1)$, but twice as many of any other even index than there are in $\mathbb{Z}[\xi]$. In fact, if $\mathfrak{a} = aM$ and $(1+i) \mid a$, then $\tilde{\mathfrak{a}} = \xi aM$ is another ideal, but both are inside $a\mathbb{Z}[\xi] = \xi a\mathbb{Z}[\xi]$ (this is due to ξ being a unit of $\mathbb{Z}[\xi]$, but not of M). Now, we have $[M : aM] = [\mathbb{Z}[\xi] : a\mathbb{Z}[\xi]]$ for all $0 \neq a \in M$ (note that for $a \in M$, aM is never an ideal of $\mathbb{Z}[\xi]$).

So, for counting the principal ideals of M, we have to remove from the zeta function of $\mathbb{Q}(\xi)$ those terms that count ideals $\mathfrak{a}$ for which $N(\mathfrak{a})$ has the form $2(2l+1)$, and to count all other even ideals twice. We thus obtain:

$$\begin{aligned}\zeta_M(s) \;&=\; (1 - 2^{-s} + 2\cdot 4^{-s}) \cdot \zeta_{\mathbb{Q}(\xi)}(s) \qquad (8)\\ &=\; 1 + \tfrac{2}{4^s} + \tfrac{2}{8^s} + \tfrac{2}{9^s} + \tfrac{2}{16^s} + \tfrac{4}{17^s} + \tfrac{2}{25^s} + \tfrac{2}{32^s} + \tfrac{4}{36^s} + \tfrac{4}{41^s} + \tfrac{2}{49^s} + \tfrac{2}{64^s} + \tfrac{8}{68^s} + \ldots\end{aligned}$$

where $\zeta_{\mathbb{Q}(\xi)}(s)$ is given explicitly in Ref. 3.

5 An example in three dimensions

Let us finally treat the module $\mathcal{M}_c = \mathbb{Z}[\tau]^3$ which is closely related to the icosahedral modules [3,5]. To do so, we have to return to the original definition of (linear) similarities, i.e. to $f(x) = \alpha R x$ and the condition $f(\mathcal{M}_c) \subset \mathcal{M}_c$. In terms of the basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_3\}$, all entries of the matrix αR must be integral, i.e. in $\mathbb{Z}[\tau]$. With αR, also αR^t must be integral, and orthogonality of R implies $(\alpha R)(\alpha R^t) = \alpha^2 \mathbb{1}$, hence $\alpha^2 \in \mathbb{Z}[\tau]$. Together with $\det(\alpha R) = \alpha^3 \in \mathbb{Z}[\tau]$, this gives $\alpha \in \mathbb{Q}(\tau)$, and $\alpha^2 \in \mathbb{Z}[\tau]$ is then only possible for $\alpha \in \mathbb{Z}[\tau]$.

With this result, αR integral means $R \in \mathrm{O}(3, \mathbb{Q}(\tau))$ and $\alpha \in \mathbb{Z}[\tau]$ must be a $\mathbb{Z}[\tau]$-multiple of the denominator of R,

$$\mathrm{den}(R) \;:=\; \gcd\{\beta \in \mathbb{Z}[\tau]_+ \mid \beta R \text{ integral}\}\,, \qquad (9)$$

which is well defined, even as a number rather than an ideal, up to units, i.e. up to $\pm\tau^m$, $m \in \mathbb{Z}$, due to unique factorization in $\mathbb{Z}[\tau]$. Note that $\pm\tau^m \mathcal{M}_c = \mathcal{M}_c$.

With $\mathcal{R} := \{\mathrm{den}(R) \cdot R \mid R \in \mathrm{O}(3, \mathbb{Q}(\tau))\}$, we characterize $\mathcal{L}(\mathcal{M}_c)$ as

$$\mathcal{L}(\mathcal{M}_c) \;=\; (\{\mathbb{Z}[\tau]^\bullet \times \mathcal{R}\}, \cdot) \qquad (10)$$

and the group of units in it turns out to be

$$\mathcal{L}_1(\mathcal{M}_c) \;=\; \{\tau^m \mid m \in \mathbb{Z}\} \times \mathrm{O}(3, \mathbb{Z}) \;\simeq\; C_\infty \times ((C_2)^3 \times_s S_3)\,. \qquad (11)$$

Note that an analogous result holds in general for (hyper-)cubic Fibonacci modules in Euclidean spaces of odd dimension.

Finally, we have to count the similarity submodules of $\mathcal{M}_c$. For a mapping $f(x) = \alpha \cdot \mathrm{den}(R) \cdot Rx$, $\alpha \in \mathbb{Z}[\tau]$, we have the index formula

$$\mathrm{ind}(f) \;=\; [\mathcal{M}_c : f(\mathcal{M}_c)] \;=\; |\mathrm{N}[\alpha]^3 \cdot \mathrm{N}[\mathrm{den}(R)]^3| \, . \tag{12}$$

The solution of the coincidence problem [2] gives us the generating function for $1/24$ times the number of $\mathrm{SO}(3, \mathbb{Q}(\tau))$ matrices R with $|\mathrm{N}[\mathrm{den}(R)]| = m$,

$$\Phi_c(s) \;=\; \frac{1 + 4^{1-s}}{1 + 4^{-s}} \cdot \frac{\zeta_{\mathbb{Q}(\tau)}(s)\zeta_{\mathbb{Q}(\tau)}(s-1)}{\zeta_{\mathbb{Q}(\tau)}(2s)} \, , \tag{13}$$

with the zeta-function $\zeta_{\mathbb{Q}(\tau)}(s)$ as given in Eq. (4).

But then, since precisely 24 different $\mathrm{SO}(3, \mathbb{Q}(\tau))$ matrices give rise to the same similarity submodule, we finally get the following Dirichlet series generating function for the number of similarity submodules of given index

$$\begin{aligned} F_{\mathcal{M}_c}(s) \;&=\; \zeta_{\mathbb{Q}(\tau)}(3s)\Phi_c(3s) \qquad (14) \\ &=\; 1 + \tfrac{9}{64^s} + \tfrac{7}{125^s} + \tfrac{11}{729^s} + \tfrac{26}{1331^s} + \tfrac{41}{4096^s} + \tfrac{42}{6859^s} + \tfrac{63}{8000^s} + \tfrac{37}{15625^s} + \tfrac{62}{24389^s} + \cdots \end{aligned}$$

Here, the factor $\zeta_{\mathbb{Q}(\tau)}(3s)$ takes care of the freedom to go from a maximal submodule M to a scaled copy, αM, with $\alpha \in \mathbb{Z}[\tau]$. Note that similar arguments apply to $\mathbb{Z}[\tau]$-extensions of face-centred and body-centred cubic lattices.

6 Concluding remarks

With the methods and examples presented here, which apply mainly to modulated structures, and with the explicit results of Refs. 3 and 5, the structure of invariant submodules and their semigroups of self-similarities should be transparent enough so that other examples can be worked out along the same lines.

Among the applications is the determination of the possible colourings of a given module if one demands that one colour occupies an invariant submodule and the others code the cosets, see Refs. 3,4,7,8,11 for examples.

An extension to four dimensions, covering the 4D cubic lattices and the Elser-Sloane quasicrystal, is also possible due to their relation to maximal orders in the skew-field of quaternions, such as Hurwitz' ring of integral quaternions or the icosian ring [9,2]. This will be reported separately.

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