Generalized Model Sets and Dynamical Systems

Martin Schlottmann

Abstract. It is shown that the dynamical systems approach to the diffraction properties of model sets can be generalized to regular model sets in arbitrary σ-compact Abelian groups with arbitrary locally compact Abelian groups as internal spaces. It is then shown that these regular model sets possess pure point diffraction spectra.

Introduction

Model sets have been of interest for physicists since the observation of crystallographically anomalous Bragg diffraction spectra with characteristic symmetries in connection with so-called quasi-crystals. Occurring in different disguise as the cut-and-project method, the klotz construction, and several other brands, which have been developed independently [**KN, DK, KKL**], it turned out to be possible to consider all these methods as special cases of a construction introduced by Meyer [**Me**], dubbed "model sets".

Although the understanding of the quasicrystal diffraction spectra has been the primary goal of the application of model sets in physics, it took until the late nineties (in comparison with the discovery of quasicrystals in 1984) before the Bragg character of the diffraction of model sets was rigorously established under fairly general conditions by Hof [**H**]. Hof used a beautiful trick invented in a different context by Dworkin [**D**] which connects the diffraction spectrum with the spectrum of a specific dynamical system constructed from the point sets.

The present paper is devoted to a further generalization of this scheme which has already proved fruitful in the study of substitution tilings [**BMS**]. The class of model sets used in physical applications is usually restricted to the case of Euclidean space as "internal space". This restriction seems, in the spirit of Meyer's book, unnecessary and even unnatural, and, according to a suggestion by Moody [**M**], it is immediately clear that the theory should extend to general locally compact Abelian groups as internal spaces. This impression is confirmed by the observation that model sets with these general internal spaces can be characterized by a natural criterion which is formulated exclusively in terms of the "direct space" [**Sch**]. The latter paper gives some preliminary results mostly concerning the geometrical properties of general model sets and the existence of relative frequencies of local

---

It is my pleasure to express my thanks to M. Baake and R. V. Moody for valuable discussions and encouragement. This work was supported by the Pacific Institute of Mathematical Sciences (PIMS) and the Natural Sciences and Engineering Research Council of Canada (NSERC).

patterns. Here we continue the development by extending Hof's analysis to the general case, the main result (Thm. 4.5) establishing the pure point nature of the dynamical spectrum, and hence also of the diffraction spectrum, of regular model set in the complete generality of model sets over locally compact Abelian groups.

The paper is organized as follows. In Section 1, we collect some notation and facts on translationally bounded measures and establish two technical lemmas which seem to fit nowhere else. We decided to generalize the direct space to locally compact groups, too, so some of the functional analysis used at this point becomes a little bit abstract; the reader uncomfortable with this additional generalization may safely assume an Euclidean space as direct space and will recognize the cited constructions when restricted to the Euclidean case. Section 2 describes the construction of a topology, more precisely a uniform structure, on sets of point sets which gives a precise meaning of two point sets being close. The study of the dynamical system generated by the action of the group on a local indistinguishability (LI) class is carried out in Section 3; here, Dworkin's argument is introduced. Finally, in Section 4, we will introduce Pleasant's "torus parametrization" of LI classes [**BHP**] and apply the preceding results to model sets.

1. Translationally bounded measures

In this section, we collect some technicalities and notations concerning translationally bounded measures on locally compact Abelian groups (LCAG).

Throughout this paper, we assume that $\mathfrak{G}$ is an LCAG which is σ-compact. As usual, the commutative group operation will be written additively. The group $\mathfrak{G}$ is supposed to be equipped with an invariant (Haar) measure which will be fixed throughout the following. We denote integration with respect to this Haar measure by $\int \ldots dt$; the Haar measure of a subset D of $\mathfrak{G}$ will be written as $|D|$.

We identify the set $\mathfrak{M}(\mathfrak{G})$ of complex regular Borel measures in $\mathfrak{G}$ with the space $\mathfrak{C}_c(\mathfrak{G})^*$ of complex continuous linear functionals on the space $\mathfrak{C}_c(\mathfrak{G})$ of complex continuous functions with compact support in $\mathfrak{G}$ which is equipped with the inductive locally convex limit topology induced by the canonical embeddings $\mathfrak{C}_K(\mathfrak{G}) \to \mathfrak{C}_c(\mathfrak{G})$ where $\mathfrak{C}_K(\mathfrak{G})$ is the space of complex continuous functions with support in K for $K \subseteq \mathfrak{G}$ compact. The vague topology on $\mathfrak{M}(\mathfrak{G})$ equals the weak* topology on $\mathfrak{C}_c(\mathfrak{G})^*$, i.e., the topology $\sigma(\mathfrak{C}_c(\mathfrak{G})^*, \mathfrak{C}_c(\mathfrak{G}))$ which is the weakest topology making all the functionals $\nu \mapsto \nu(\varphi)$ continuous ($\varphi \in \mathfrak{C}_c(\mathfrak{G})$).

The identification of $\mathfrak{M}(\mathfrak{G})$ and $\mathfrak{C}_c(\mathfrak{G})^*$ generates some ambiguity in the functional notation of measures: for a Borel set A, $\nu(A)$ is the measure of A, whereas, for a function φ, $\nu(\varphi)$ is the integral of φ with respect to ν. In the following, the intended meaning will always be clear from the context, and we will get rid of a lot of integral signs this way.

For $t \in \mathfrak{G}$, $\varphi \in \mathfrak{C}_c(\mathfrak{G})$, we denote by τ_t the translate of φ by t, i.e., $\tau_t\varphi(x) = \varphi(x-t)$ ($x \in \mathfrak{G}$); similarly, if $\nu \in \mathfrak{M}(\mathfrak{G})$, we let $\tau_t\nu(\varphi) = \nu(\tau_{-t}\varphi)$ ($\varphi \in \mathfrak{C}_c(\mathfrak{G})$). Analogously, $\varphi^*(x) := \overline{\varphi(-x)}$ (the complex conjugate of $\varphi(-x)$), and $\nu^*(\varphi) := \overline{\nu(\varphi^*)}$. The convolution of a function $\varphi \in \mathfrak{C}_c(\mathfrak{G})$ with a measure ν is the continuous function $(\varphi * \nu)(t) = \nu(\tau_t\overline{\varphi^*})$. This matches the usual definition of convolution if ν is absolutely continuous (with respect to Haar measure) and represented by its Radon-Nikodym density.

A measure ν is defined to be translationally bounded iff, for every $\varphi \in \mathfrak{C}_c(\mathfrak{G})$, the set $\{ \tau_t\nu(\varphi) \mid t \in \mathfrak{G} \}$ is bounded. By the Banach-Steinhaus theorem [**T**, Thm.

33.2], this is equivalent to the condition that the set $\{ \tau_t \nu \mid t \in \mathfrak{G} \}$ is relatively compact in the vague topology; it is, furthermore, equivalent to the condition that the set $\{ \tau_t \nu \mid t \in \mathfrak{G} \}$ is strongly bounded, i.e., that, for every bounded subset B of $\mathfrak{C}_c(\mathfrak{G})$, the set $\{ \tau_t \nu(\varphi) \mid t \in \mathfrak{G},\ \varphi \in B \}$ is bounded. Because we assume $\mathfrak{G}$ to be σ-compact, we can finally conclude that ν is translationally bounded iff, for every compact $K \subseteq \mathfrak{G}$, the set $\{ |\nu|(t+K) \mid t \in \mathfrak{G} \}$ is bounded, where $|\nu|$ is the variation of ν. We will use these different characterizations of translational boundedness interchangeably.

Several results of the following sections will deal with averaging of translationally bounded measures. For this, we need "nice" sequences of compact sets over which the averaging is carried out. We choose generalized van Hove sequences as defined below for this purpose.

If C and K are non-empty compact subsets of $\mathfrak{G}$, we denote

$$\partial^K C \;:=\; (C+K)\backslash C^\circ \cup ((\overline{\mathfrak{G}\backslash C} - K) \cap C)\,. \tag{1.1}$$

The compact set $\partial^K C$ can be superficially interpreted as the set of points which have a distance measured by K from the "boundary of C"; however, this "boundary" cannot, in general, be taken in the topological sense, as in the example of a discrete $\mathfrak{G}$ the topological boundary would be always empty.

A sequence $(D_n)_{n\in\mathbb{N}}$ of compact subsets of $\mathfrak{G}$ of positive volume is a generalized *van Hove sequence* iff, for every compact $K \subseteq \mathfrak{G}$,

$$\lim_{n\to\infty} \frac{|\partial^K D_n|}{|D_n|} \;=\; 0\,. \tag{1.2}$$

The meaning of (1.2) is that the surface/bulk ratio becomes sufficiently negligible in the limit $n \to \infty$.

