DYNAMICAL SYSTEMS ON TRANSLATION BOUNDED MEASURES: PURE POINT DYNAMICAL AND DIFFRACTION SPECTRA

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Abstract. Certain topological dynamical systems are considered that arise from actions of σ-compact locally compact Abelian groups on compact spaces of translation bounded measures. Such a measure dynamical system is shown to have pure point dynamical spectrum if and only if its diffraction spectrum is pure point.

1. Introduction

This paper deals with certain dynamical systems build from measures on σ-compact locally compact Abelian groups. These dynamical systems give rise to two spectra: the dynamical spectrum and the diffraction spectrum. After introducing the dynamical systems and discussing their basic topological features, we will focus on studying the relationship between these two spectra. Particular attention will be paid to the case where one of the spectra is pure point. This will be shown to happen if and only if the other is pure point as well (read on for details and a discussion of related results.)

The motivation for our study comes from physics and, more precisely, from the study of solids with long range aperiodic order and crystal-like diffraction spectrum. Such solids are known as genuine quasicrystals. The existence of quasicrystals is now a well-established and widely accepted experimental fact. Even if discussions about the precise structures will still be going on for a while, the common feature of aperiodicity has opened a new chapter of crystallography and solid state research.

The original discovery of quasicrystals [34, 18] was somewhat accidental and only possible through one of their most striking features, namely their sharp Bragg diffraction with point symmetries that are not possible for 3-dimensional crystals (such as n-fold rotation axes with $n = 8, 10, 12$, or icosahedral symmetry); see [1] for a summary and [2] for a guide to the literature. These experimental results called for a mathematical explanation and created a subject now often referred to as mathematical diffraction theory.

Mathematical diffraction theory is concerned with the Fourier transform of the autocorrelation measure (or Patterson measure) of a given translation bounded (possibly complex) measure ω. Here, ω is the mathematical idealisation of the physical structure of a solid or, more generally, of any state of matter. In its simplest form, ω is just a Dirac comb, i.e., a collection of (possibly weighted) point measures which mimic the positions of the atoms (and their scattering strengths). The autocorrelation measure γ_{ω} of ω (see below for a precise definition) is then a positive definite measure. Its Fourier transform $\widehat{\gamma_{\omega}}$ is a positive measure, called the diffraction measure, which models the outcome of a diffraction experiment; see [10] for background material and details on the physical justification of this approach.

Now, given this setting, one of the most obvious questions to address is that for (the characterisation of all) examples of measures ω with a diffraction measure $\widehat{\gamma_\omega}$ that is pure point, i.e., consists of point measures only.

This was addressed by Bombieri and Taylor in [8]. However, no rigorous answer could be given at that time. Soon after, Hof [16] showed that structures obtained from the cut-and-project formalism [20] possess a pure point diffraction spectrum under rather decent assumptions, and Solomyak started a rather systematic study of substitution dynamical systems with pure point spectrum in [36]. By now, large classes of examples are known [5, 21], also beyond the class of ordinary projection sets [6]. Moreover, Schlottmann was able to free the cut-and-project formalism from basically all special properties of Euclidean space [32] and to establish that all regular model sets are pure point diffractive [33], see also [25] for a summary on model sets.

A cornerstone in many of these considerations is the use of ergodic theory and the so-called Dworkin argument [11] (see [17, 33] as well). This argument links the diffraction spectrum to the dynamical spectrum. It can be used to infer pure point diffraction spectrum from pure point dynamical spectrum. These investigations heavily depend on the underlying point sets being point sets of finite local complexity (FLC). This, however, is not necessary, as becomes clear from two alternative approaches, one for general Dirac combs and measures on the basis of almost periodicity by Moody and one of the authors [5], and the other for so-called deformed model sets by Bernuau and Duneau [9].

Thus, at the moment, there is a considerable gap between the cases that can be treated by the method of almost periodicity of measures [13, 5] and those using ergodic theory and requiring FLC together with unique ergodicity. It is the primary aim of this article to narrow this gap. This will be achieved by thoroughly analysing the link between the diffraction spectrum and the dynamical spectrum given by the Dworkin argument.

The analysis carried out below will also be a crucial ingredient in a forthcoming paper of ours [4] which investigates the stability of pure point diffraction. Namely, we will set up a perturbation theory for pure point diffraction by studying deformations of dynamical systems with pure point dynamical spectrum. Particular emphasis will be put on deformed model sets and iso-spectral deformations of Delone sets. Note that the deformation of model sets almost immediately leads to point sets which violate FLC.

Let us now discuss our results in more detail. The first step in our approach is to choose a setting of measures rather than point sets. Defining appropriate dynamical systems with measures on a locally compact Abelian group will free us from essentially all restrictions mentioned. This setting is presented in Section 3, where also the relevant topological questions are discussed. The relationship between our measure dynamical systems and point dynamical systems is investigated in Section 4. It is shown that the measure dynamical systems enclose the usual Delone dynamical systems. This section introduces a topology on Delone sets (and actually all closed subsets of the group) with very nice compactness properties. The results generalise and strengthen the corresponding considerations of [33, 22] and may be of independent interest in further studies of point sets not satisfying FLC. The extension of diffraction theory and the Dworkin argument (as developed for point sets in, e.g., [11, 33]) to our setting is achieved in Section 5.

The main result of our paper is summarised in Theorem 7 in Section 7. It states that, under some rather mild assumptions,

- pure point dynamical spectrum is equivalent to pure point diffraction spectrum.

This generalises the main results of Lee, Moody and Solomyak [21] in at least three ways: It is not restricted to dynamical systems arising from Delone sets, and in fact not even to dynamical systems arising from point sets. It does not need any condition of finite local complexity. It does not need an ergodicity assumption for the invariant measure involved.

Let us mention that a generalisation of the main result of [21] sharing the last two features had already been announced by Gouéré [14, 15], within the framework of point processes and Palm measures, see his recent work [14] for a study of this framework as well. His result applies to the point dynamical systems studied below in Section 4. Thus, there is some overlap between his result an ours. However, in general, our setting, methods and results are quite different from his, as we leave the scenario of point sets. In fact, the measure theoretical setting seems very adequate and natural in view of the physical applications, where point sets are only a somewhat crude approximation of the arrangements of scatteres.

Let us also mention that, in general, the diffraction spectrum and the dynamical spectrum can be of different type, as has been investigated by van Enter and Międkisz in [12].

Our proof of the equivalence of the two notions of pure pointedness relies on two results which are of interest in their own right. These results are

- an abstract characterisation of pure point dynamical spectrum for arbitrary topological dynamical systems,
- a precise interpretation of the Dworkin argument.

Here, the abstract characterisation is achieved in Theorem 1 in Section 2. Roughly speaking, it states that a system has pure point spectrum once it has a lot of point spectrum. The precise interpretation of the Dworkin argument is given in Theorem 6 in Section 6. It says that the diffraction measure is a spectral measure for a suitable subrepresentation of the translation action at hand.

The relationship between suitable subrepresentation with pure point spectrum and the original representation can actually be analysed in some more detail. To do so, we take a second look at the abstract theory in Section 8. Namely, we discuss how the group of all eigenvalues and the continuity of eigenfunctions is already determined by the set of eigenvalues and continuity of eigenfunctions asssociated to a suitable subrepresentation with pure point spectrum.

This material is rather general and may be of independent interest. Here, we apply it to our topological measure dynamical systems. This gives a criterion for the continuity of the eigenfunctions. More importantly, it shows that the group of all eigenvalues is generated by the support of the diffraction spectrum. The validity of such a result was brought to our attention by R. V. Moody for the case of point dynamical systems satisfying FLC [26].

The material presented above, and the abstract strategy to prove our main result, can be adopted to study a measurable framework (as opposed to a topological one). This will be analysed in the future.

2. An abstract criterion

In this section, we introduce some notation and provide a simple result which lies at the heart of our considerations. It is rather general and might be useful in other situations as well.

Let $\varOmega$ be a compact topological space (by which we mean to include the Hausdorff property) and G be a locally compact Abelian (LCA) group which is σ-compact. Let

$$\alpha\colon G \times \varOmega \longrightarrow \varOmega \tag{1}$$

be a continuous action of G on $\varOmega$, where, of course, $G \times \varOmega$ carries the product topology (later on, we will specify it via $\alpha_t(P) = t + P$ for $P \subset G$ and $t \in G$). Then, $(\varOmega, \alpha)$ is called a *topological dynamical system*. The set of continuous functions on $\varOmega$ will be denoted by $C(\varOmega)$. Let m be a G-invariant probability measure on $\varOmega$ and denote the corresponding set of square integrable functions on $\varOmega$ by $L^2(\varOmega, m)$. This space is equipped with the inner product $\langle f, g\rangle := \int \overline{f(\omega)} g(\omega) \,\mathrm{d}m(\omega)$. The action α induces a unitary representation T of G on $L^2(\varOmega, m)$ in the obvious way, namely $T^t h$ is given by $(T^t h)(\omega) := h(\alpha_{-t}(\omega))$. Whenever we want to emphasise the dependence of the inner product and the unitary representation on the chosen invariant measure m, we write $\langle f, g\rangle_m$ instead of $\langle f, g\rangle$ and T_m rather than T.

The dual group of G is denoted by $\widehat{G}$, and the pairing between a character $\hat{s} \in \widehat{G}$ and an element $t \in G$ is written as $(\hat{s}, t)$, which, of course, is a number on the unit circle, compare [30, Ch. 4] for background material.

A non-zero $h \in L^2(\varOmega, m)$ is called an *eigenvector* (or eigenfunction) of T if there exists an $\hat{s} \in \widehat{G}$ with $T^t h = (\hat{s}, t) h$ for every $t \in G$. The closure (in $L^2(\varOmega, m)$) of the linear span of all eigenfunctions of T will be denoted by $\mathcal{H}_{pp}(T)$.

The following is a variant (and an extension) of a result from [21].

Lemma 1. *Let $(\varOmega, \alpha)$ be a topological dynamical system with an invariant measure m. Then, $\mathcal{H}_{pp}(T) \cap C(\varOmega)$ is a subalgebra of $C(\varOmega)$ which is closed under complex conjugation and contains all constant functions. Similarly, $\mathcal{H}_{pp}(T) \cap L^\infty(\varOmega, m)$ is a subalgebra of $L^\infty(\varOmega, m)$ that is closed under complex conjugation and contains all constant functions.*

Proof. We only show the statement about $\mathcal{H}_{pp}(T) \cap C(\varOmega)$. The other result can be shown in the same way.

The set $\mathcal{H}_{pp}(T) \cap C(\varOmega)$ is a vector space because it is the intersection of two vector spaces. Moreover, every constant (non-vanishing) function is obviously continuous and an eigenvector of T (with eigenvalue 1, i.e., with the trivial character $(\hat{s}, t) \equiv 1$).

It remains to be shown that $\mathcal{H}_{pp}(T) \cap C(\varOmega)$ is closed under complex conjugation and under forming products.

Closedness under complex conjugation: Let f be an eigenfunction of T to, say, $\hat{s}$. Then, $\overline{f}$ is an eigenfunction of T to the character $\hat{s}^{-1}$. Here, of course, the inverse $\hat{s}^{-1}$ of $\hat{s} \in \widehat{G}$ is given by $t \mapsto \overline{(\hat{s}, t)}$, where the bar denotes complex conjugation. Using this, it is not hard to see that $\mathcal{H}_{pp}(T)$ is closed under complex conjugation. As this is true of $C(\varOmega)$ as well, we see that the intersection $\mathcal{H}_{pp}(T) \cap C(\varOmega)$ is closed under complex conjugation.