The existence of van Hove sequences in compactly generated Abelian groups is easily seen from the structure theorem for these groups which states that every such group is of the form $\mathfrak{G} = \mathbb{Z}^d \times \mathbb{R}^{d'} \times \mathfrak{K}$ for nonnegative integers d, d', and a compact group $\mathfrak{K}$, see [**HR**, Thm. 9.6]; then, a van Hove sequence is given by

$$D_n \;=\; \{ (\vec{m},\vec{x},k) \in \mathbb{Z}^d \times \mathbb{R}^{d'} \times \mathfrak{K} \mid |m_i| \le n,\ |x_j| \le n \}\,. \tag{1.3}$$

In the case of a general σ-compact LCAG $\mathfrak{G}$, the group $\mathfrak{G}$ can be written as a union of countably many relatively compact open subsets V_n. Because every $\overline{V_n}$ generates a compactly generated open subgroup $\mathfrak{G}_n$ of $\mathfrak{G}$, we can find a compact subset $D_n \subseteq \mathfrak{G}$ of positive volume such that $|\partial^{\overline{V_n}} D_n|/|D_n| < 1/n$. Then, the sequence $(D_n)_{n\in\mathbb{N}}$ is a van Hove sequence in $\mathfrak{G}$.

The benefit of van Hove sequences lies in the following lemma.

Lemma 1.1. *Let $(D_n)_{n\in\mathbb{N}}$ be a van Hove sequence in $\mathfrak{G}$, and ν a translationally bounded measure.*

1) *For every compact K,*

$$\lim_{n\to\infty} \frac{|\nu|(\partial^K D_n)}{|D_n|} \;=\; 0 \tag{1.4}$$

2) *The sequence $(\frac{|\nu|(D_n)}{|D_n|})_{n\in\mathbb{N}}$ is bounded.*

3) *If φ is a continuous function with compact support, then*

$$\lim_{n\to\infty} \frac{1}{|D_n|}\Big((\varphi * \nu)(D_n) - \int \varphi(t)dt \cdot \nu(D_n) \Big) \;=\; 0 \tag{1.5}$$

PROOF:

1) There is a compact subset C of $\mathfrak{G}$ and a discrete closed $L \subseteq \mathfrak{G}$ such that $\mathfrak{G} = L + C$ and such that different $t + C$ $(t \in L)$ overlap at most in a set of zero Haar measure. For example, take D_1 from (1.3) for C and an appropriate discrete subgroup for L in the case of a compactly generated $\mathfrak{G}$; the general case is handled by doing this for an open compactly generated subgroup and translating into cosets.

   Because ν is translationally bounded, there is a constant $c \geq 0$ such that $|\nu|(t + C) \leq c \cdot |t + C|$ independently of $t \in L$. For every n, let $W_n := \bigcup \{ t+C \mid t \in L, (t+C) \cap \partial^K D_n \neq \emptyset \}$; this set is the union of finitely many $t + C$ and $\partial^K D_n \subseteq W_n \subseteq \partial^{K+C-C} D_n$. Then $|\nu|(W_n) \leq c \cdot |W_n|$, therefore, $|\nu|(\partial^K D_n) \leq c \cdot |\partial^{K+C-C} D_n|$, and (1.4) follows because $(D_n)_{n\in\mathbb{N}}$ is a van Hove sequence.
2) In the same fashion, we find W_n with $|\nu|(W_n) \leq c \cdot |W_n|$ and $D_n \subseteq W_n \subseteq D_n \cup \partial^{C-C} D_n$; the assertion follows from 1).
3) One calculates:

$$
\begin{aligned}
&|(\varphi * \nu)(D_n) - \int \varphi(t)dt \cdot \nu(D_n)| \\
&= \ |\int \int_{D_n - s} \varphi(t)dt\nu(ds) - \int_{D_n} \int \varphi(t)dt\nu(ds)| \\
&= \ |\int_{\mathfrak{G}\setminus D_n} \int_{D_n - s} \varphi(t)dt\nu(ds) + \int_{D_n} \int_{\mathfrak{G}\setminus(D_n - s)} \varphi(t)dt\nu(ds)| \\
&\leq \ \int_{(D_n+K)\setminus D_n} \int |\varphi(t)|dt|\nu|(ds) + \int_{(\mathfrak{G}\setminus D_n - K)\cap D_n} \int |\varphi(t)|dt|\nu|(ds) \\
&\leq \ \int |\varphi(t)|dt \cdot |\nu|(\partial^K D_n)
\end{aligned}
$$

   for K containing the support of φ. □

The point in introducing van Hove sequences is that later on, when considering autocorrelation measures, we would like to form the convolution of translationally bounded measures with themselves, a usually ill-defined operation. Instead, one does that with finite portions of these measures located on members of van Hove sequences and normalizes by the volume of the van Hove set. More precisely: for a measure ν, we denote the trace of ν on a measurable subset D by ν_D which is defined by the expression

$$\nu_D(\varphi) \ := \ \int_D \varphi \, d\nu \, . \tag{1.6}$$

A measure γ is said to be the (unique) *autocorrelation* of the translationally bounded measure ν iff, for every van Hove sequence $(D_n)_{n\in\mathbb{N}}$, the sequence $\frac{1}{|D_n|}(\nu_{D_n})^* * \nu_{D_n}$ converges vaguely to γ. Because ν_{D_n} has compact support, the formation of this convolution does not pose a problem. One major problem will be to establish that certain translationally bounded measures arising from certain point sets in $\mathfrak{G}$ have such autocorrelation measures. As a tool for this, we note

LEMMA 1.2. *If μ, ν are translationally bounded and $(D_n)_{n\in\mathbb{N}}$ is a van Hove sequence, then we have, in the vague topology,*

$$\frac{1}{|D_n|}(\mu_{D_n} * \nu_{D_n} - \mu * \nu_{D_n}) \longrightarrow 0 \qquad (n \to \infty)\,. \tag{1.7}$$

PROOF: A short calculation shows that, for $\varphi \in \mathfrak{C}_c(\mathfrak{G})$ with support in a compact subset K,

$$|(\mu_{D_n} * \nu_{D_n})(\varphi) - (\mu * \nu_{D_n})(\varphi)| \;\le\; \int_{\partial^K D_n} (|\varphi| * |\mu|)(t)|\nu|(dt)\,. \tag{1.8}$$

Because μ is translationally bounded, the expression on the right side is smaller than $c\cdot|\nu|(\partial^K D_n)$ for a suitable constant c. Now, the lemma follows from (1.5). □

If a measure ν has an autocorrelation measure γ, then the latter is clearly positive definite (cf. [**AG**, p. 23]). In particular, it has a well-defined Fourier transform $\hat{\gamma}$ which is a positive measure on the dual group $\hat{\mathfrak{G}}$ (for the definition of Fourier transform for unbounded measures the reader may refer to [**AG**, p. 8]). Because in diffraction physics the latter describes the kinematic diffraction pattern of a structure modelled with a measure ν, we will call this Fourier transform $\hat{\gamma}$ the *diffraction measure* of ν. (The reader interested in a more detailed description of the connection between diffraction and Fourier transform may consult any textbook on crystallography, e.g, [**C**].) Note that in our general setup there is no well-defined possibility to build the Fourier transform of ν itself even if ν is fairly regular.

As a positive measure, $\hat{\gamma}$ can be decomposed according to Lebesgue's decomposition theorem. We are particularly interested in the point part of $\hat{\gamma}$ because this is the part which corresponds with so-called Bragg diffraction pattern, an important tool in the analysis of ordered structures [**C**].

2. Uniform structure on sets of point sets

In the following, we denote by $\mathfrak{D}$ the set of all closed and discrete subsets of $\mathfrak{G}$. Our first task is to define a topology, more specifically a uniformity (cf. [**K**]) on $\mathfrak{D}$ which makes it a complete uniform Hausdorff space. For this, we have to define what it means for two point sets, P_1, P_2, to be close; basically P_1 will be close to P_2 if, after a small translation, these two coincide on a large compact region.

In the following, we will use the following notations for operations on sets of ordered pairs:

$$U^{-1} \;:=\; \{\,(x,y) \mid (y,x) \in U\,\} \tag{2.1}$$

$$U \circ V \;:=\; \{\,(x,z) \mid \exists\, y : (x,y) \in U \text{ and } (y,z) \in V\,\} \tag{2.2}$$

$$U[x] \;:=\; \{\, y \mid (x,y) \in U\,\}. \tag{2.3}$$

Now, we set, for compact $K \subseteq \mathfrak{G}$ and a neighbourhood V of 0:

$$U_{K,V} \;:=\; \{\,(P_1,P_2) \in \mathfrak{D}\times\mathfrak{D} \mid \exists\, s \in V :\; P_1 \cap K = (s + P_2) \cap K\,\}\,. \tag{2.4}$$

It is immediately verified that $(P,P) \in U_{K,V}$, $U_{K_1\cup K_2, V_1\cap V_2} \subseteq U_{K_1,V_1} \cap U_{K_2,V_2}$; furthermore, for a compact neighbourhood $W \subseteq -V$ of 0, $U_{K+W,W} \subseteq (U_{K,V})^{-1}$; if we take a compact neighbourhood W of 0 such that $W + W \subseteq V$, we calculate $U_{K-W,W} \circ U_{K-W,W} \subseteq U_{K,V}$. This shows that the set of $U_{K,V}$ as defined in (2.4) forms a base of a uniform structure [**K**] on $\mathfrak{D}$; the topology on $\mathfrak{D}$ induced by this uniformity is given by taking the sets $U_{K,V}[P]$ as a base of neighbourhoods of $P \in \mathfrak{D}$.

This uniform structure separates points: if $(P_1, P_2) \in U_{K,V}$ for all K, V, then, for any K: $P_1 \cap K \subseteq \overline{P_2} = P_2$, hence $P_1 \subseteq P_2$. By symmetry, $P_1 = P_2$. Hence $\mathfrak{D}$ becomes a uniform Hausdorff space.

From the definition of the topology on $\mathfrak{G}$, it is easy to conclude that the action

$$\mathfrak{G} \times \mathfrak{D} \longrightarrow \mathfrak{D}, \quad (t, P) \mapsto t + P \tag{2.5}$$

is continuous (in both variables simultaneously). What is more, its restriction to $K \times \mathfrak{D}$ is uniformly continuous for every compact subset K of $\mathfrak{G}$ (one calculates that, if $K, K' \subseteq \mathfrak{G}$ compact, V, V' are neighbourhoods of 0 with $V' + V' \subseteq V$, then, if $t_1, t_2 \in K$ with $t_1 - t_2 \in V'$ and $P_1, P_2 \in \mathfrak{D}$ with $(P_1, P_2) \in U_{K'-K,V'}$, then $(t_1 + P_1, t_2 + P_2) \in U_{K',V}$).

Proposition 2.1. *$\mathfrak{D}$ is complete in the uniformity defined by eq.* (2.4).

Proof: Let $(P_i)_{i \in I}$ be a Cauchy net in $\mathfrak{D}$ for some index set I directed by a relation $\leq$; we show that a subnet converges in $\mathfrak{D}$. Without loss of generality, we may assume that there is a compact $K_0 \subseteq \mathfrak{G}$ such that $P_i \cap K_0 \neq \emptyset$, for all $i \in I$, because otherwise we can find a subnet converging to $\emptyset \in \mathfrak{D}$. We pick one $t_i \in P_i \cap K_0$ for each $i \in I$; passing to a subnet, we can assume that t_i converge to some t. It is clear that $(-t_i + P_i)_{i \in I}$ form a Cauchy net in $\mathfrak{D}$ and convergence of $(-t_i + P_i)_{i \in I}$ entails convergence of $(P_i)_{i \in I}$. Hence, we may assume that $0 \in P_i$ for all $i \in I$.

We may, furthermore, assume that, for some compact neighbourhood V_0 of 0 and some $i_0 \in I$, every $P_i \cap V_0$ equals some $(s_i + P_{i_0}) \cap V_0 = s_i + P_{i_0} \cap (-s_i + V_0)$ for suitably chosen $s_i \in V_0$. As $0 \in P_i \cap V_0$, this s_i is in $(-P_{i_0}) \cap V_0$, therefore all $P_i \cap V_0 \subseteq P_{i_0} \cap (V_0 - V_0) - P_{i_0} \cap (V_0 - V_0)$ which is a finite set. We can, therefore, choose some neighbourhood V_1 of 0 such that $P_i \cap V_1 = \{0\}$ for all $i \in I$.

As a consequence, if $K \subseteq \mathfrak{G}$ is compact and contains 0, and if $(P_i, P_j) \in U_{K,V_1}$, then necessarily $P_i \cap K = P_j \cap K$. The Cauchy property of the net $(P_i)_{i \in I}$ can therefore be strengthened: for every compact $K \subseteq \mathfrak{G}$, there is some $i_K \in I$ such that, if $i, j \geq i_K$, then $P_i \cap K = P_j \cap K$.

It is now clear that $(P_i)_{i \in I}$ converges to the closed discrete set P as defined by $P := \bigcup \{\, P_{i_K} \cap K \mid K \subseteq \mathfrak{G} \text{ compact} \,\}$. ☐

Of particular interest are those point sets whose $\mathfrak{G}$-orbits are precompact and which therefore, after closure, form well-behaved dynamical systems. These sets have a nice geometrical characterization.

Definition 2.2. A point set P in $\mathfrak{G}$ is of *finite local complexity* (FLC) iff $P - P$ is closed and discrete.

The term "finite local complexity" is justified by property 3) in the following proposition which basically states that, up to translations, there are only finitely many patterns of any given finite size in P.

Proposition 2.3. *Let $P \in \mathfrak{D}$. The following are equivalent:*

1) *P is of finite local complexity,*
2) *The $\mathfrak{G}$-orbit $[P]_{\mathfrak{G}} := \{\, t + P \mid t \in \mathfrak{G} \,\}$ of P is precompact.*
3) *For every compact $K \subseteq \mathfrak{G}$, there is a compact $K' \subseteq \mathfrak{G}$ such that, for every $t \in \mathfrak{G}$, there is some $t' \in K'$ with $(t + P) \cap K = (t' + P) \cap K$.*

PROOF:

1)⇒2) For any compact $K \subseteq \mathfrak{G}$ and any neighbourhood V of 0, the set $F_K := (P-P) \cap (K-K)$ is finite, because $P-P$ is closed and discrete. We can, furthermore, find a finite set $E_K \subseteq \mathfrak{G}$ such that $K \subseteq E_K + V$. A simple calculation shows that $t+P \in \bigcup \{ U_{K,V}^{-1}[s+F] \mid s \in E_K,\ F \subseteq F_K \}$, for any $t \in \mathfrak{G}$: if $(t+P) \cap K = \emptyset$, there is nothing to show; otherwise, choose $p \in P$ such that $t+p \in K$, find $s \in E_K$ and $r \in V$ such that $r+s = t+p$ and observe $(t+P) \cap K = (r+s+(P-p)) \cap K = r+s+(P-p) \cap (K-(s+r)) = (r+s+F) \cap K$ for $F := (P-p) \cap (K-(s+r)) \subseteq (P-P) \cap (K-K)$, i.e., $t+P \in U_{K,V}^{-1}[s+F]$. The assertion follows.

2)⇒3) If $[P]_{\mathfrak{G}}$ is precompact, for given K we choose some compact neighbourhood V of 0 and $t_1 \dots t_l \in \mathfrak{G}$ such that, given $t \in \mathfrak{G}$, we have $t+P \in U_{K,V}[t_i+P]$, for some suitable t_i. This means $(t+P) \cap K = (s+t_i+P) \cap K$, for some $s \in V$. Then, $K' := \bigcup_{i=1}^{l} (t_i + V)$ fulfils the requirement of 3).

3)⇒1) Let K be compact, K' chosen for K according to 3). We may assume that $0 \in K$. Let $q \in (P-P) \cap K$, for example $q_1, q_2 \in P$ with $q = q_1 - q_2$. Then, we find $t' \in K'$ such that $q_1 - q_2 \in (-q_2+P) \cap K = (t'+P) \cap K = P \cap (K-t') + t'$, and, because $0 \in (-q_2+P) \cap K$, $-t'$ is an element of P, and therefore $q = q_1 - q_2 \in P \cap (K-K') - P \cap (-K')$, which is a finite set. □

In the rest of this paper, unless stated otherwise, we will deal only with sets $P \subseteq \mathfrak{G}$ of finite local complexity (FLC sets). The closure of the $\mathfrak{G}$-orbit $[P]_{\mathfrak{G}}$ in $\mathfrak{D}$ will be denoted by X_P. A consequence of the previous proposition is that, for an FLC set P, the topological space X_P is a compact Hausdorff space, because it is the closure of a precompact set in a complete uniform Hausdorff space.

Proposition 2.3 is an example of the intimate connection of geometrical properties of P with certain topological properties of the space X_P; further examples of this will show up in the sequel. In order to study the topology of X_P, it is useful to determine the algebra $\mathfrak{C}(X_P)$ of complex continuous functions on X_P. We note that, under the present assumptions, P may have arbitrarily large holes, i.e., compact regions of arbitrary size in $\mathfrak{G}$ which contain no points of P. We don't want to exclude this class of sets from the start, because it contains interesting examples like the set of visible points in a lattice (cf. [**A**]). In these cases, the empty set $\emptyset$ will be an element of X_P. In the following description of $\mathfrak{C}(X_P)$, this turns out to be rather inconvenient, so we consider the locally compact space $X_P \backslash \{\emptyset\}$ and the algebra $\mathfrak{C}_0(X_P \backslash \{\emptyset\})$ of complex continuous functions vanishing at ∞ on $X_P \backslash \{\emptyset\}$ instead. Of course, if $\emptyset \notin X_P$, then $\mathfrak{C}(X_P) = \mathfrak{C}_0(X_P \backslash \{\emptyset\})$; on the other hand, if $\emptyset \in X_P$, then $\mathfrak{C}_0(X_P \backslash \{\emptyset\})$ can be identified with the algebra of those continuous functions on X_P which vanish at $\emptyset$.