Closedness under products: This is shown in Lemma 3.7 in [21] in the case that m is not only translation invariant but also ergodic. To adopt their argument to the case at hand, we note that every eigenfunction can be approximated arbitrarily well (in $L^2(\varOmega, m)$) by bounded eigenfunctions via a simple cut-off procedure. More precisely, if f is an eigenfunction, then $|f|$ is an α-invariant function. Therefore, for an arbitrary $N > 0$, the function

$$f^N(\omega) \; := \; \begin{cases} f(\omega), & |f(\omega)| \leq N \\ 0\,, & \text{otherwise} \end{cases} \tag{2}$$

is again an eigenfunction (with the same $\hat{s}$ as f). Apparently, the f^N converge to f in $L^2(\Omega, m)$ as $N \to \infty$.

After this preliminary consideration, we can conclude the proof following [21]: Let two functions $f, g \in \mathcal{H}_{pp}(T) \cap C(\Omega)$ be given. Then, fg belongs to $C(\Omega)$. It remains to be shown that it belongs to $\mathcal{H}_{pp}(T)$ as well.

Choose $\varepsilon > 0$ arbitrarily. Observe that $\|g\|_\infty < \infty$, as $g \in C(\Omega)$ with Ω compact. Since f is in $\mathcal{H}_{pp}(T)$, there exists a finite linear combination $f' = \sum a_i f_i$ of eigenfunctions of T with

$$\|f - \sum a_i f_i\|_2 \;\leq\; \frac{\varepsilon}{\|g\|_\infty}\,.$$

By the preliminary consideration around Eq. (2), we can assume that all f_i are bounded functions. Thus, in particular, $\|f'\|_\infty < \infty$.

Similarly, choose another finite linear combination $g' = \sum b_j g_j$ of bounded functions g_j in $\mathcal{H}_{pp}(T)$ with

$$\|g - \sum b_j g_j\|_2 \;\leq\; \frac{\varepsilon}{\|f'\|_\infty}\,.$$

Then,

$$\|fg - f'g'\|_2 \;\leq\; \|f'\|_\infty \, \|g - g'\|_2 + \|g\|_\infty \, \|f - f'\|_2 \;\leq\; 2\varepsilon.$$

The proof is complete by observing that $f'g'$ is in $\mathcal{H}_{pp}(T)$ because the product of bounded eigenfunctions is again a bounded eigenfunction. □

With Lemma 1, the following result is a rather direct consequence of the Stone-Weierstrass Theorem.

Theorem 1. *Let (Ω, α) be a topological dynamical system with invariant probability measure m. Then, the following assertions are equivalent.*

(a) *T has pure point spectrum, i.e., $\mathcal{H}_{pp}(T) = L^2(\Omega, m)$.*

(b) *There exists a subspace $\mathcal{V} \subset \mathcal{H}_{pp}(T) \cap C(\Omega)$ which separates points.*

Proof. (a) $\Longrightarrow$ (b): This is clear, as one can take $\mathcal{V} = \mathcal{H}_{pp}(T) \cap C(\Omega) = C(\Omega)$.
(b) $\Longrightarrow$ (a): By Lemma 1, $\mathcal{H}_{pp}(T) \cap C(\Omega)$ is an algebra which is closed under complex conjugation and contains the constant functions. It also separates points as it contains a subspace, $\mathcal{V}$, with this property by (b). Thus, we can apply the Stone-Weierstrass Theorem (compare [27, Thm. 4.3.4]), to conclude that $\mathcal{H}_{pp}(T) \cap C(\Omega)$ is dense in $C(\Omega)$. By standard measure theory, see [27, Prop. 6.4.11], $\mathcal{H}_{pp}(T)$ is then dense in $L^2(\Omega, m)$ as well. As $\mathcal{H}_{pp}(T)$ is closed, statement (a) follows. □

3. Measure dynamical systems

For the remainder of the paper, let G be a fixed σ-compact LCA group with identity 0. Integration with respect to Haar measure is denoted by $\int_G \ldots \,\mathrm{d}t$, and the measure of a subset D of G is denoted by $|D|$. The vector space of complex valued continuous functions on G with compact support is denoted by $C_c(G)$. It is made into a locally convex space by the inductive limit topology, as induced by the canonical embeddings

$$C_K(G) \;\hookrightarrow\; C_c(G)\,, \quad K \subset G \text{ compact.}$$

Here, $C_K(G)$ is the space of complex valued continuous functions on G with support in K, which is equipped with the usual supremum norm $\|.\|_\infty$. The support of $\varphi \in C_c(G)$ is denoted by $\mathrm{supp}(\varphi)$.

The dual $C_c(G)^*$ of the locally convex space $C_c(G)$ is denoted by $\mathcal{M}(G)$. The space $\mathcal{M}(G)$ carries the vague topology. This topology equals the weak-$*$ topology of $C_c(G)^*$, i.e., it is the weakest topology which makes all functionals $\mu \mapsto \mu(\varphi)$, $\varphi \in C_c(G)$, continuous. The definition of the inductive limit topology and a direct calculation yield

$$\sup\{|\mu(\psi)| : |\psi| \leq \varphi\} \; < \; \infty$$

for every $\mu \in \mathcal{M}(G)$ and all $\varphi \in C_c(G)_+ := \{\varphi \in C_c(G) : \varphi(t) \geq 0 \text{ for all } t \in G\}$. Therefore, [27, Thm. 6.5.6] together with its proof shows that the mapping

$$C_c(G)_+ \longrightarrow \mathbb{R}\,, \quad \varphi \mapsto \sup\{|\mu(\psi)| : \psi \in C_c(G,\mathbb{R}) \text{ with } |\psi| \leq \varphi\}$$

can be extended to a linear functional on $C_c(G,\mathbb{R}) := \{\varphi \in C_c(G) : \varphi(t) \in \mathbb{R} \text{ for all } t \in G\}$, which is uniquely determined. This functional can further be extended uniquely to a linear functional on $C_c(G) = C_c(G,\mathbb{C})$. This functional is denoted by $|\mu|$ and called the *total variation* of μ.

The total variation $|\mu|$ is positive, i.e., $|\mu|(\varphi) \geq 0$ for all $\varphi \in C_c(G)_+$. In particular, by [27, Thm. 6.3.4], $|\mu|$ can be identified with a measure on the σ-algebra of Borel sets of G that satisfies

- $|\mu|(K) < \infty$ for every compact set $K \subset G$,
- $|\mu|(A) = \sup\{|\mu|(K) : K \subset A, K \text{ compact}\}$ for every Borel set $A \subset G$.

As G is σ-compact, we furthermore have

- $|\mu|(A) = \inf\{|\mu|(B) : A \subset B, B \text{ open}\}$ for every Borel set $A \subset G$

by [27, Prop. 6.3.6], i.e., $|\mu|$ is an (unbounded) regular Borel measure, in line with the Riesz-Markov Theorem [29, Thm. IV.18].

Finally, we note that there exists, by [27, Thm. 6.5.6], a measurable function $u \colon G \longrightarrow \mathbb{C}$ with $|u(t)| = 1$ for $|\mu|$-almost every $t \in G$ such that

$$\mu(\varphi) \; = \; \int_G \varphi \, u \, \mathrm{d}|\mu| \qquad \text{for all } \varphi \in C_c(G). \tag{3}$$

This polar decomposition permits us to identify the elements of $\mathcal{M}(G)$ with the regular complex Borel measures on G, which is the Riesz-Markov Theorem for this situation.

For later use, we also introduce some notation concerning Fourier transforms and convolutions, compare [31, 7] for details. The Fourier transform of a quantity q will always be denoted by $\widehat{q}$. For $\varphi, \psi \in C_c(G)$, we define the convolution $\varphi * \psi$ by $\big(\varphi * \psi\big)(t) := \int_G \varphi(s)\psi(t-s)\,\mathrm{d}s$ and the function $\widetilde{\varphi} \in C_c(G)$ by $\widetilde{\varphi}(t) := \overline{\varphi(-t)}$. For $\mu \in \mathcal{M}(G)$ and $\varphi \in C_c(G)$, the convolution $\varphi * \mu$ is the function given by $\big(\varphi * \mu\big)(t) := \int_G \varphi(t-s)\,\mathrm{d}\mu(s)$. For two convolvable measures $\mu, \nu \in \mathcal{M}(G)$, the convolution $\mu * \nu$ is the element of $\mathcal{M}(G)$ given by $\big(\mu * \nu\big)(\varphi) := \int_G \int_G \varphi(s+t)\,\mathrm{d}\mu(s)\,\mathrm{d}\nu(t)$ for $\varphi \in C_c(G)$; the measures $\widetilde{\mu}$ and $\overline{\mu}$ are defined by $\widetilde{\mu}(\varphi) := \overline{\mu(\widetilde{\varphi})}$ and $\overline{\mu}(\varphi) := \overline{\mu(\overline{\varphi})}$, respectively. For $\mu \in \mathcal{M}(G)$ and a measurable set $B \subset G$, we denote the restriction of μ to B by μ_B. Finally, for $x \in G$, we define the measure δ_x to be the normalised point measure at x.

We will consider actions of G on spaces consisting of measures on G. The relevant set of measures will be defined next.

Definition 1. *Let $C > 0$ and a relatively compact open set V in G be given. A measure $\mu \in \mathcal{M}(G)$ is called (C,V)-translation bounded if $|\mu|(t+V) \leq C$ for all $t \in G$. It is simply called translation bounded if there exist C, V such that it is (C,V)-translation bounded. The* *set of all (C,V)-translation bounded measures is denoted by $\mathcal{M}_{C,V}(G)$ and the set of all translation bounded measures by $\mathcal{M}^\infty(G)$.*

The vague topology on $\mathcal{M}(G)$ has very nice features when restricted to the translation bounded measures.

Theorem 2. *Let $C > 0$ and a relatively compact open set V in G be given. Then, $\mathcal{M}_{C,V}(G)$ is a compact Hausdorff space. If G is second countable, $\mathcal{M}_{C,V}(G)$ is metrisable.*

To prove the theorem, we start with the following simple result from measure theory.

Proposition 1. *Let $\mu \in \mathcal{M}(G)$ and a relatively compact open set V in G be given. Then, $|\mu|(V) = \sup\{|\mu(\varphi)| : \varphi \in C_c(G)$ with $\operatorname{supp}(\varphi) \subset V$ and $\|\varphi\|_\infty \le 1\}$.*

Proof. Denote the supremum in the statement by S. The polar decomposition of Eq. (3) gives

$$|\mu(\varphi)| \;=\; \Big| \int_G \varphi\, u \,\mathrm{d}|\mu| \Big| \;\le\; \int_G |\varphi|\, |u| \,\mathrm{d}|\mu| \;=\; \int_V |\varphi| \,\mathrm{d}|\mu| \;\le\; |\mu|(V)$$

for every $\varphi \in C_c(G)$ with $\operatorname{supp}(\varphi) \subset V$ and $\|\varphi\|_\infty \le 1$, hence $S \le |\mu|(V)$ is obvious.