In the following, we fix some FLC set $P_0 \in \mathfrak{D}$. First, we introduce some notation.

DEFINITION 2.4. For $t \in \mathfrak{G}$ and compact $K \subseteq \mathfrak{G}$ with $(t+P_0) \cap K \neq \emptyset$, we set

$$T_{K,t}(P) := \{ s \in \mathfrak{G} \mid (s+P) \cap K = (t+P_0) \cap K \}, \tag{2.6}$$

for any $P \in X_{P_0}$. Furthermore, if $\varphi \in \mathfrak{C}_c(\mathfrak{G})$ (continuous functions on $\mathfrak{G}$ with compact support), we define the following function on X_{P_0}:

$$\varphi_{K,t}(P) := \sum_{s \in T_{K,t}(P)} \varphi(-s) \qquad (P \in X_{P_0}) \tag{2.7}$$

Later on, the case $T_{\{0\},-t} = -P$ for $t \in P$ will be of particular interest. Obviously, $T_{K,t}$ is a subset of a translate of $-P$, therefore, $\varphi_{K,t}$ is well-defined as, for each P, the sum in (2.7) has only finitely many nonzero terms. Furthermore, if $\emptyset \in X_{P_0}$, then $\varphi_{K,t}(\emptyset) = 0$. In view of the definition of the topology on X_{P_0} we see easily that $\varphi_{K,t}$ is continuous, because, if P_1 and P_2 coincide on a large enough compact set, then $T_{K,t}(P_1)$ and $T_{K,t}(P_2)$ coincide on the support of φ; on the other hand, if $P_1 = s + P_2$ for a small s then $\varphi_{K,t}(P_1)$ and $\varphi_{K,t}(P_2)$ can differ only by a small amount (see (2.7)).

The promised description of $\mathfrak{C}_0(X_{P_0}\backslash\{\emptyset\})$ is given in the following proposition.

Proposition 2.5. *The set of all functions of the form* (2.7) *forms a total subset of* $\mathfrak{C}_0(X_{P_0}\backslash\{\emptyset\})$, *i.e. every complex continuous function on* X_{P_0} *vanishing at* $\emptyset$ *can be uniformly approximated by linear combinations of suitable* $\varphi_{K,t}$.

Proof: We use the Stone-Weierstraß approximation theorem [**La**, Ch. III, §1, Thm. 1.4]. For this, we have to show that, 1) if $\emptyset \neq P_1 \neq P_2$, there are φ, K, and t such that $\varphi_{K,t}(P_1) = 1$ and $\varphi_{K,t}(P_2) = 0$ (separation property); and 2) the set of linear combinations of $\varphi_{K,t}$'s form an subalgebra of $\mathfrak{C}(X_{P_0})$.
1) We choose $q_0 \in P_1 \Delta P_2$ (symmetric difference), $q_1 \in P_1$ and a neighbourhood V of 0 in G such that $(q_1 - V) \cap (P_1 \cup P_2) = \{q_1\}$. We take a $\varphi \in \mathfrak{C}_c(\mathfrak{G})$ with support in V and $\varphi(0) = 1$. Let $K = \{q_0, q_1\}$ and choose $t \in \mathfrak{G}$ such that $P_1 \cap K = (t + P_0) \cap K$. Now, if $s \in V$ and $(s + P_i) \cap K = P_1 \cap K$ for $i = 1$ or $i = 2$, then necessarily $s = 0$ (by the choice of V). By construction, $P_1 \cap K \neq P_2 \cap K$. Therefore, $\varphi_{K,t}$ has the asserted properties.
2) We have to show that the product of two functions of the form (2.7) is a linear combination of suitable $\varphi_{K,t}$'s. Let, for $i = 1, 2$,

$$\Phi^i(P) \;=\; \sum_{s \in T_{K_i,t_i}(P)} \varphi^i(s) \qquad (P \in X_{P_0}) \tag{2.8}$$

(skipping the sign under φ which is irrelevant for the moment). Setting $F_i := (t_i + P_0) \cap K_i$ $(i = 1, 2)$ which are finite sets, we calculate:

$$\Phi^1(P)\Phi^2(P) \;=\; \sum_{s_1, s_2} \varphi^1(s_1)\varphi^2(s_2) \tag{2.9}$$

where the sum is taken over all s_1, s_2 fulfilling $(s_i + P) \cap K_i = F_i$ $(i = 1, 2)$. Note that such a pair (s_1, s_2) necessarily fulfils $s_1 - s_2 \in F_1 - F_2 + P_0 - P_0$.

Let K be compact containing the supports of both, φ^1 and φ^2, in its interior; then, $F := (F_1 - F_2 + P_0 - P_0) \cap (K - K)$ is finite, and we find a compact neighbourhood V of 0 with $(F - F) \cap (V + V) \subseteq \{0\}$.

Because we can always decompose φ^i into a sum of continuous functions with arbitrarily small supports in K, we may assume without loss of generality that $\mathrm{supp}(\varphi^i) - \mathrm{supp}(\varphi^i) \subseteq V$; then $V' - V' \subseteq V + V$ for $V' := \mathrm{supp}(\varphi^1) - \mathrm{supp}(\varphi^2)$ and, hence, $V' \cap F \subseteq \{s_0\}$ for some $s_0 \in \mathfrak{G}$. As a consequence, any pair (s_1, s_2) contributing at all to the sum (2.9) must fulfil $s_1 - s_2 = s_0$.

If we now set $F_0 := F_1 \cup (s_0 + F_2)$ and $K_0 := K_1 \cup (s_0 + K_2)$, then $s_1 - s_2 \in V'$ and $(s_i + P) \cap K_i = F_i$ for $i = 1, 2$ if and only if $s_1 - s_2 = s_0$ and $(s_1 + P) \cap K_0 = F_0$. Now, either there is no $t_0 \in \mathfrak{G}$ such that $(t_0 + P_0) \cap K_0 = F_0$, and then $\Phi^1\Phi^2$ vanishes, or we can choose such a t_0 and see that $\Phi^1\Phi^2 = \psi_{K_0,t_0}$ setting $\psi(s) = \varphi^1(-s)\varphi^2(-s - s_0)$. $\square$

The functions of the form (2.7) can be expressed in terms of convolution. For this, we associate to every discrete closed point set $P \subseteq \mathfrak{G}$ the uniquely determined measure ν^P defined by

$$\nu^P(\varphi) := \sum_{t \in P} \varphi(t) \qquad (\varphi \in \mathfrak{C}_c(\mathfrak{G})), \tag{2.10}$$

i.e., $\nu^P = \sum_{t \in P} \delta_t$ where δ_t is the unit point measure concentrated at t. It is clear that ν^P is translationally bounded for FLC sets P. For $K \subseteq \mathfrak{G}$ compact, $t, t' \in \mathfrak{G}$, $\varphi \in \mathfrak{C}_c(\mathfrak{G})$, and $P \in X_{P_0}$, we calculate,

(2.11)

$$\varphi_{K,t}(t' + P) \;=\; \sum_{s \in T_{K,t}(P)} \varphi(t' - s) \;=\; \nu^{T_{K,t}(P)}(\tau_{t'}\overline{\varphi^*}) \;=\; (\varphi * \nu^{T_{K,t}(P)})(t'),$$

a formula which will come in handy when $\varphi_{K,t}$ is studied on a single translation orbit.

3. The dynamical system of an FLC set

We now take a closer look at the dynamical system $(X_P, \mathfrak{G})$ given by the canonical action (2.5) of $\mathfrak{G}$ on X_P. As usual, we assume P to be an FLC set.

Let us first give a geometric description of the elements of X_P. By definition, $P' \in X_P$ iff P' can be approximated by translates $t + P$ in the uniform topology defined in (2.4). Expanding the definition, we see that $P' \in X_P$ iff, for every compact $K \subseteq \mathfrak{G}$, there is a $t \in \mathfrak{G}$ such that $P' \cap K = (t + P) \cap K$.

This is one half of the condition of "local indistinguishability" of P' and P. That is, P' and P are locally indistinguishable iff $P' \in X_P$ and $P \in X'_P$, i.e., $X_P = X'_P$. In terms of dynamical systems: $(X_P, \mathfrak{G})$ is minimal iff every $P' \in X_P$ is locally indistinguishable from P, or, equivalently, iff the equivalence class of P with respect to local indistinguishability, the so-called LI class $[P]_{LI}$ of P, coincides with X_P. The last condition clearly is equivalent to compactness of $[P]_{LI}$.

Now, there is a nice geometric interpretation of compactness of $[P]_{LI}$. For this, we introduce the simplified notation

$$T_K(P) := \{\, t \in \mathfrak{G} \mid (t + P) \cap K = P \cap K \,\}. \tag{3.1}$$

(This is a special case of (2.6) for $P = P_0$ and $t_0 = 0$.)

Recall that a subset S of $\mathfrak{G}$ is relatively dense iff there is a compact $K \subseteq \mathfrak{G}$ such that $S + K = \mathfrak{G}$. We have now the following equivalences:

Proposition 3.1. *For an FLC set P, the following statements are equivalent:*

1) *$T_K(P)$ is relatively dense for every compact K*
2) *$X_P = [P]_{LI}$*
3) *P is repetitive , i.e., for every compact $K \subseteq \mathfrak{G}$, there is a compact $K' \subseteq \mathfrak{G}$ such that, for every $t_1, t_2 \in \mathfrak{G}$, there is an $s \in K'$ with $(t_1 + P) \cap K = (s + t_2 + P) \cap K$.*

Proof: For any compact neighbourhood V of 0, one calculates that $T_K(P) \subseteq \{\, t \in \mathfrak{G} \mid (t + P, P) \in U_{K,V} \,\} \subseteq T_K(P) - V$. Therefore, 1) is true iff P is an almost periodic point of the dynamical system $(X_P, \mathfrak{G})$. As already noted, 2) holds iff $(X_P, \mathfrak{G})$ is minimal. Therefore, 1)$\Leftrightarrow$2) follows from general theorems on dynamical systems [**E**, Prop. 2.5, p. 10], sometimes referred to as Gottschalk's theorem.

Furthermore, 1) is a special case of 3), so 3)$\Rightarrow$1).

Assume now that 1) holds and let $K \subseteq \mathfrak{G}$ be compact. Because P is an FLC set, with Prop. 2.3 3) we obtain a compact K_1 such that for all $t_1 \in \mathfrak{G}$, there is $t' \in K_1$ with $(t + P) \cap K = (t' + P) \cap K$. According to assumption 1) we find a compact K_2 such that, for every $t_2 \in \mathfrak{G}$, there is some $t'' \in K_2$ with $(t''+t_2+P)\cap(K-K_1) = P\cap(K-K_1)$. Because $-t_1+K \subseteq K-K_1$, we calculate $(t_2 + t' + t'' + P) \cap K = (t_1 + P) \cap K$. As $t_1 + t_2 \in K_1 + K_2$, 3) is fulfilled for $K' := K_1 + K_2$. □

We now move from the topological aspects of the dynamical system $(X_{P_0}, \mathfrak{G})$ to the study of measure theoretic properties. The following theorem connects unique ergodicity of $(X_{P_0}, \mathfrak{G})$ to a property which admits a direct geometric interpretation. Let $(D_n)_{n\in\mathbb{N}}$ be a fixed van Hove sequence in $\mathfrak{G}$.

THEOREM 3.2. *The dynamical system $(X_{P_0}, \mathfrak{G})$ is uniquely ergodic iff, for every compact $K \subseteq \mathfrak{G}$ and every $t_0 \in \mathfrak{G}$, there is a constant c_{K,t_0} such that*

$$\lim_{n\to\infty} \frac{1}{|D_n|}\nu^{T_{K,t_0}(P)}(D_n) \;=\; c_{K,t_0}\,, \tag{3.2}$$

in the vague topology, independently of $P \in X_{P_0}$.

In this case, (3.2) *holds for every van Hove sequence in $\mathfrak{G}$, the constants c_{K,t_0} being independent of the choice of the sequence.*

PROOF: Because of (1.5) in connection with (2.11), the condition (3.2) is equivalent to

$$\lim_{n\to\infty} \frac{1}{|D_n|}\int_{D_n} \varphi_{K,t_0}(t+P)dt \;=\; c_{K,t_0}\int \varphi(t)dt\,, \tag{3.3}$$

for all $\varphi \in \mathfrak{C}_c(\mathfrak{G})$.

Because the set of φ_{K,t_0} is total in $\mathfrak{C}(X_{P_0})$ (Proposition 2.5), by a 3ϵ argument we see that the existence and independence of P of the limit in (3.3) for all φ_{K,t_0} is equivalent to the existence and independence of P of

$$\lim_{n\to\infty} \frac{1}{|D_n|}\int_{D_n} \Phi(t+P)dt\,, \tag{3.4}$$

for every $\Phi \in \mathfrak{C}(X_{P_0})$. By the pointwise ergodic theorem for continuous functions, this is equivalent to unique ergodicity of the dynamical system $(X_{P_0}, \mathfrak{G})$. (For the pointwise ergodic theorem for continuous functions, see, e.g., [**F**, Thm. 3.5]; the theorem given there is immediately generalizable to the present situation).

In particular, in the uniquely ergodic case, the left side of (3.3) equals $\mu(\varphi_{K,t_0})$ independently of the choice of the van Hove sequence, where we write μ for the uniquely determined $\mathfrak{G}$-invariant probability measure on X_{P_0}. □

The expression $\nu^{T_{K,t_0}(P)}(D_n)$ is just the number of points of $T_{K,t_0}(P)$ which are located inside D_n. Theorem 3.2, therefore, states that $(X_{P_0}, \mathfrak{G})$ is uniquely ergodic iff all $T_{K,t_0}(P)$ have a well-defined average density (or frequency) independently of $P \in X_{P_0}$ when averaged over a van Hove sequence. In view of the definition (2.6), the frequency of $T_{K,t_0}(P)$ can be interpreted as the frequency of the finite pattern $(t_0 + P_0) \cap K$ in P. Hence, informally stated,

COROLLARY 3.3. *The dynamical system $(X_{P_0}, \mathfrak{G})$ is uniquely ergodic iff the frequencies of finite patterns exist in every $P \in X_{P_0}$ and are independent of the particular choice of P.* □

One can show that frequencies of finite patterns exist and coincide for arbitrary $P \in X_{P_0}$ for a given van Hove sequence iff they exist and coincide only in P_0, but for arbitrary van Hove sequences. This way, one can connect this Corollary with similar results in [**LP**].

In the uniquely ergodic case, we can prove the existence of the autocorrelation measure by an argument due to Dworkin [**D**]. We denote the uniquely determined $\mathfrak{G}$-invariant probability measure on X_{P_0} by μ. The action of $\mathfrak{G}$ on X_{P_0} induces a strongly continuous unitary representation u_t of $\mathfrak{G}$ on $L^2(\mu)$ by

$$(u_t\xi)(s) \;:=\; \xi(s-t) \qquad (\xi \in L^2(\mu), s, t \in \mathfrak{G})\,. \tag{3.5}$$

We denote the scalar product on $L^2(\mu)$ by $\langle \cdot, \cdot \rangle_\mu$.

THEOREM 3.4. *If $(X_{P_0}, \mathfrak{G})$ is uniquely ergodic, the autocorrelation of $P \in X_{P_0}$,*

$$\gamma^P \;:=\; \lim_{n\to\infty} \frac{1}{|D_n|}(({\nu^P}_{D_n})^* * {\nu^P}_{D_n})\,, \tag{3.6}$$

exists in the vague sense and is independent of the particular choice of P and the van Hove sequence $(D_n)_{n\in\mathbb{N}}$.

Furthermore, for $\varphi, \psi \in \mathfrak{C}_c(\mathfrak{G})$, the equation

$$(\psi^* * \varphi * \gamma^P)(t) \;=\; \langle \psi_{\{0\},-t_0}, u_{-t}\varphi_{\{0\},-t_0}\rangle_\mu \tag{3.7}$$

holds for any $t_0 \in P_0$, where $\psi_{\{0\},-t_0}$ and $\varphi_{\{0\},-t_0}$ are continuous functions on X_{P_0} according to (2.7).

PROOF: Choose some $P \in X_{P_0}$. In order to make the notation more readable, we let $\nu := \nu^P$. Using (1.7) three times, we see that we have only to show that

$$\frac{1}{|D_n|}((\nu^*)_{D_n} * \nu)$$

converges vaguely to some limit independently of P.