Conversely, let $|\mu|_V$ denote the restriction of $|\mu|$ to V. Then, $C_c(V)$ is dense in $L^1(V, |\mu|_V)$ by standard arguments, compare [27, Prop. 6.4.11]. In particular, there exists a sequence (φ_n) in $C_c(V)$ with

$$\varphi_n \xrightarrow{n\to\infty} \overline{u} \cdot 1_V \qquad \text{in } L^1(V, |\mu|_V)\,.$$

We may replace (φ_n) by a suitable subsequence, still called (φ_n) for simplicity, which converges also pointwise, for $|\mu|_V$-almost all $t \in V$. Defining the continuous cut-off function $h\colon \mathbb{C} \longrightarrow \mathbb{C}$ by $h(z) = z/\max(1, |z|)$, one then sees that the sequence $(h \circ \varphi_n)$ converges pointwise almost everywhere to $\overline{u} \cdot 1_V$ and consists of uniformly bounded functions. Hence it converges also in $L^1(V, |\mu|_V)$ to $\overline{u} \cdot 1_V$. This means that we may assume $\|\varphi_n\|_\infty \le 1$ without loss of generality.

Clearly, since u is bounded, we also have $u\varphi_n \xrightarrow{n\to\infty} u\overline{u} \cdot 1_V$ in $L^1(V, |\mu|_V)$. So, one obtains

$$\begin{aligned} S \;&\ge\; \lim_{n\to\infty} |\mu(\varphi_n)| \;=\; \lim_{n\to\infty} \Big| \int_V \varphi_n\, u \,\mathrm{d}|\mu| \Big| \;=\; \Big| \int_G |u|^2\, 1_V \,\mathrm{d}|\mu| \Big| \\ &=\; \Big| \int_G 1_V \,\mathrm{d}|\mu| \Big| \;=\; \big| |\mu|(V) \big| \;=\; |\mu|(V) \end{aligned}$$

which proves the proposition. □

Proof of Theorem 2. By definition of $\mathcal{M}_{C,V}(G)$, for each $\varphi \in C_c(G)$, there exists a radius $R(\varphi) > 0$ such that $\mu(\varphi) \in \overline{B_{R(\varphi)}}$ for every $\mu \in \mathcal{M}_{C,V}(G)$, where B_r is the (open) ball of radius r around 0 and $\overline{B_r}$ its closure. Thus, we can consider $\mathcal{M}_{C,V}(G)$ as a subspace of $\varPi := \prod_{\varphi \in C_c(G)} \overline{B_{R(\varphi)}}$ equipped with the product topology via the embedding

$$j\colon \mathcal{M}_{C,V}(G) \hookrightarrow \varPi\,, \quad \big(j(\mu)\big)(\varphi) := \mu(\varphi).$$

As $\varPi$ is obviously a compact Hausdorff space, this shows immediately that $\mathcal{M}_{C,V}(G)$ is relatively compact and Hausdorff. It remains to be shown that $j(\mathcal{M}_{C,V}(G))$ is closed. This is a direct consequence of Definition 1 together with Proposition 1.

The statement about metrisability is standard: if G is second countable, there exists a countable dense subset $\{\varphi_n : n \in \mathbb{N}\}$ in $C_c(G)$. Then,

$$d\colon \mathcal{M}_{C,V}(G) \times \mathcal{M}_{C,V}(G) \longrightarrow \mathbb{R}, \quad d(\mu,\nu) \;:=\; \sum_{n\in\mathbb{N}} 2^{-n} \frac{|\mu(\varphi_n) - \nu(\varphi_n)|}{1 + |\mu(\varphi_n) - \nu(\varphi_n)|}$$

gives a metric on $\mathcal{M}_{C,V}(G)$ which generates the topology. $\square$

Having discussed the topology of $\mathcal{M}(G)$, we can now introduce the topological dynamical systems associated to subsets of $\mathcal{M}(G)$. To do so, we will use the obvious action α of G on $\mathcal{M}(G)$ given by

$$\alpha\colon\, G \times \mathcal{M}(G) \longrightarrow \mathcal{M}(G)\,, \quad (t,\mu) \mapsto \alpha_t(\mu) := \delta_t * \mu,$$

or, more explicitly, $\big(\alpha_t(\mu)\big)(\varphi) = \int_G \varphi(t+s)\,\mathrm{d}\mu(s)$. We use the same symbol for the action as in Eq. (1), since misunderstandings are unlikely, and we will usually write $\alpha_t\mu$ for $\alpha_t(\mu)$.

This action is compatible with the topological structure of G and $\mathcal{M}(G)$.

Proposition 2. *Let $C > 0$ and a relatively compact open set $V \subset G$ be given. Then, the action $\alpha\colon G\times\mathcal{M}_{C,V}(G) \longrightarrow \mathcal{M}_{C,V}(G)$ is continuous, where $G\times\mathcal{M}_{C,V}(G)$ carries the product topology.*

Proof. Let (t_ι,μ_ι) be a net in $G \times \mathcal{M}_{C,V}(G)$ converging to (t,μ). We have to show that the net $(\alpha_{t_\iota}(\mu_\iota))$ converges to $\alpha_t(\mu)$, i.e., we have to check that

$$\int \varphi(s+t_\iota)\,\mathrm{d}\mu_\iota(s) \;\longrightarrow\; \int \varphi(s+t)\,\mathrm{d}\mu(s)\,, \quad \text{for all } \varphi \in C_c(G).$$

By $t_\iota \longrightarrow t$, there exists an index ι_0 and a compact set K such that the support of φ and the supports of all φ_ι with $\iota \geq \iota_0$ are contained in K. Moreover, as $\mu_\iota \in \mathcal{M}_{C,V}(G)$ for every ι, there exists a constant C' with $|\mu|(K) \leq C'$ as well as $|\mu_\iota|(K) \leq C'$ for all ι. Now, the desired statement follows easily from

$$\begin{aligned} |\mu_\iota(\varphi(.+t_\iota)) - \mu(\varphi(.+t))| \;&\leq\; |\mu_\iota(\varphi(.+t_\iota)) - \mu_\iota(\varphi(.+t))| + |\mu_\iota(\varphi(.+t)) - \mu(\varphi(.+t))| \\ &\leq\; |\mu_\iota|(C)\|\varphi(.+t_\iota) - \varphi(.+t)\|_\infty + |\mu_\iota(\varphi(.+t)) - \mu(\varphi(.+t))|, \end{aligned}$$

as both terms on the right hand side tend to zero for $t_\iota \longrightarrow t$ and $\mu_\iota \longrightarrow \mu$. $\square$

The dynamical systems we are interested in are defined as follows.

Definition 2. (Ω,α) *is called a dynamical system on the translation bounded measures on* G (TMDS) *if there exist a constant $C > 0$ and a relatively compact open $V \subset G$ such that Ω is a closed subset of $\mathcal{M}_{C,V}(G)$ that is invariant under the G-action α.*

Remarks. (a) The α invariant subsets of $\mathcal{M}(G)$ are called *translation stable sets* in [13]. Thus, a TDMS is just a closed translation stable subset of $\mathcal{M}_{C,V}(G)$.
(b) The space Ω of a TMDS is always compact by Theorem 2 and the action α is continuous by Proposition 2. Thus, a TMDS is a topological dynamical system in the sense of Section 2.
(c) The considerations of this section (and those of the next) do not use commutativity of the underlying group G. They immediately extend to arbitrary σ-compact locally compact groups. But since we need harmonic analysis later on, we stick to Abelian groups in this paper.

4. Point dynamical systems

This section has two aims. Firstly, we present an abstract topological framework which allows us to treat point dynamical systems which are not of finite local complexity. Secondly, we show how these systems fit into our setting of measure dynamical systems. As for the first aim, we actually introduce a topology on the set of all closed subsets of G. For the case of

$\mathbb{R}^d$, this topology has already been studied by Stollmann and one of the authors in [22]. Our extension to arbitrary locally compact groups is strongly influenced by the investigation of Schlottmann [33] (which, however, is restricted to FLC systems).

We start by defining the relevant sets of points.

Definition 3. *Let G be a σ-compact* LCA *group, and V an open neighbourhood of 0 in G.*

(a) *A subset Λ of G is called V-discrete if every translate of V contains at most one point of Λ. The set of all V-discrete subsets of G is denoted by $\mathcal{D}_V(G)$.*
(b) *A subset of G is called uniformly discrete if it is V-discrete for some V. The set of all uniformly discrete subsets of G is denoted as $\mathcal{UD}(G)$.*
(c) *The set of all discrete and closed subsets of G will be denoted by $\mathcal{D}(G)$.*
(d) *The set of all closed subsets of G is denoted as $\mathcal{C}(G)$.*
(e) *A subset Λ of G is called relatively dense if there is a compact K with $G \subset \Lambda + K$.*

Note that a uniformly discrete subset of G is closed. As it presents no extra difficulty, we will actually topologise not only $\mathcal{D}(G)$ but rather the larger set $\mathcal{C}(G)$. This will be done by providing a suitable uniformity (see [19, Ch. 6] for details on uniformities). Namely, for $K \subset G$ compact and V a neighbourhood of 0 in G, we set

$$U_{K,V} \;:=\; \{(P_1,P_2) \in \mathcal{C}(G) \times \mathcal{C}(G) : P_1 \cap K \subset P_2 + V \text{ and } P_2 \cap K \subset P_1 + V\}.$$

It is not hard to check that

$$(P,P) \in U_{K,V},\;\; U_{K,V} = U_{K,V}^{-1},\;\; U_{K_1\cup K_2, V_1\cap V_2} \subset U_{K_1,V_1} \cap U_{K_2,V_2},\;\; U_{K-W,W} \circ U_{K-W,W} \subset U_{K,V}$$

for V a neighbourhood of 0, W a compact neighbourhood of 0 with $W + W \subset V$, K in G compact, and P any closed subset of G. Here, on sets U, U_1, U_2 consisting of ordered pairs, we define $U^{-1} := \{(y,x) : (x,y) \in U\}$ and

$$U_1 \circ U_2 \;:=\; \{(x,z) : \exists\, y \in G \text{ with } (x,y) \in U_1 \text{ and } (y,z) \in U_2\}.$$

This guarantees that $\{U_{K,V} : K \text{ compact}, V \text{ open with } 0 \in V\}$ generates a uniformity, and hence a topology on $\mathcal{C}(G)$ via the neighbourhoods

$$U_{K,V}(P) \;:=\; \{Q : (Q,P) \in U_{K,V}\}\,, \quad P \in \mathcal{C}(G)\,.$$

Note that we could equally well generate the same uniformity with V running through compact neighbourhoods of $0 \in G$.

Definition 4. *The topology defined above is called the local rubber topology* (LRT).

This topology essentially means that two sets P_1, P_2 are close if they "almost" agree on large compact sets.

Fundamental properties of the LRT are given in the following result (see [22] for an earlier result on $\mathbb{R}^d$).

Theorem 3. *With the* LRT, *the set $\mathcal{C}(G)$ of closed subsets of G is a compact Hausdorff space. If the topology of G has a countable base then $\mathcal{C}(G)$ is metrizable.*

Proof. The set $\mathcal{C}(G)$ is Hausdorff, as the intersection of all $U_{K,V}$ contains only the diagonal $\{(P,P) : P \in \mathcal{C}(G)\}$, see [19, Ch. 6].