Let φ and ψ be some elements of $\mathfrak{C}_c(\mathfrak{G})$. We want to show that

$$\frac{1}{|D_n|}(\psi^* * \varphi * (\nu^*)_{D_n} * \nu))(t) \;=\; \frac{1}{|D_n|}((\nu^*)_{D_n} * (\varphi * \nu))(\tau_t\overline{\psi})$$

converges pointwise to a function on $\mathfrak{G}$ which is independent of P. Using (1.7) again, we are reduced to show that

$$\frac{1}{|D_n|}(\nu^* * (\varphi * \nu)_{D_n})(\tau_t\overline{\psi}) \;=\; \frac{1}{|D_n|}((\psi * \nu)^* * (\varphi * \nu)_{D_n})(t)$$

converges (independently of P). We choose some $t_0 \in P_0$; let $\Phi := \varphi_{\{0\},-t_0}$ and $\Psi := \psi_{\{0\},-t_0}$ and observe that $\Phi(t+P) = (\varphi * \nu)(t)$ and $\Psi(t+P) = (\psi * \nu)(t)$ according to (2.11).

$$\begin{aligned}
\frac{1}{|D_n|}((\psi * \nu)^* * (\varphi * \nu)_{D_n})(t) \quad &= \quad \frac{1}{|D_n|}\int_{D_n} (\psi * \nu)^*(t-s)(\varphi * \nu)(s)ds \\
&= \quad \frac{1}{|D_n|}\int_{D_n} \overline{\Psi(s-t+P)}\Phi(s+P)ds \\
&\overset{n\to\infty}{\longrightarrow} \quad \int \overline{\Psi(-t+P')}\Phi(P')\mu(dP') \\
&= \quad \langle \Psi, u_{-t}\Phi\rangle_\mu
\end{aligned} \tag{3.8}$$

using the pointwise ergodic theorem for continuous functions.

In particular, $\frac{1}{|D_n|}((\nu_{D_n})^* * \nu)$ converges pointwise on the subspace of $\mathfrak{C}_c(\mathfrak{G})$ generated by functions of the form $\varphi * \psi$ which is a total set in $\mathfrak{C}_c(\mathfrak{G})$; furthermore, the limit is independent of $P \in X_{P_0}$. A short calculation shows that, for arbitrary $\varphi \in \mathfrak{C}_c(\mathfrak{G})$, $|((\nu_{D_n})^* * \nu)(\varphi)| \leq \nu(D_n) \cdot \sup_{t \in \mathfrak{G}}(|\varphi| * \nu)(t)$. With Lemma 1.1, we can conclude that $\frac{1}{|D_n|}((\nu_{D_n})^* * \nu)$ is weakly and, therefore, strongly bounded. Hence, convergence on a total set entails convergence on all of $\mathfrak{C}_c(\mathfrak{G})$. It follows also that the vague limit is independent of $P \in X_{P_0}$.

Equation (3.7) is now a simple consequence of (3.8). □

Equation (3.7) connects the unitary representation u_t of $\mathfrak{G}$ and the autocorrelation γ^P. In particular, we can say more about the Fourier transform of γ^P, i.e., the diffraction properties of P_0 in terms of (3.7). By Stone's (or SNAG) theorem [**RN**, § 140], there is a projection valued measure E (the spectral resolution of u_t) on the dual group $\hat{\mathfrak{G}}$ of $\mathfrak{G}$ such that, for every $\zeta, \xi \in L^2(\mu)$, the function $t \mapsto \langle \zeta, u_t \xi \rangle_\mu$ is the Fourier transform of the measure $A \mapsto \langle \zeta, E(A)\xi \rangle_\mu$ (we identify $\mathfrak{G}$ and the dual of $\hat{\mathfrak{G}}$). If we allow for the sign in (3.7) we obtain that, for $\varphi, \psi \in \mathfrak{C}_c(\mathfrak{G})$, the inverse Fourier transform of $A \mapsto \langle \psi_{\{0\},-t_0}, E(A)\varphi_{\{0\},-t_0} \rangle_\mu$ is the function $\psi^* * \varphi * \gamma^P$. Hence (cf. [**AG**]),

$$\overline{\hat{\psi}}\hat{\varphi}\hat{\gamma}^P \;=\; \langle \psi_{\{0\},-t_0}, E(\cdot)\varphi_{\{0\},-t_0} \rangle_\mu \tag{3.9}$$

Because in general the functions of the form $\varphi_{\{0\},-t_0}$ do not form a total set in the Hilbert space $L^2(\mu)$ (cf. 2.5), we do not get a one-to-one correspondence between the diffraction measure $\hat{\gamma}^P$ and the dynamical spectrum E. For example, it is not clear that $\hat{\gamma}^P$ has a non-trivial discrete component solely on the grounds that E has such a component, if the latter turns out to be the case. But, from (3.9), we can certainly conclude that $\hat{\gamma}^P$ is pure point if E is pure point; this will be used in the next section. Of course, it is equally possible to conclude that $\hat{\gamma}^P$ has no non-trivial discrete component if E has none.

4. The torus parametrization

In the present section we show how, for a particular class of FLC sets, the diffraction spectrum can be analyzed more deeply. This class encompasses the so-called model sets (also known as cut-and-project sets).

Let us assume that $P_0 \neq \emptyset$ is a repetitive FLC subset of $\mathfrak{G}$, fixed throughout this section. The subgroup of $\mathfrak{G}$ generated by $P_0 - P_0$ will be denoted by L. Without loss of generality, we may assume that $P_0 \subset L$. We assume, furthermore, (with loss of generality) that there is a LCAG $\mathfrak{G}_{\text{int}}$ and an algebraic homomorphism

$$L \longrightarrow \mathfrak{G}_{\text{int}}\,, \quad t \mapsto t^* \tag{4.1}$$

with the properties

- **S1** The subgroup $\tilde{L} := \{\, (t, t^*) \in \mathfrak{G} \times \mathfrak{G}_{\text{int}} \mid t \in L \,\}$ of $\mathfrak{G} \times \mathfrak{G}_{\text{int}}$ is a lattice in $\mathfrak{G} \times \mathfrak{G}_{\text{int}}$ (i.e., $\tilde{L}$ is discrete and $(\mathfrak{G} \times \mathfrak{G}_{\text{int}})/\tilde{L}$ is compact),
- **S2** P_0^* is relatively compact.

If such a map * exists, we can, without loss of generality, assume that, additionally, the following conditions are fulfilled:

- **S3** L^* is dense in $\mathfrak{G}_{\text{int}}$,
- **S4** $W_0 := \overline{P_0^*}$ (closure of P_0^* in $\mathfrak{G}_{\text{int}}$) has no non-trivial translation invariance (i.e., $\{\, c \in \mathfrak{G}_{\text{int}} \mid c + W_0 = W_0 \,\} = \{0\}$),

(otherwise, form suitable subgroups and quotients). A map * with these properties, following [**M**], will be called a *star map* for P_0. Such a star map will be called *regular* iff, additionally,

S5 The boundary of W_0 as defined in **S4** has zero Haar measure.

We assume that we have fixed some star map for P_0 fulfilling **S1**-**S4**. We now construct a $\mathfrak{G}$-map β from the LI class onto $\mathfrak{T} := (\mathfrak{G} \times \mathfrak{G}_{\text{int}})/\tilde{L}$ (the so-called "torus parametrization" [**BHP**] of X_{P_0}). Let

$$\pi: \; \mathfrak{G} \times \mathfrak{G}_{\text{int}} \to \mathfrak{T} \tag{4.2}$$

be the canonical projection.

Lemma 4.1. *If $P \in X_{P_0}$ such that $P \subseteq L$, then the set*

$$\bigcap \{ \, t^* - W_0 \mid t \in P \, \} \tag{4.3}$$

contains exactly one element $c_P \in \mathfrak{G}_{int}$, and $\overline{P^*} = c_P + W_0$.

Proof: Let $q_0 \in P$ be fixed, $K_0 \subseteq \mathfrak{G}$ compact such that $q_0 \in P \cap K_0$. For every compact $K \supseteq K_0$, we can find a $t_K \in L$ such that $P \cap K = (t_K + P_0) \cap K$. Then, $q_0 \in t_K + P_0$, therefore $t_K \in q_0 - P_0$, hence $t_K^* \in q_0^* - W_0$. As $q_0^* - W_0$ is compact, we can find a subnet $(t_{K_i})_{i \in I}$ of $(t_K)_{(K \text{compact})}$ such that its image under $*$ converges to some $c \in \mathfrak{G}_{\text{int}}$. Any $q \in P$ is contained in some K_i, therefore $q \in t_{K_i} + P_0$, i.e., $q^* \in t_{K_i}^* + W_0$. Going to the limit we see that $q^* \in c + W_0$ which implies that c is an element of the intersection (4.3). It is clear that any such c must fulfil $P^* \subseteq c + W_0$.

Let $W := \overline{P^*}$. Because $W \subseteq c + W_0$, it is compact. Up to now, we have not used the fact that W_0 fulfils condition **S4**, therefore, by symmetry, we can conclude that, for some $c' \in \mathfrak{G}_{\text{int}}$, we have $P_0^* \subseteq c' + W$. Therefore, $W_0 \subseteq c' + W \subseteq c' + c + W_0$, hence $W_0 \subseteq c' + c + W_0$. Because W_0 is compact, this can only be if $W_0 = c' + c + W_0$. As a consequence, $W = c + W_0$. We can now use condition **S4** in order to conclude that c is uniquely determined. □

Obviously, for $P \in X_{P_0}$ with $P \subseteq L$, and for $t \in L$, we have $c_{t+P} = t^* + c_P$. For every $P \in X_{P_0}$, there is some $s \in \mathfrak{G}$ with $s + P \subseteq L$; if $s + P \subseteq L$ and $s' + P \subseteq L$, then $s - s'$ must be in L. What we have just shown is that the following is well-defined.

Definition 4.2. Define

$$\beta: \; X_{P_0} \to \mathfrak{T}\,, \quad P \mapsto \beta(P) := \pi((s, c_{s+P})) \tag{4.4}$$

for s chosen such that $s + P \subseteq L$.

It is clear that $\beta(t + P) = t + \beta(P)$ for every $t \in \mathfrak{G}$, hence β is a $\mathfrak{G}$-map (note that we have a canonical action of $\mathfrak{G}$ on $\mathfrak{T}$).

Proposition 4.3. *β is a continuous map of X_{P_0} onto $\mathfrak{T}$.*

Proof: Let $P \in X_{P_0}$ and choose $s \in \mathfrak{G}$ with $s + P \subseteq L$. Let, furthermore, V' be an open neighbourhood of c_{s+P} in $\mathfrak{G}_{\text{int}}$. Lemma 4.1 now shows us that $\{c_{s+P}\} = \bigcap \{ \, t^* - W_0 \mid t \in s + P \, \}$, thus $\bigcap \{ \, (t^* - W_0) \backslash V' \mid t \in s + P \, \} = \emptyset$. As every $(t^* - W_0) \backslash V'$ is compact, there is a finite subset F of $s + P$ such that $\bigcap \{ \, (t^* - W_0) \backslash V' \mid t \in F \, \} = \emptyset$. In particular, there is a compact set $K \subseteq \mathfrak{G}$ with $\bigcap \{ \, (t^* - W_0) \backslash V' \mid t \in (s + P) \cap K \, \} = \emptyset$, so $\bigcap \{ \, t^* - W_0 \mid t \in (s + P) \cap K \, \} \subseteq V'$.

From this it follows that, if $t \in \mathfrak{G}$, $P' \in X_{P_0}$, and $(s+t+P') \cap K = (s+P) \cap K$, then necessarily $c_{s+t+P'} \in V'$.

For any neighbourhood V of 0 in $\mathfrak{G}$, we can conclude that, if $(P', P) \in U_{-s+K,V}$ (cf. (2.4)), then $(s+t, c_{s+t+P'}) \in (s+V) \times (c_{s+P} + V')$, where we have chosen $t \in V$ such that $(t+P') \cap (-s+K) = P \cap (-s+K)$. The continuity of β follows from the observation that $\beta(P) = \pi((s, c_{s+P}))$ and $\beta(P') = \pi((s+t, c_{s+t+P'}))$.

Because X_{P_0} is compact, in order to show that β is onto it suffices to show that its image is dense in $\mathfrak{T}$. This follows if we note that the orbit of P_0 under $\mathfrak{G}$ is mapped onto the orbit of 0 in $\mathfrak{T}$ under the action of $\mathfrak{G}$. The latter is given by $\pi(\mathfrak{G} \times \{0\}) = \pi(\mathfrak{G} + \tilde{L})$ which is easily seen to be dense in $\mathfrak{T}$ in view of **S3**. $\square$

Before we continue to study the dynamical system $(X_{P_0}, \mathfrak{G})$, we would like to note a nice consequence of the previous proposition.

Corollary 4.4. *If defined as in* **S4**, W_0 *equals the closure of its interior.*

Proof: Because L is generated by $P_0 - P_0$, L^* is generated by $(W_0 - W_0) \cap L^*$. Then, $\mathfrak{G}_{\text{int}}$ must be compactly generated because of **S3**. As a consequence, $\mathfrak{G} \times \mathfrak{G}_{\text{int}}$ is σ-compact, because $\mathfrak{G}$ was supposed to be σ-compact from the beginning. Because $\tilde{L}$ is discrete, it must be countable. Therefore, L^* is countable.

By Baire's category theorem, we can choose an element c of $\mathfrak{G}_{\text{int}}$ which is not in $L^* - \partial W_0$. Then, $(c + \partial W_0) \cap L^* = \emptyset$. From the surjectivity of β we get the existence of some $P \in X_{P_0}$ with $c_P = c$, hence $\overline{P^*} = c + W_0$. By the choice of c, P^* must be contained in the interior of $c + W_0$. $\square$

In the following, let us fix a $\mathfrak{G}$-invariant probability measure μ on X_{P_0}; on $\mathfrak{T}$, we have of course a unique $\mathfrak{G}$-invariant probability measure, namely the normalized Haar measure (observe that, because $\mathfrak{G}$-orbits are dense in $\mathfrak{T}$, every $\mathfrak{G}$-invariant measure is automatically also $\mathfrak{T}$-invariant; as a consequence, $\mathfrak{G}$-invariant subsets of $\mathfrak{T}$, such as $\pi(\mathfrak{G} + \partial W_0)$, have Haar measure 0 or 1). The $\mathfrak{G}$-mapping β from X_{P_0} onto $\mathfrak{T}$ induces a $\mathfrak{G}$-mapping

$$\beta_\sharp : \; L^2(\mathfrak{T}) \to L^2(\mu) \quad \xi \mapsto \xi \circ \beta \tag{4.5}$$

According to the above remark, the image measure $\beta\mu$ must coincide with the Haar probability measure on $\mathfrak{T}$. Therefore, $\beta_\sharp$ is an isometry of $L^2(\mathfrak{T})$ into $L^2(\mu)$; we may identify $L^2(\mathfrak{T})$ with the corresponding subspace of $L^2(\mu)$.

As a consequence, in the dynamical spectrum of $(X_{P_0}, \mathfrak{G})$, there is always a component which is isomorphic to the dynamical spectrum of $(\mathfrak{T}, \mathfrak{G})$. The latter is easy enough to determine: the dual $\hat{\mathfrak{T}}$ forms a total set of eigenfunctions in $L^2(\mathfrak{T})$. As remarked at the end of the last section, this does not say much about the diffraction spectrum of X_{P_0}. However, in the special case of regular model sets, the mapping $\beta_\sharp$ will turn out to be surjective, and, as a consequence, in this case the diffraction spectrum of $(X_{P_0}, \mathfrak{G})$ must be pure point.

A *regular model set* is, by definition, a repetitive FLC set P_0 which (after a translation) has a regular star map (4.1) such that

$$\{\, t \in L \mid t^* \in {W_0}^\circ \,\} \;\subseteq\; P_0 \,. \tag{4.6}$$

We assume now that P_0 is a regular model set and that a regular star map (4.1) is chosen in this fashion (in particular, we assume **S5**, i.e. that ∂W has Haar measure 0). Then, if $P \in X_{P_0}$ and $P \subseteq L$, $W := \overline{P^*}$,

$$\{\, t \in L \mid t^* \in W^\circ \,\} \;\subseteq\; P \,. \tag{4.7}$$

In order to see this, take an approximative net $(t_{K_i}+P_0)_{i\in I}$ as in the proof of Lemma 4.1, i.e., $(t_{K_i}+P_0)\cap K_i = P\cap K_i$ for all $i\in I$, the K_i cover $\mathfrak{G}$, and $t^*_{K_i}$ converge to some $c\in\mathfrak{G}_{\rm int}$. If $t\in L$ and $t^*\in c+{W_0}^\circ$, i.e., $c\in t^*-{W_0}^\circ$, then there is some $i\in I$ with $t\in K_i$ and $t^*_{K_i}\in t^*-{W_0}^\circ$, hence $t\in(t_{K_i}+P_0)\cap K_i$. By construction, $t\in P$.