We next show completeness: Let $(P_\iota)_{\iota \in I}$ be a Cauchy net in $\mathcal{C}(G)$, where I is an index set directed by $\leq$, compare [19, Ch. 2].

We have to show that the Cauchy net converges to a closed subset of G, hence an element of $\mathcal{C}(G)$. To this end, we introduce the set P of those $x \in G$ such that, for every neighbourhood V of 0, there exists $\iota_{x,V} \in I$ with

$$(x+V) \cap P_\iota \neq \varnothing \quad \text{for all } \iota \geq \iota_{x,V}. \tag{4}$$

This P will turn out to be the set of limit points of our Cauchy net.

We first show that P is closed: Let (y_κ) be a net in P converging to $y \in G$ and let V be an arbitrary neighboorhood of 0. Then, there exists a neighbourhood W of 0 with $W = -W$ and $W + W \subset V$. Since $y_\kappa \longrightarrow y$, there exists a κ' with $y_{\kappa'} \in y + W$. As $y_{\kappa'} \in P$, we have $(y_{\kappa'} + W) \cap P_\iota \neq \varnothing$ for all $\iota \geq \iota_{y_{\kappa'},V}$. Putting this together, we infer $(y+V) \cap P_\iota \neq \varnothing$ for all $\iota \geq \iota_{y_{\kappa'},V}$. Thus, y belongs indeed to the set P as defined by (4).

Next, let a compact K in G and a neighbourhood V of 0 be given. We have to provide a $\iota_{K,V}$ with $P \cap K \subset P_\iota + V$ for all $\iota \geq \iota_{K,V}$. Let W be a symmetric neighbourhood of 0 with $W + W \subset V$. As P is closed, the set $P \cap K$ is compact. Thus, there exists a finite set $D \subset P$, with $P \cap K \subset D + W$. As D is finite and belongs to P, there exists an index ι_D, with $D \subset P_\iota + W$ for every $\iota \geq \iota_D$. Then,

$$P \cap K \ \subset \ D + W \ \subset \ P_\iota + W + W \ \subset \ P_\iota + V$$

for every $\iota \geq \iota_D$. Thus, with $\iota_{K,V} = \iota_D$, the desired statement holds.

To find the needed converse, we start with an intermediate result.

Claim. *Let $C \subset G$ be compact. If $P_\kappa \cap C \neq \varnothing$ for all $\kappa \geq \kappa_0$ and some $\kappa_0 \in I$, then also $P \cap C \neq \varnothing$.*

Proof of the Claim. Assume the opposite, and choose $c \in C$ arbitrary. Then, there exists a neighbourhood V_c of 0 such that for every κ, there exists a $\kappa^* \geq \kappa$ with

$$(c + V_c) \cap P_{\kappa^*} \ = \ \varnothing. \tag{5}$$

Choose a compact symmetric neighbourhood W_c of 0 with $W_c + W_c \subset V_c$ and let κ_c be chosen such that $(P_\iota, P_\kappa) \subset U_{c+W_c,W_c}$ for all $\iota, \kappa \geq \kappa_c$. Then, a short moments thought and (5) give

$$(c + W_c) \cap P_\kappa \ = \ \varnothing\,, \quad \text{for all } \kappa \geq \kappa_c.$$

As C is compact, we can then find $c_1, \ldots, c_n \in C$ and a κ_0 such that

$$C \ \subset \ \bigcup_{i=1}^{n} (c_i + W_{c_i}) \quad \text{and} \quad (c_i + W_{c_i}) \cap P_\kappa \ = \ \varnothing$$

for every $1 \leq i \leq n$ and all $\kappa \geq \kappa_0$. In particular, $C \cap P_\kappa = \varnothing$ for all $\kappa \geq \kappa_0$, contradicting the assumption of the claim. □

So, finally, let a compact K in G and a neighbourhood V of 0 be given. Assume, without loss of generality, that V is compact and symmetric. As (P_ι) is a Cauchy net, there exists a $\iota_{K,V}$ with $(P_\iota, P_\kappa) \in U_{K,V}$ for all $\iota, \kappa \geq \iota_{K,V}$. Consider $\iota \geq \iota_{K,V}$ and choose an arbitrary $q \in P_\iota \cap K$. Then,

$$(q + V) \cap P_\kappa \ \neq \ \varnothing$$

for every $\kappa \geq \iota_{K,V}$, as V is symmetric and $P_\iota \cap K \subset P_\kappa + V$ by $(P_\iota, P_\kappa) \in U_{K,V}$. Thus, the assumptions of the Claim are satisfied and we can infer the existence of a $p \in P$ with

$p \in q+V$. In particular, invoking the symmetry of V once more, we have $q \in p+V \subset P+V$. As $q \in P_\iota \cap K$ was arbitrary, we infer

$$P_\iota \cap K \;\subset\; P+V.$$

Put together, these considerations show existence of a closed P (namely that defined by (4)) and an index $\iota_{K,V}$ such that $(P_\iota, P) \subset U_{K,V}$ for all $\iota \geq \iota_{K,V}$, whenever a compact K and a neighbourhood V of 0 are given. This proves the desired completeness statement.

We next show compactness of $\mathcal{C}(G)$. As $\mathcal{C}(G)$ is complete, it suffices to show that it is precompact. This means, for any given K compact and V an open neighbourhood of 0 in G, we have to provide a natural number n and $P_i \in \mathcal{C}(G)$, $i = 1, \ldots, n$, such that

$$\mathcal{C}(G) \;\subset\; \bigcup_{i=1}^{n} U_{K,V}(P_i).$$

Since $U_{K,V}(P) \supset U_{K,V\cap(-V)}(P)$ for all $P \in \mathcal{C}(G)$ and $V \cap (-V)$ is symmetric, we can assume, without loss of generality, that V is symmetric (i.e., $V = -V$). As K is compact, there exists a finite set $D \subset K$ with

$$K \subset D+V.$$

For $P \in \mathcal{C}(G)$, define

$$D(P) \;:=\; \{x \in D : (x+V) \cap P \neq \varnothing\}.$$

Then, $D(P)$ is a subset of D. Moreover, by construction and the symmetry of V, the set P belongs to $U_{K,V}(D(P))$, i.e., the inclusions $P \cap K \subset D(P)+V$ and $D(P) \cap K \subset P+V$ hold. As $P \in \mathcal{C}(G)$ was arbitrary, we find $\mathcal{C}(G) \subset \bigcup_{i=1}^{n} U_{K,V}(D_i)$, where D_i, $i = 1, \ldots, n$, is an enumeration of the power set of D.

The statement about metrisability is a direct consequence of [19, Thm. 6.13] and the remark thereafter. □

Having topologised $\mathcal{C}(G)$, and thus $\mathcal{UD}(G)$ as well, we can now introduce our point dynamical systems. The natural action of G on $\mathcal{C}(G)$ by translation will also be denoted by α. Explicitly, we define $\alpha_t(P) = t+P$ for $P \in \mathcal{C}(G)$, where $t+P = \{t+x : x \in P\}$ as usual.

Definition 5. *Let Ω be a subset of G and α the translation action just defined.*

- (a) *The pair (Ω, α) is called a set dynamical system if Ω is a closed subset of $\mathcal{C}(G)$ which is invariant under α.*
- (b) *A set dynamical system (Ω, α) is called a point dynamical system if Ω is a subset of $\mathcal{D}_V(G)$ for some open neighbourhood V of 0.*
- (c) *A point dynamical system (Ω, α) is called a Delone dynamical system, if every element of Ω is a relatively dense subset of G.*

It follows from Theorem 3 that a set dynamical system is indeed a topological dynamical system in the sense of Section 2.

A special way of obtaining set dynamical systems is the following: Choose $P \in \mathcal{C}(G)$. Then, the LRT-closure $X(P)$ of the orbit $\{\alpha_t(P) : t \in G\}$ of P in $\mathcal{C}(G)$ is a closed α-invariant subset of $\mathcal{C}(G)$, hence compact. Thus, $(X(P), \alpha)$ is a set dynamical system. We say that P and P' are *locally indistinguishable* if $P \subset X(P')$ and $P' \subset X(P)$, hence if $X(P) = X(P')$. The set of all P' which are locally indistinguishable form P is called the RLI class of P, written as RLI(P). Here, as before, the letter R stands for "rubber".

Given these notions, we can characterise minimality of a set dynamical system in the following way, which extends [33, Prop. 3.1] to our setting.

Proposition 3. *Let $P \in \mathcal{C}(G)$ be given. Then the following assertions are equivalent.*

(a) *The set dynamical system $(X(P), \alpha)$ is minimal.*
(b) *$X(P) = \mathrm{RLI}(P)$.*
(c) *The set $\{t \in G : (t + P, P) \in U_{K,V}\}$ is relatively dense in G for all compact $K \subset G$ and all open neighbourhoods V of 0.*
(d) *The set P is repetitive, i.e., for every K compact and every open neighbourhood V of 0, there is a compact set $C = C(K, V) \subset G$ such that for $t_1, t_2 \in G$, there is an $s \in C$ with $(t_1 + P, s + t_2 + P) \in U_{K,V}$.*

Proof. The equivalence of (a) and (b) is clear by the definition of minimality and the local indistinguishability class. This is also known as Gottschalk's Theorem, compare [28, Thm. 4.1.2]. The remaining assertions can be shown by mimicking and slightly extending the proof of [33, Prop. 3.1]. Since we do not use these results later on, we skip further details. □

Having discussed point dynamical systems, we can now relate them to special dynamical systems on measures. The connection relies on the map

$$\delta \colon \mathcal{UD}(G) \longrightarrow \mathcal{M}^{\infty}(G), \quad \delta(\varLambda) := \sum_{x \in \varLambda} \delta_x ,$$

Note that δ is indeed a map into $\mathcal{M}^{\infty}(G)$, as every $\varLambda \in \mathcal{UD}(G)$ is uniformly discrete.

Lemma 2. *The map $\delta \colon \mathcal{UD}(G) \longrightarrow \mathcal{M}^{\infty}(G)$ is injective, continuous and compatible with the action of G. The inverse $\delta^{-1} \colon \delta(\mathcal{UD}(G)) \longrightarrow \mathcal{UD}(G)$ is continuous as well.*

Proof. Injectivity and compatibility with the action of G are immediate. In particular, using $\delta_{t+x} = \delta_t * \delta_x$, one can check that

$$\delta(\alpha_t(\varLambda)) = \delta(t + \varLambda) = \delta_t * \delta(\varLambda) = \alpha_t(\delta(\varLambda)) .$$

To show continuity of δ, we have to show $\delta(\varLambda_\iota)(\varphi) \longrightarrow \delta(\varLambda)(\varphi)$ for all $\varphi \in C_c(G)$, whenever $\varLambda_\iota \longrightarrow \varLambda$. Let V be an open neighbourhood of 0 in G such that $\varLambda \in \mathcal{D}_V(G)$. Let $\varphi \in C_c(G)$ be given, so $\mathrm{supp}(\varphi)$ is compact. By a simple partition of unity argument, we can now write $\varphi = \sum_{j=1}^{n} \varphi_j$, where each φ_j, $j = 1, \dots, n$, has its support in a set of the form $W + t_j$, with $W + W \subset V$. Now, for such φ_j, the convergence $\delta(\varLambda_\iota)(\varphi_j) \longrightarrow \delta(\varLambda)(\varphi_j)$ follows easily. This yields the desired convergence for φ. The continuity of δ^{-1} can be shown similarly. □

As can be seen from simple examples, $\delta(\mathcal{UD}(G))$ is, in general, not closed in $\mathcal{M}^{\infty}(G)$ and the LRT is not the same as the vague topology on $\delta(\mathcal{UD}(G))$. For example, if (x_n) is a sequence in G with $x_n \longrightarrow 0$ and $x_n \neq 0$, then, $\varLambda_n := \{0, x_n\} \in \mathcal{UD}(G)$ with $\varLambda_n \longrightarrow \{0\}$ in the LRT. However, $\delta(\varLambda_n) \longrightarrow 2\delta_0$ in the vague topology, and $2\delta_0$ does not belong to $\delta(\mathcal{UD}(G))$. Nevertheless, the following still holds.

Proposition 4. *Let V be an open neighbourhood of 0 in G. Then, $\mathcal{D}_V(G)$ is compact in the* LRT *and $\delta(\mathcal{D}_V(G))$ is compact in the vague topology.*

Proof. By compactness of $\mathcal{C}(G)$ and continuity of δ, it suffices to show that $\mathcal{D}_V(G)$ is closed. Since V is open, this is easy. □

Our main result on the relationship of point dynamical systems and the framework of measure dynamical systems reads as follows.

Theorem 4. *If* (Ω, α) *is a point dynamical system, the restriction* $\delta|_\Omega \colon \Omega \longrightarrow \delta(\Omega)$ *of* δ *to* Ω *establishes a topological conjugacy between the dynamical systems* (Ω, α) *and* $(\delta(\Omega), \alpha)$.

Proof. By Lemma 2, $\delta|_\Omega$ is an injective and continuous map which is compatible with the group action. As Ω is compact, so is $\delta(\Omega)$. By a standard argument [27, Prop. 1.6.8], $\delta|_\Omega$ is then a homeomorphism between (Ω, α) and $(\delta(\Omega), \alpha)$. Together, this establishes the topological conjugacy. □

We finish this section by briefly discussing the relationship of the LRT and the topology usually considered for Delone dynamical systems with the FLC property. A thorough discussion of the latter topology has been given in [33]. This discussion actually gives a topology on the closed subsets of G (though this is not explicitly noted in [33]). This topology will be called the local matching topology (LMT).

The definition of the LMT in [33] shows immediately that the LRT is coarser than the LMT. Thus, the identity

$$id\colon (\mathcal{C}(G), \mathrm{LMT}) \longrightarrow (\mathcal{C}(G), \mathrm{LRT}), \quad P \mapsto P$$

is continuous. This yields the following result, which essentially shows that our way of topologising the uniformly discrete sets coincides with the usual topology when restricted to sets of finite local complexity.

Proposition 5. *Let* Ω *be a subset of* $\mathcal{C}(G)$. *If* Ω *is compact in the* LMT, *then* Ω *is compact in the* LRT *as well, and the two topologies agree on* Ω.

Proof. The restriction $id_\Omega \colon (\Omega, \mathrm{LMT}) \longrightarrow (\Omega, \mathrm{LRT})$ of the identity to Ω is continuous. Thus, as (Ω, LMT) is compact, so is its image (Ω, LRT). Now, continuity of the inverse is standard, cf. [27, Prop. 1.6.8]. Thus, the two topologies agree. □

5. The diffraction spectrum

In this section, we extend the basics of diffraction theory as developed for Delone dynamical systems in [3, 5, 11, 16, 17, 32, 33] to our measure dynamical systems. More precisely, we show the existence of the autocorrelation measure by a limiting procedure provided certain ergodicity assumptions hold.

To phrase our results, we need two more pieces of notation. Firstly, recall from [33] that a sequence (B_n) of compact subsets of G is called a *van Hove sequence* if

$$\lim_{n\to\infty} \frac{|\partial^K B_n|}{|B_n|} \; = \; 0$$

for all compact $K \subset G$. Here, for compact B, K, the "K-boundary" $\partial^K B$ of B is defined as

$$\partial^K B \; := \; \overline{((B+K)\setminus B)} \cup (\overline{G\setminus B} - K) \cap B,$$

where the bar denotes the closure. The existence of van Hove sequences for all σ-compact LCA groups is shown in [33, p. 249], see also Section 3.3 and Theorem (3.L) of [37, Appendix]. Moreover, every van Hove sequence is a Følner sequence, i.e., $|B_n \triangle (B_n + K)|/|B_n| \xrightarrow{n\to\infty} 0$, for every compact set $K \subset G$, where $\triangle$ denotes the operator for the symmetric difference of two sets, compare Section 3.2 of [37, Appendix]; for a partial converse, consult Theorem (3.K) of the same reference.

Secondly, for $\varphi \in C_c(G)$ and $\mu \in \mathcal{M}(G)$, we define

$$f_\varphi(\mu) \; := \; \big(\varphi * \mu\big)(0) \; = \; \int_G \varphi(-s) \, \mathrm{d}\mu(s).$$

This gives a way to "push" functions from $C_c(G)$ to $C(\Omega)$. In fact, up to the sign, f_φ is just the canonical embedding of $C_c(G)$ into its bidual $\mathcal{M}(G)^*$. Basic features of the map $\varphi \mapsto f_\varphi$ are gathered in the following lemma.

Lemma 3. *Recall the definition $\alpha_t \mu = \delta_t * \mu$ for measures μ, and set $\beta_t(\varphi) = \delta_t * \varphi$ for functions φ. Then one has:*

(a) *The function $f_\varphi \colon \mathcal{M}(G) \longrightarrow \mathbb{C}$, $\mu \mapsto f_\varphi(\mu)$ is continuous, for all $\varphi \in C_c(G)$.*
(b) *If (Ω, α) is a* TMDS, *the map $f \colon C_c(G) \longrightarrow C(\Omega)$, $\varphi \mapsto f_\varphi$, is linear, continuous and compatible with the action of G in that $f_\varphi(\alpha_t \mu) = f_{\beta_t(\varphi)}(\mu)$.*

Proof. (a) For $\varphi \in C_c(G)$, we have $f_\varphi(\mu) = \mu(\varphi_-)$, where $\varphi_-(t) = \varphi(-t)$. Thus, continuity of f_φ is immediate from the definition of the topology on $\mathcal{M}(G)$.

(b) Linearity of the map f is obvious. To show continuity of f, recall that $C_c(G)$ is equipped with the inductive limit topology induced from the embeddings $C_K(G) \hookrightarrow C_c(G)$ with $K \subset G$ compact. Thus, it suffices to show the continuity of the map

$$f_K \colon \, C_K(G) \; \hookrightarrow \; C_c(G) \, , \quad \varphi \mapsto f_\varphi$$

for every compact K in G. Thus, let (φ_ι) be a net in $C_K(G)$ converging to $\varphi \in C_K(G)$. Then, $\|\varphi_\iota - \varphi\|_\infty \longrightarrow 0$, and $\mathrm{supp}(\varphi), \mathrm{supp}(\varphi_\iota) \subset K$ for all ι. As $\Omega \subset \mathcal{M}_{C,V}(G)$ with suitable C, V, this easily implies $f_{\varphi_\iota} \longrightarrow f_\varphi$. Finally, a direct calculation shows $f_\varphi(\alpha_t \mu) = f_{\beta_t(\varphi)}(\mu)$. □

Remark. The lemma is particularly interesting as there does not seem to exist any canonical map from Ω to G or from G to Ω in our setting (let alone a map which is compatible with the corresponding group actions). However, if one views a function φ as the Radon-Nikodym derivative of a measure that is absolutely continuous with respect to the Haar measure of G, the action α_t induces β_t as defined.

Now, our result on existence of the autocorrelation function reads as follows.

Theorem 5. *Let α be the translation action of G on $\mathcal{M}^\infty(G)$ as introduced above.*

(a) *If (Ω, α) is a uniquely ergodic* TMDS, *there exists a translation bounded measure γ on G such that the sequence $(\frac{1}{|B_n|} \widetilde{\omega_{B_n}} * \omega_{B_n})$ converges, in the vague topology, to γ for every van Hove sequence (B_n) and every $\omega \in \Omega$. Moreover, the equation $\big(\widetilde{\varphi} * \psi * \gamma\big)(t) = \langle f_\varphi, T^t f_\psi \rangle$ holds for arbitrary $\varphi, \psi \in C_c(G)$ and $t \in G$.*
(b) *Let G have a topology with countable base. Let (Ω, α) be a* TMDS *with ergodic probability measure m. Then, there exists a translation bounded measure γ on G such that the sequence $(\frac{1}{|B_n|} \widetilde{\omega_{B_n}} * \omega_{B_n})$ converges, in the vague topology, to γ for m-almost every $\omega \in \Omega$, whenever (B_n) is a van Hove sequence along which the Birkhoff ergodic theorem holds. Moreover, the equation $\big(\widetilde{\varphi} * \psi * \gamma\big)(t) = \langle f_\varphi, T^t f_\psi \rangle$ holds for arbitrary $\varphi, \psi \in C_c(G)$ and $t \in G$.*

Remarks. (a) Note that every LCA group with a countable base of the topology admits a van Hove sequence along which the Birkhoff ergodic theorem holds, as follows from recent results of Lindenstrauss [23], see also Tempelman's monograph [37], in particular its Appendix, for background material. More precisely, every van Hove sequence is a Følner sequence, and thus contains a so-called tempered subsequence with the desired property, compare [23]. Note also that G second countable implies σ-compactness as well as metrisability of G.
(b) The theorem generalises the corresponding results of [11, 33, 17].

To prove Theorem 5, we need some preparation in form of the following results.

Lemma 4. *Let D be a dense subset of $C_c(G)$. Let $C > 0$ and a relatively compact open V in G be given. If (μ_ι) is a net of measures in $\mathcal{M}_{C,V}(G)$ such that $\mu_\iota(\varphi)$ converges for every $\varphi \in D$, then there exists a translation bounded measure $\mu \in \mathcal{M}(G)$ such that (μ_ι) converges vaguely to μ.*

Proof. As D is dense, every μ in $\mathcal{M}(G)$ is uniquely determined by its values on D. Thus, all converging subnets of (μ_ι) have the same limit. As $\mathcal{M}_{C,V}(G)$ is compact by Theorem 2, there exist converging subnets. Putting this together, we arrive at the desired statement. □

Lemma 5 (Lemma 1.2 in [33]). *Let μ, ν be translation bounded measures on G and (B_n) be a van Hove sequence. Then, in the vague topology, $\lim_{n\to\infty}(\mu_{B_n} * \nu_{B_n} - \mu * \nu_{B_n}) = 0$.* □

Lemma 6 (Lemma 1.1 (b) in [33]). *Let (B_n) be a van Hove sequence in G and μ a translation bounded measure. Then, the sequence $\big(|\mu|(B_n)/|B_n|\big)$ is bounded.* □