The main thing to note is that now the torus parametrization β becomes 1-1 on a huge portion of X_{P_0}. Namely, if c is chosen such that $(c+\partial W_0)\cap L^* = \emptyset$, then, by (4.7), if $P\in X_{P_0}$, $P\subseteq L$, $c+W_0 = \overline{P^*} =: W$, we have

$$P \;=\; \{\, t\in L \mid t^*\in W^\circ \,\}\,. \tag{4.8}$$

As a consequence, P is uniquely determined by c. We call the elements of X_{P_0} which are translates of such P *generic* ; let $X^g_{P_0}$ be the set of all generic elements of X_{P_0}. By construction, β is injective on $X^g_{P_0}$, and $\beta(X^g_{P_0})$ and $\beta(X_{P_0}\backslash X^g_{P_0})$ are disjoint. The image of $X^g_{P_0}$ under β is the complement of the set $\pi(\mathfrak{G}+\partial W_0)$, a null set w.r.t Haar measure. (This shows that the dynamical system $(X_{P_0},\mathfrak{G})$ is an "almost 1-1 extension" of a minimal rotation action on a "torus", a result that was also obtained, in the Euclidean case, in [**FHK**].) This has an important consequence:

THEOREM 4.5. *Let P_0 be a regular model set. Then*

1) *The dynamical system of P_0 is uniquely ergodic (informally: frequencies of finite patterns exist uniformly, cf. Section 3).*
2) *The dynamical spectrum is isomorphic to the dynamical spectrum of $(\mathfrak{T},\mathfrak{G})$; in particular, it is pure point (as a consequence, the diffraction measure of P_0 is pure point).*

PROOF:

1) We show that our $\mathfrak{G}$-invariant probability measure μ is uniquely determined. As observed above, its image under β must be the Haar probability measure which we will temporarily denote by h. We note that $X^g_{P_0}$ is measurable as pre-image of a measurable set under β. For a measurable $A\subseteq X^g_{P_0}$, the image $\beta(A)$ is measurable, too (to see this, look at relatively closed subsets which generate the σ-algebra of measurable subsets of $X^g_{P_0}$). Furthermore, $\beta^{-1}\beta(A)=A$. We obtain $\mu(X^g_{P_0}) = \mu(\beta^{-1}\beta(X^g_{P_0})) = \beta\mu(\beta(X^g_{P_0})) = h(\beta(X^g_{P_0})) = 1$. By the same calculation, for arbitrary measurable $A\subseteq X_{P_0}$, $\mu(A) = \mu(A\cap X^g_{P_0}) = h(\beta(A\cap X^g_{P_0}))$. Now, the right hand side is independent of the choice of the invariant measure μ.
2) As noted above, $X^g_{P_0}$ has measure 1; it is, of course, $\mathfrak{G}$-invariant. Therefore, $L^2(\mu\downarrow_{X^g_{P_0}})$ is $\mathfrak{G}$-isomorphic to $L^2(\mu)$. The former is, via $\beta\downarrow_{X^g_{P_0}}$, $\mathfrak{G}$-isomorphic to $L^2(h\downarrow_{X^g_{P_0}})$, which in turn is $\mathfrak{G}$-isomorphic to $L^2(\mathfrak{T})$. The assertion follows. □

Let us put the carrier of the spectrum in a more familiar form. We can identify $\hat{\mathfrak{T}}$ with the annihilator $\tilde{L}^\perp = \{\,\chi\in(\mathfrak{G}\times\mathfrak{G}_{\rm int})^\wedge \mid \chi(g)=1\ (g\in\tilde{L})\,\}$ of $\tilde{L}$ in the dual of $\mathfrak{G}\times\mathfrak{G}_{\rm int}$. The latter can be identified with $\hat{\mathfrak{G}}\times\hat{\mathfrak{G}}_{\rm int}$. Sometimes, $\tilde{L}^\perp$ is called the *reciprocal* or *dual* lattice of $\tilde{L}$. For $\chi\in\hat{\mathfrak{G}}\times\hat{\mathfrak{G}}_{\rm int}$, let $\chi_\|$ be the restriction of χ to $\mathfrak{G}$; $\chi_\|$ can be identified with the projection image of χ under the canonical projection of $\hat{\mathfrak{G}}\times\hat{\mathfrak{G}}_{\rm int}$ onto the first coordinate. Hence, the dynamical spectrum of

$(\mathfrak{T}, \mathfrak{G})$ and, under the conditions of the above theorem, of $(X_{P_0}, \mathfrak{G})$ is concentrated on the projection image of the reciprocal lattice of $\tilde{L}$ in $\hat{\mathfrak{G}}$.

Conclusion

The work presented here settles the Fourier analysis and diffraction properties of general regular model set. However, it seems possible to explore the techniques further in order to investigate an even more general class of point sets: namely those sets which are repetitive and uniquely ergodic and admit a regular star map in the fashion of (4.1) without necessarily being model sets themselves. In this case, the analysis in the preceding Section carries as far as (4.5) which injects the L^2 of the "torus" which carries a discrete spectrum into the L^2 of the dynamical system. Therefore, the *dynamical* spectrum will contain a pure point component, and one might be tempted to conclude that the diffraction spectrum must contain a pure point component as well. Unfortunately, the argumentation in Section 4 does not allow to transfer this result to the diffraction spectrum; although it seems very likely to be true, I was not able to rigorously establish that the part of the L^2 of the dynamical system containing this pure point component is not orthogonal to the part generated by the continuous functions contributing to (3.9).

Another question which, although perhaps not of much practical relevance, keeps puzzling me is the question what happens if the boundary of the window has non-zero Haar measure. In this case, the techniques presented above break down completely; for example, the "torus" parametrization remains injective on a null set only which renders it next to useless. It would be of interest to know whether such a pattern can still be uniquely ergodic.

References

[A] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York (1976).

[AG] L. Argabright and J. Gil de Lamadrid, *Fourier Analysis of Unbounded Measures on Locally Compact Abelian Groups*, Memoirs of the AMS, vol. 145, Amer. Math. Soc., Providence, RI (1974).

[BMS] M. Baake, R. V. Moody and M. Schlottmann, *Limit-(quasi)periodic point sets as quasicrystals with p-adic internal spaces*, J. Phys. **A 31** (1998), 5755–5765.

[BHP] M. Baake, J. Hermisson and P. A. B. Pleasants, *The torus parametrization of quasiperiodic LI classes*, J. Phys. **A 30** (1997), 3029–3056.

[C] J. M. Cowley, *Diffraction Physics*, 3rd ed., North-Holland, Amsterdam (1995).

[DK] M. Duneau and A. Katz, *Quasiperiodic patterns*, Phys. Rev. Lett. **54** (1985), 2688–2691.

[D] S. Dworkin, *Spectral theory and X-ray diffraction*, J. Math. Phys. **34** (1993), 2965–2967.

[E] R. Ellis, *Lectures on Topological Dynamics*, Benjamin, New York (1969).

[FHK] A. Forrest, J. Hunton and J. Kellendonk, *Projection quasicrystals I: Toral rotations*, SFB 288 preprint no. 340, Berlin (1998).

[F] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, Princeton (1978).

[HR] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, 2nd ed., Springer, New York (1979).

[H] A. Hof, *Uniform distribution and the projection method*, in: *Quasicrystals and Discrete Geometry*, ed. J. Patera, Fields Institute Monographs, vol. 10, Amer. Math. Soc., Providence, RI (1998), pp. 201–206.

[KKL] P. A. Kalugin, A. Y. Kitayev and L. S. Levitov, *6-dimensional properties of AlMn alloy*, J. Physique Lett. **46** (1985), L601–L607.

[K] J. L. Kelley, *General Topology*, Van Nostrand, Princeton (1955); reprint: Springer, New York (1975).

[KN] P. Kramer and R. Neri, *On periodic and non-periodic space fillings of E^n obtained by projection*, Acta Cryst. **A 40** (1984), 580–587; and: Acta Cryst. **A 41** (1985), 619 (Erratum).

[La] S. Lang, *Real and Functional Analysis*, Springer, New York (1993).

[LP] W. F. Lunnon and P. A. B. Pleasants, *Quasicrystallographic tilings*, J. Math. Pures Appl. **66** (1987), 217–263.

[Me] Y. Meyer, *Algebraic Numbers and Harmonic Analysis*, North-Holland, Amsterdam (1972).

[M] R. V. Moody, *Meyer sets and their duals*, in: *The Mathematics of Long-Range Aperiodic Order*, ed. R. V. Moody, NATO ASI Series C 489, Kluwer, Dordrecht (1997), pp. 403–41.

[RW] C. Radin and M. Wolff, *Space tilings and local isomorphism*, Geom. Dedicata **42** (1992), 355–560.

[RN] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, 2nd ed., Ungar, New York (1955); reprint: Dover, New York (1990).

[Sch] M. Schlottmann, *Cut-and-project sets in locally compact Abelian groups*, in: *Quasicrystals and Discrete Geometry*, ed. J. Patera, Fields Institute Monographs, vol. 10, Amer. Math. Soc., Providence, RI (1998), pp. 247–264.

[So1] B. Solomyak, *Dynamics of self-similar tilings*, Ergod. Th. & Dyn. Syst. **17** (1997), 695–738.

[So2] B. Solomyak, *Spectrum of dynamical systems arising from Delone sets*, in: *Quasicrystals and Discrete Geometry*, ed. J. Patera, Fields Institute Monographs, vol. 10, Amer. Math. Soc., Providence, RI (1998), pp. 265–275.

[T] F. Treves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York (1967).

Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

E-mail address: martin@miles.math.ualberta.ca