Proof of Theorem 5. (a) As $\omega \in \mathcal{M}_{C,V}(G)$, Lemma 6 and a short calculation give a constant $C' > 0$ and and a relatively compact open $V' \subset G$ such that the sequence $\big((\widetilde{\omega_{B_n}} * \omega_{B_n})/|B_n|\big)$ is contained in $\mathcal{M}_{C',V'}$. Moreover, the set $\{\varphi * \psi : \varphi, \psi \in C_c(G)\}$ is dense in $C_c(G)$ by standard arguments involving approximate units [31]. Thus, by Lemma 4 and Lemma 5, it suffices to show $\lim_{n\to\infty} \frac{1}{|B_n|}\big(\widetilde{\varphi} * \psi * \widetilde{\omega_{B_n}} * \omega_{B_n}\big)(t) = \langle f_\varphi, T^t f_\psi\rangle$ for arbitrary $\varphi, \psi \in C_c(G)$ and $t \in G$. By Lemma 5, it suffices to show

$$\lim_{n\to\infty} \frac{1}{|B_n|}\big(\widetilde{\varphi} * \psi * \widetilde{\omega} * \omega_{B_n}\big)(t) \;=\; \langle f_\varphi, T^t f_\psi\rangle.$$

This follows by unique ergodicity and a Dworkin type calculation [11, 33, 21]. As the details are somewhat more involved than in the case of Delone sets, we include a sketch for the convenience of the reader. We define $Z_n := \big(\widetilde{\varphi} * \psi * \widetilde{\omega} * \omega_{B_n}\big)(t)$. Then,

$$Z_n \;=\; \int_G \big(\widetilde{\varphi} * \psi\big)(t-u)\, \mathrm{d}(\widetilde{\omega} * \omega_{B_n})(u) \;=\; \int_G \int_G \int_G \overline{\varphi}(v-t+r)\psi(v-s) 1_{B_n}(s)\, \mathrm{d}v\, \mathrm{d}\widetilde{\omega}(r)\, \mathrm{d}\omega(s),$$

where 1_{B_n} denotes the characteristic function of B_n. Using Fubini's Theorem and sorting the terms we arrive at

$$Z_n \;=\; \int_G \left(\int_G \overline{\varphi}(v-t+r)\, \mathrm{d}\widetilde{\omega}(r)\right) \left(\int_G \psi(v-s) 1_{B_n}(s)\, \mathrm{d}\omega(s)\right) \mathrm{d}v\,.$$

We will now study the two terms in brackets. A short calculation shows $\int_G \overline{\varphi}(v-t+r)\, \mathrm{d}\widetilde{\omega}(r) = \overline{f_\varphi(\alpha_{t-v}\omega)}$. As for the other term, we consider the difference function

$$D(v) \;:=\; \int_G \psi(v-s)\, 1_{B_n}(s)\, \mathrm{d}\omega(s) - \int_G \psi(v-s)\, \mathrm{d}\omega(s)\, 1_{B_n}(v)\,.$$

Let K be a compact set with $K = -K$, $0 \in K$ and $\operatorname{supp}(\psi) \subset K$. Then, it is not hard to see that the difference $D(v)$ (and in fact each of its terms alone) vanishes for $v \notin B_n + K$. Similarly, one can show that $D(v)$ is supported in $\partial^K B_n$. Apparently, $|D(v)|$ is bounded above by $C := 2\|\psi\|_\infty \sup\{|\omega|(t + \operatorname{supp}(\psi)) : t \in G\} < \infty$. As (B_n) is a van Hove sequence, we conclude that

$$0 \;\le\; \frac{1}{|B_n|}\int_G \Big| D(v)\overline{f_\varphi(\alpha_{t-v}\omega)}\Big| \,\mathrm{d}v \;\le\; C\|f_\varphi\|_\infty \frac{|\partial^K B_n|}{|B_n|} \;\xrightarrow{n\to\infty}\; 0\,.$$

Noting that $\int_G \psi(v-s)\,\mathrm{d}\omega(s) = f_\psi(\alpha_{-v}\omega)$, and putting these considerations together, we arrive at

$$\lim_{n\to\infty}\frac{Z_n}{|B_n|} \;=\; \lim_{n\to\infty}\frac{1}{|B_n|}\int_G \overline{f_\varphi(\alpha_{t-v}\omega)}\, f_\psi(\alpha_{-v}\omega)\, 1_{B_n}(v)\,\mathrm{d}v\,.$$

This yields

$$\begin{aligned}\lim_{n\to\infty}\frac{Z_n}{|B_n|} \;&=\; \int_\Omega \overline{f_\varphi(\alpha_t\omega)}f_\psi(\omega)\,\mathrm{d}m(\omega) \;=\; \int_\Omega \overline{f_\varphi(\omega)}f_\psi(\alpha_{-t}\omega)\,\mathrm{d}m(\omega)\\ &=\; \int_\Omega \overline{f_\varphi(\omega)}(T^t f_\psi)(\omega)\,\mathrm{d}m(\omega) \;=\; \langle f_\varphi, T^t f_\psi\rangle\,,\end{aligned}$$

where we used the pointwise ergodic theorem for continuous functions on a uniquely ergodic system in the first step and α-invariance of m in the second step. (Note that this ergodic theorem only relies on compactness of the underlying space Ω and does not require separability of G. This can easily be seen by going through a proof of this thereom as presented e.g. in Theorem 6.19 of [38].)

(b) This can be seen similarly: After replacing the pointwise ergodic theorem for uniquely ergodic systems by the Birkhoff ergodic theorem, the considerations of (a) can be carried through to show that, for each function $\widetilde{\varphi} * \psi$, there exists a set $\Omega_{\varphi,\psi} \subset \Omega$ of full measure such that

$$\frac{1}{|B_n|}\big(\widetilde{\varphi} * \psi * \widetilde{\omega_{B_n}} * \omega_{B_n}\big)(0) \;\xrightarrow{n\to\infty}\; \langle f_\varphi, f_\psi\rangle$$

for every $\omega \in \Omega_{\varphi,\psi}$. As G is second countable, there exists a countable set D in $C_c(G)$ such that D and $\{\widetilde{\varphi} * \psi : \varphi, \psi \in D\}$ are dense in G. Thus, there is a set $\Omega_0 \subset \Omega$ of full measure such that

$$\frac{1}{|B_n|}\big(\widetilde{\varphi} * \psi * \widetilde{\omega_{B_n}} * \omega_{B_n}\big)(0) \;\xrightarrow{n\to\infty}\; \langle f_\varphi, f_\psi\rangle$$

for all $\varphi, \psi \in D$ and all $\omega \in \Omega_0$. By the density of $\{\widetilde{\varphi} * \psi : \varphi, \psi \in D\}$ in $C_c(G)$ and Lemma 4, the vague convergence of $\big(\widetilde{\omega_{B_n}} * \omega_{B_n}\big)/|B_n|$ towards a translation bounded measure γ with $\big(\widetilde{\varphi} * \psi * \gamma\big)(0) = \langle f_\varphi, f_\psi\rangle$ for all $\varphi, \psi \in D$ follows. This gives the desired vague convergence.

It remains to show the last part of the statement: As D is dense in $C_c(G)$ and f is a continuous map, the formula

$$\big(\widetilde{\varphi} * \psi * \gamma\big)(0) \;=\; \langle f_\varphi, f_\psi\rangle$$

does not only hold for $\varphi, \psi \in D$ but for arbitrary $\varphi, \psi \in C_c(G)$. For $t \in G$, this implies

$$\big(\widetilde{\varphi} * \psi * \gamma\big)(t) \;=\; \big(\delta_{-t} * \widetilde{\varphi} * \psi * \gamma\big)(0) \;=\; \big(\widetilde{\varphi} * (\delta_{-t} * \psi) * \gamma\big)(0) \;=\; \langle f_\varphi, f_{\delta_{-t} * \psi}\rangle\,.$$

By Lemma 3, we have

$$f_{\delta_{-t}*\psi}(\omega) \;=\; f_{\beta_{-t}(\psi)}(\omega) \;=\; f_\psi(\alpha_{-t}\omega) \;=\; T^t f_\psi(\omega)\,.$$

Thus, we can conclude

$$\big(\widetilde{\varphi} * \psi * \gamma\big)(t) \;=\; \langle f_\varphi, f_{\delta_{-t}*\psi}\rangle \;=\; \langle f_\varphi, T^t f_\psi\rangle\,.$$

This finishes the proof. □

Remark. By Lemma 5, the convergence of $|B_n|^{-1}\widetilde{\omega_{B_n}} * \omega_{B_n}$ towards γ discussed in the previous theorem implies convergence of $|B_n|^{-1}\widetilde{\omega}_{B_n} * \omega_{B_n}$ towards γ as well.

The measure

$$\gamma \;=\; \gamma_\omega \;=\; \lim_{n\to\infty} \frac{1}{|B_n|}\,\widetilde{\omega_{B_n}} * \omega_{B_n}$$

appearing in the theorem is called the *autocorrelation measure* (or just *autocorrelation*) of $\omega \in \Omega$. It is obviously positive definite, and hence transformable. By Bochner's Theorem, compare [7, Ch. I.4], its Fourier transform is then a positive measure on $\widehat{G}$, called the *diffraction measure* of $\omega \in \Omega$. We will have to say more about autocorrelation and diffraction measures in the next section.

6. Relating diffraction and dynamical spectrum

In this section, we show that the diffraction spectrum is equivalent to the spectrum of a certain subrepresentation of T. This type of result is implicit in essentially every work using the so-called Dworkin argument [11, 16, 33, 35]. However, it seems worthwhile to make this connection explicit. In fact, this is one of the two cornerstones of our approach to the characterisation of pure pointedness, the other being Theorem 1.

We start by giving a closed formula for the autocorrelation measure. This closed formula does not rely on any ergodicity assumptions. Thus, via this formula, an autocorrelation measure can be attached to any TMDS with an invariant probability measure m. We should like to mention that this is inspired by recent work of Gouéré [14], which gives a closed formula in the context of Palm measures and point processes.

Proposition 6. *Let (Ω,α) be a* TMDS *with invariant probability measure m. Let a function $\sigma \in C_c(G)$ be given with $\int_G \sigma(t)\,\mathrm{d}t = 1$. For $\varphi \in C_c(G)$, define*

$$\gamma_{\sigma,m}(\varphi) \;:=\; \int_\Omega \int_G f_\varphi(\alpha_{-t}\overline{\omega})\sigma(t)\,\mathrm{d}\omega(t)\,\mathrm{d}m(\omega).$$

This leads to the following assertions.

(a) *The map $\gamma_{\sigma,m}\colon C_c(G) \longrightarrow \mathbb{C}$ is continuous, i.e., $\gamma_{\sigma,m} \in \mathcal{M}(G)$.*
(b) *For $\varphi,\psi \in C_c(G)$, the equation $\big(\widetilde{\varphi} * \psi * \gamma_{\sigma,m}\big)(t) = \langle f_\varphi, T^t f_\psi\rangle$ holds.*
(c) *The measure $\gamma_{\sigma,m}$ does not depend on $\sigma \in C_c(G)$, provided $\int_G \sigma\,\mathrm{d}t = 1$.*
(d) *The measure $\gamma_{\sigma,m}$ is positive definite.*

Proof. Note that $f_\varphi(\alpha_s\overline{\omega}) = \overline{f_{\overline{\varphi}}(\alpha_s(\omega))}$.
(a) Obviously, $|f_\varphi(\overline{\omega})| \le \|\varphi\|_\infty \sup\{|\omega|(t+\mathrm{supp}(\varphi)) : t \in G\}$. As σ has compact support, $\gamma_{\sigma,m}(\varphi)$ is then finite. Moreover, $\varphi_\iota \longrightarrow \varphi$ implies $\overline{\varphi_\iota} \longrightarrow \overline{\varphi}$ which, in turn, yields $\overline{f_{\overline{\varphi_\iota}}} \longrightarrow \overline{f_{\overline{\varphi}}}$ by continuity of f. As σ has compact support and $\Omega \subset \mathcal{M}_{C,V}(G)$, this gives

$$\int_G \overline{f_{\overline{\varphi_\iota}}}(\alpha_{-t}(\omega))\sigma(t)\,\mathrm{d}\omega(t) \;\longrightarrow\; \int_G \overline{f_{\overline{\varphi}}(\alpha_{-t}(\omega))}\sigma(t)\,\mathrm{d}\omega(t)\,,$$

uniformly on $\varOmega$. The desired continuity statement follows.
(b) We first show the statement for $t = 0$. To this aim, we define

$$Z \; := \; \big(\widetilde{\varphi} * \psi * \gamma_{\sigma,m}\big)(0) \; = \; \int_G \big(\widetilde{\varphi} * \psi\big)(-s) \, \mathrm{d}\gamma_{\sigma,m}(s) \; = \; \int_\varOmega \int_G \overline{f_{\widetilde{\widetilde{\varphi} * \psi}}}(\alpha_{-t}\omega) \sigma(t) \, \mathrm{d}\omega(t) \, \mathrm{d}m(\omega).$$

Then, we can calculate

$$\begin{aligned} Z \;&=\; \int_\varOmega \int_G \overline{\int_G \overline{(\widetilde{\varphi} * \psi)(s-t)} \, \mathrm{d}\omega(s)} \, \sigma(t) \, \mathrm{d}\omega(t) \, \mathrm{d}m(\omega) \\ &=\; \int_\varOmega \int_G \int_G \int_G \widetilde{\varphi}(u) \psi(s-t-u) \, \mathrm{d}u \; \mathrm{d}\overline{\omega}(s) \sigma(t) \, \mathrm{d}\omega(t) \, \mathrm{d}m(\omega) \\ &=\; \int_\varOmega \int_G \int_G \int_G \widetilde{\varphi}(u+s) \psi(-t-u) \, \mathrm{d}u \; \mathrm{d}\overline{\omega}(s) \sigma(t) \, \mathrm{d}\omega(t) \, \mathrm{d}m(\omega) \\ &=\; \int_G \int_\varOmega \int_G \overline{f_\varphi(\alpha_u \omega)} \psi(-t-u) \sigma(t) \, \mathrm{d}\omega(t) \, \mathrm{d}m(\omega) \, \mathrm{d}u, \end{aligned}$$

where we used the translation invariance of the Haar measure in the second last step, and Fubini's Theorem and $\int_G \widetilde{\varphi}(u+s) \, \mathrm{d}\overline{\omega}(s) = \overline{f_\varphi(\alpha_u \omega)}$ in the last step. By the invariance of m, and Fubini's Theorem together with $\int \sigma(t) \, \mathrm{d}t = 1$, this gives:

$$\begin{aligned} Z \;&=\; \int_G \int_\varOmega \int_G \overline{f_\varphi(\omega)} \psi(-t-u) \sigma(t) \, \mathrm{d}(\alpha_{-u}\omega)(t) \, \mathrm{d}m(\omega) \, \mathrm{d}u \\ &=\; \int_G \int_\varOmega \int_G \overline{f_\varphi(\omega)} \psi(-t) \sigma(t-u) \, \mathrm{d}\omega(t) \, \mathrm{d}m(\omega) \, \mathrm{d}u \\ &=\; \int_\varOmega \overline{f_\varphi(\omega)} f_\psi(\omega) \, \mathrm{d}m(\omega) \;\;=\;\; \langle f_\varphi, f_\psi \rangle \, . \end{aligned}$$

The case of arbitrary $t \in G$ can now be treated by mimicking the last part of the proof of part (b) of Theorem 5.
(c) This is immediate from (b) and (a) as $\{\widetilde{\varphi} * \psi : \varphi, \psi \in C_c(G)\}$ is dense in $C_c(G)$.
(d) This is a direct consequence of (b). □

Part (b) of the Lemma shows, in particular, that the measure $\gamma_{\sigma,m}$ equals the autocorrelation measure introduced in the last section if m is ergodic. Part (d) shows that $\gamma_{\sigma,m}$ is positive definite. Thus, by Bochner's Theorem, see [7], its Fourier transform is a positive measure on the dual group $\widehat{G}$.

Definition 6. *Let $(\varOmega, \alpha)$ be a* TMDS *with invariant probability measure m. The measure $\gamma_m := \gamma_{\sigma,m}$, where $\sigma \in C_c(G)$ with $\int_G \sigma \, \mathrm{d}t = 1$, is called the autocorrelation measure of the dynamical system $(\varOmega, \alpha)$ with invariant measure m. Its Fourier transform $\widehat{\gamma_m}$ is called the diffraction measure of the dynamical system $(\varOmega, \alpha)$ with invariant measure m.*

We summarise the preceding considerations in the following lemma, where we use $\psi_{\_}$ for the function defined by $\psi_{\_}(t) = \psi(-t)$.

Lemma 7. *Let $(\varOmega, \alpha)$ be a* TMDS *with invariant probability measure m. Then, there exists a unique measure on G assigning the value $\langle f_\varphi, f_\psi \rangle$ to the function $\overline{\varphi} * \psi_{\_}$, for $\varphi, \psi \in C_c(G)$. This measure is the autocorrelation measure γ_m of $(\varOmega, \alpha)$.*

Proof. Uniqueness is clear as the set $\{\overline{\varphi} * \psi\_ : \varphi, \psi \in C_c(G)\}$ is dense in $C_c(G)$. Existence follows from Proposition 6, as

$$\gamma_m(\overline{\varphi} * \psi\_) = \big(\widetilde{\varphi} * \psi * \gamma_m\big)(0) = \langle f_\varphi, f_\psi \rangle.$$

This proves the lemma. □

Remark This definition of the autocorrelation and the diffraction of a dynamical system is to be compared with the corresponding objects of a single measure (namely an element in Ω) studied in the last section. In the latter case, one faces the problem of its dependence of the measure m, or of the averaging sequence (B_n). It is reasonable, both mathematically and physically, to replace this by the objects defined in Definition 6, at least for most aspects of the spectral theory connected with it.

Having cast the diffraction measure in an abstract context, we will now briefly discuss the basic quantities in the spectral theory of dynamical systems: Let (Ω, α) be a TMDS. By Stone's Theorem (compare [24, Sec. 36D]), there exists a projection valued measure

$$E_T \colon \text{Borel sets on } \widehat{G} \longrightarrow \text{Projections on } L^2(\Omega, m)$$

with

$$\langle f, T^t f \rangle \;=\; \int_{\widehat{G}} (\widehat{s}, t) \,\mathrm{d}\langle f, E_T(\widehat{s}) f \rangle \;=\; \int_{\widehat{G}} (\widehat{s}, t) \,\mathrm{d}\rho_f(\widehat{s}) \,,$$

where ρ_f is the measure on $\widehat{G}$ defined by $\rho_f(B) := \langle f, E_T(B) f \rangle$. It is then not hard to see that T has pure point spectrum (in the sense defined in Section 2) if and only if all the measures ρ_f, $f \in L^2(\Omega, m)$, are pure point measures.

Note that every closed T-invariant subspace $\mathcal{V}$ of $L^2(\Omega, m)$ gives rise to a representation $T|_{\mathcal{V}}$ of G on $\mathcal{V}$ by restricting the representation T to $\mathcal{V}$. The spectral family of $T|_{\mathcal{V}}$ will be denoted by $E_{T|_{\mathcal{V}}}$. With the canonical inclusion $i_{\mathcal{V}} : \mathcal{V} \longrightarrow L^2(\Omega, m)$ and projection $P_{\mathcal{V}} \colon L^2(\Omega, m) \longrightarrow \mathcal{V}$, we obviously have

$$T|_{\mathcal{V}} \;=\; P_{\mathcal{V}}\, T\, i_{\mathcal{V}} \quad \text{and} \quad E_{T|_{\mathcal{V}}} = P_{\mathcal{V}}\, E_T\, i_{\mathcal{V}}.$$

In our setting, a translation invariant subspace appears naturally. This is discussed next.

Lemma 8. *Let (Ω, α) be a* TMDS *with invariant probability measure m. The set of functions $\mathcal{U}_0 := \{f_\varphi : \varphi \in C_c(G)\}$ is a translation invariant subspace of $L^2(\Omega, m)$, and so is its closure.*

Proof. The first part of the statement follows from Lemma 3 (b). The second part of the statement is then immediate. □

Definition 7. *Let $\mathcal{U}$ be the closure of the space $\mathcal{U}_0$ from Lemma* 8 *in $L^2(\Omega, m)$.*

Before we can give a precise version of the relationship between $\widehat{\gamma_m}$ and T we need one more definition.

Definition 8. *Let ρ be a measure on $\widehat{G}$ and S be an arbitrary unitary representation of G on $L^2(\Omega, m)$. Then, ρ is called a* spectral measure *for S if the following holds for all Borel sets B: $E_S(B) = 0$ if and only if $\rho(B) = 0$.*

Now, the relationship between $\widehat{\gamma_m}$ and $T = T_m$ can be phrased as follows.

Theorem 6. *Let (Ω, α) be a* TMDS *with invariant probability measure m. Then, the measure $\widehat{\gamma_m}$ is a spectral measure for the restriction $T|_{\mathcal{U}}$ of T to $\mathcal{U}$.*

Proof. Let B be a Borel set in $\widehat{G}$. Then, obviously, $E_{T|_{\mathcal{U}}}(B) = 0$ if and only if we have $0 = \langle f_\varphi, E_{T|_{\mathcal{U}}}(B) f_\varphi\rangle$ for every $\varphi \in C_c(G)$. As $\langle f_\varphi, E_T(B) f_\varphi\rangle = \langle f_\varphi, E_{T|_{\mathcal{U}}}(B) f_\varphi\rangle$ for every $\varphi \in C_c(G)$, by the very definition of $\mathcal{U}$ and $E_{T|_{\mathcal{U}}}$, we infer that $E_{T|_{\mathcal{U}}}(B) = 0$ if and only if

$$\langle f_\varphi, E_T(B) f_\varphi\rangle \; = \; 0\,, \quad \text{for every } \varphi \in C_c(G)\,.$$

Now, the (inverse) Fourier transform (on $\widehat{G}$) of ρ_{f_φ} is $t \mapsto \langle f_\varphi, T^t f_\varphi\rangle$ as discussed above. By Lemma 6, we have $\langle f_\varphi, T^t f_\varphi\rangle = \big(\widetilde{\varphi} * \varphi * \gamma_m\big)(t)$. Thus, taking the Fourier transform (on G), we infer $\rho_{f_\varphi} = |\widehat{\varphi}|^2 \widehat{\gamma_m}$, compare [7, 13]. In particular, this implies

$$\langle f_\varphi, E_T(B) f_\varphi\rangle \; = \; \int_B |\widehat{\varphi}|^2 \,\mathrm{d}\widehat{\gamma_m}\,.$$

These considerations show that $E_{T|_{\mathcal{U}}}(B) = 0$ if and only if $0 = \int_B |\widehat{\varphi}|^2 \,\mathrm{d}\widehat{\gamma_m}$ for every function $\varphi \in C_c(G)$. Thus, it remains to be shown that $\widehat{\gamma_m}(B) = 0$ if and only if $0 = \int_B |\widehat{\varphi}|^2 \,\mathrm{d}\widehat{\gamma_m}$ for every $\varphi \in C_c(G)$. The only if part is clear. As for the converse, recall that the image of $L^1(G, \mathrm{d}t)$ under the Fourier transform separates points in $\widehat{G}$, see [31]. As $C_c(G)$ is dense in in $L^1(G, \mathrm{d}t)$, the same holds for the image of $C_c(G)$ under the Fourier transform. Therefore, for every $\widehat{s} \in \widehat{G}$, there exists a $\varphi \in C_c(G)$ with $\widehat{\varphi}(\widehat{s}) \neq 0$. Thus, $\widehat{\gamma_m}(B \cap K) = 0$ for every compact K is a direct consequence of $0 = \int_B |\widehat{\varphi}|^2 \,\mathrm{d}\widehat{\gamma_m}$ for every $\varphi \in C_c(G)$. As $\widehat{\gamma_m}$ is regular $\widehat{\gamma_m}(B) = 0$ follows. □

7. The main result

In this section, we state and prove our main result. It shows equivalence of pure point diffraction and pure point dynamical spectrum for rather general measure theoretic dynamical systems.

Theorem 7. *Let (Ω, α) be a* TMDS *with invariant probability measure m. Let T_m be the associated unitary representation of G by translation operators and $\widehat{\gamma_m}$ the associated diffraction measure. The following assertions are now equivalent.*

(a) *The diffraction measure $\widehat{\gamma_m}$ is pure point.*
(b) *The representation T_m has pure point spectrum.*

Proof of Theorem 7. (a) $\Longrightarrow$ (b). This is a consequence of Theorem 1. More precisely, we will show that the vector space

$$\mathcal{V} \; := \; \{f_\varphi : \varphi \in C_c(G)\}$$

satisfies assertion (b) of this theorem:

As $\widehat{\gamma_m}$ has pure point spectrum by (a), Theorem 6 gives that $T|_{\mathcal{U}}$ has pure point spectrum, where $\mathcal{U}$ is the closure of $\mathcal{V}$. Thus, in particular, f_φ belongs to $\mathcal{H}_{pp}(T)$ for every $\varphi \in C_c(G)$. As every element of the form f_φ is continuous by Lemma 3, we see that $\mathcal{V}$ is indeed a subspace of $\mathcal{H}_{pp}(T) \cap C(\Omega)$.

It remains to be shown that $\mathcal{V}$ separates points. Let ω_1 and ω_2 be two different points of Ω. Then, ω_1 and ω_2 are different measures on G. Therefore, there exists a $\varphi \in C_c(G)$ with $\omega_1(\varphi) \neq \omega_2(\varphi)$. This implies $f_{\varphi_-}(\omega_1) \neq f_{\varphi_-}(\omega_2)$ with $\varphi_-(t) := \varphi(-t)$.

(b) $\Longrightarrow$ (a). This is immediate from Theorem 6. □

8. Spectral properties determined by subrepresentations

The ideas of the preceeding sections can be refined to give some further information on how spectral properties of T are determined by spectral properties of $T_{\mathcal{U}}$. This concerns the continuity of the eigenfunctions, and the set of eigenvalues. While the TMDS are the application we have in mind here, the underlying result can be phrased rather abstractly.

We need a special concept on "density of a subspace with respect to multiplication". This is defined next.

Definition 9. *A subspace $\mathcal{V}$ of $L^2(\Omega, m)$ is said to satisfy condition* MD *if the set of products $f_1 \cdot \ldots \cdot f_n$ with $n \in \mathbb{N}$, $f_i \in \mathcal{V} \cap L^\infty(\Omega, m)$ or $\overline{f_i} \in \mathcal{V} \cap L^\infty(\Omega, m)$, $i = 1, \ldots, n$, is total in $L^2(\Omega, m)$.*

Theorem 8. *Let (Ω, α) be a topological dynamical system over G with α-invariant measure m. Let $\mathcal{V}$ be a closed T-invariant subspace of $L^2(\Omega, m)$ satisfying* MD. *If $T|_{\mathcal{V}}$ has pure point spectrum, then the following assertions hold:*

(a) *T has pure point spectrum.*
(b) *The group of eigenvalues of T is generated by the set of eigenvalues of $T|_{\mathcal{V}}$.*
(c) *If $\mathcal{V}$ has a basis consisting of continuous eigenfunctions of $T|_{\mathcal{V}}$, then $L^2(\Omega, m)$ has a basis consisting of continuous eigenfunctions of T, provided the multiplicity of each eigenvalue of T is at most countably infinite.*

Remarks. (a) The countability assumption in (c) is trivially satisfied if the Hilbert space $L^2(\Omega, m)$ is separable, (which holds, e.g., if Ω is metrisable). It is also satisfied if α is ergodic with respect to m. In this case, the multiplicity of each eigenvalue is one.
(b) While we have stated the theorem for topological dynamical systems, its proof does not use the topology on Ω. It can therefore be carried over without changes to give a corresponding result for measurable actions of G on a measure space Ω.

Proof. (a)/(b) Let $\mathcal{S}_1$ be an orthonormal basis of $\mathcal{V}$ consisting of eigenfunctions of $T|_{\mathcal{V}}$. Set

$$\mathcal{S}_2 \;:=\; \{f^N : f \in \mathcal{S}_1 \text{ or } \overline{f} \in \mathcal{S}_1, N \in \mathbb{N}\},$$

where, for $N \in \mathbb{N}$ and a function f, the function f^N is defined in (2). As mentioned there, f^N is again an eigenfunction of T. However, f^N need not belong to $\mathcal{V}$. Let $\mathcal{S}_3$ be the set of finite products of elements of $\mathcal{S}_2$. In particular, all elements of $\mathcal{S}_3$ are bounded functions, and the same is true of all finite linear combinations of elements of $\mathcal{S}_3$.

Claim. *Every finite product $f_1 \cdot \ldots \cdot f_n$ with $n \in \mathbb{N}$, and f_i or $\overline{f_i}$ in $\mathcal{V} \cap L^\infty(\Omega, m)$, can be approximated arbitrarily well (in $L^2(\Omega, m)$) by finite linear combinations of elements of $\mathcal{S}_3$.*

Proof of the Claim. This is shown by induction. The case $n = 1$ is simple, as $\mathcal{S}_1$ is an orthonormal basis of $\mathcal{V}$. Assume that the claim holds for fixed $n \in \mathbb{N}$. As in Lemma 1, we use again a variant of Lee, Moody and Solomyak [21]. Let $\varepsilon > 0$ be given. By the induction assumption, there exists a finite linear combination g of elements of $\mathcal{S}_3$ with

$$\|f_1 \cdot \ldots \cdot f_n - g\|_2 \;\leq\; \frac{\varepsilon}{\|f_{n+1}\|}\,.$$

Here, g is a bounded function (as all functions in $\mathcal{S}_3$ are bounded). Thus, there exists a finite linear combination h of elements in $\mathcal{S}_3$ with

$$\|f_{n+1} - h\|_2 \leq \frac{\varepsilon}{\|g\|_\infty}.$$

Now, the proof of the claim can be finished as in Lemma 1. □

The claim shows that $\mathcal{S}_3$ is total in $L^2(\Omega, m)$ as the products appearing in its statement are total in $L^2(\Omega, m)$ by the density assumption MD. Now, obviously, the elements of $\mathcal{S}_3$ are eigenfunctions of T and the corresponding eigenvalues are just the group generated by the eigenvalues of $T|_{\mathcal{V}}$. This proves (a) and (b).

To prove (c), we consider a basis $\mathcal{S}_1$ of $\mathcal{V}$ consisting of continuous eigenfunctions of $T|_{\mathcal{V}}$ and define

$$\mathcal{S}_4 \; := \; \{f_1 \cdot \ldots \cdot f_n : n \in \mathbb{N},\; f_i \in \mathcal{S}_1 \text{ or } \overline{f_i} \in \mathcal{S}_1\}\,.$$

As above, one can show that $\mathcal{S}_4$ is total in $L^2(\Omega, m)$. Apparently, the elements in $\mathcal{S}_4$ are continuous eigenfunctions of T. Moreover, by general principles, eigenfunctions belonging to different eigenvalues are orthogonal. We now apply the Gram-Schmidt orthogonalisation procedure in each eigenspace, compare [27, Sec. 3.1.13]. This is possible as the multiplicity of each eigenspace is at most countably infinite by (c). As a result we obtain a basis of eigenfunctions which are continuous. (Note that Gram-Schmidt deals only with finite sums in each step and therefore does not destroy continuity.) □

The preceeding considerations can be applied to any TMDS. This will breifly be discussed next. To apply Theorem 8, we need the following reformulation of results of the previous sections.

Proposition 7. *Let (Ω, α) be a* TMDS *with invariant probability measure m, let $\mathcal{U}$ be the space introduced in Definition* 7 *and denote by 1_Ω the characteristic function of Ω. Then, the subspace $\mathcal{V} := \mathcal{U} + \{c\,1_\Omega : c \in \mathbb{C}\}$ is closed, invariant and satisfies the density assumption* MD.

Proof. Denote the subspace $\{c\,1_\Omega : c \in \mathbb{C}\}$ by $\mathcal{S}$. As $\mathcal{S}$ is one-dimensional and $\mathcal{U}$ is closed, $\mathcal{V} = \mathcal{U} + \mathcal{S}$ is closed as well. As $\mathcal{U}$ and $\mathcal{S}$ are T-invariant, so is $\mathcal{V}$. It remains to be shown that MD is satisfied. This is a consequence of the Stone-Weierstrass Theorem as $\{f_\varphi : \varphi \in C_c(G)\}$ separates points (see the proof of Theorem 7). □

It is possible to base the proof of our main result, Theorem 7, on Theorem 8 and the previous Proposition. Here, our focus is in a somewhat different direction.

Theorem 9. *Let (Ω, α) be a* TMDS *with invariant probability measure m, T_m be the corresponding unitary representation of G by translation operators, and $\widehat{\gamma_m}$ the associated diffraction measure. Let $\mathcal{U}$ be the space of functions defined in Definition* 7*. If $\widehat{\gamma_m}$ is a pure point measure, the following assertions hold.*

(a) *The group of eigenvalues of T_m is generated by the set of points in $\widehat{G}$ of positive $\widehat{\gamma_m}$ measure, i.e., the points $\hat{s}$ with $\widehat{\gamma_m}(\{\hat{s}\}) > 0$.*
(b) *If $\mathcal{U}$ has a basis of continuous eigenfunctions of T_m, then so has $L^2(\Omega, m)$, provided the multiplicity of each eigenvalue is at most countably infinite.*

Proof. As $\widehat{\gamma_m}$ is a pure point measure, $T|_{\mathcal{U}}$ has pure point spectrum by Theorem 6. Set $\mathcal{V} := \mathcal{U} + \{c\,1_\Omega : c \in \mathbb{C}\}$. As 1_Ω is obviously a (continuous) eigenfunction of T (to the eigenvalue 1), $T_{\mathcal{V}}$ has pure point spectrum as well. Moreover, by Proposition 7, $\mathcal{V}$ is invariant, closed and satisfies MD. Thus, the assumptions of Theorem 8 are satisfied, and the theorem follows. □

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