A Guide to Mathematical Quasicrystals

MICHAEL BAAKE

Institut für Theoretische Physik, Universität Tübingen,
Auf der Morgenstelle 14, D-72076 Tübingen, Germany

1. Introduction

The discovery of alloys with long-range orientational order and sharp diffraction images of non-crystallographic symmetry [65, 35] has initiated an intensive investigation of the possible structures and physical properties of such systems. Although there were various precursors, both theoretically and experimentally [72], it was this renewed and amplified interest that established a new branch of solid state physics, and also of discrete geometry. It is usually called the theory of quasicrystals, even though it also covers ordered structures more general than those with pure Bragg diffraction spectrum.

It is now rather common to think of the regime between crystallographic and amorphous systems as an interesting area with a hierarchy of ordered states. This was not so some fifteen years ago, and it is the purpose of this contribution, and of the book as a whole, to introduce some of the ideas and methods that are needed to handle this new zoo. In particular, I will summarize some mathematical and conceptual issues connected with it, with special emphasis on proper equivalence concepts. This is more important than it might appear at first sight, because non-periodic order shows both new features and new hazards – and it is worthless to talk about a property of one specific structure if it is lost for others that are locally indistinguishable.

To develop some of these ideas, one has to start with a valid idealization of the physical structures one has in mind. Since we are interested in solids of some relevant size here, it is reasonable to replace their atomic arrangements by suitable *infinite* point sets. These should be *uniformly discrete* (i.e., there should be a uniform minimal distance between the points) and, usually, they should be *relatively dense* (i.e., there is a maximal hole). Sets with this property are called *Delone sets* and are widely used for this purpose.

In analogy to ordinary crystallography, many people prefer to think in terms of cells or *tiles* [56]. Here, one may start from a (usually finite) number of proto-tiles that fit together to tile space without gaps or overlaps. If we now decorate the tiles by finitely many points (giving the atomic positions, say), we return to a Delone set. Vice versa, given a Delone set Λ, we can perform the Voronoi construction that attaches to each point $x \in \Lambda$ the region of all points of ambient space that are closer to x than to any other point of Λ. This way, we come back to a tiling (whose dual, the so-called Delone tiling [66] is an even better candidate). Under an additional, but rather mild, condition, namely that $\Lambda - \Lambda$ is discrete[1], it will actually show only finitely many different tiles, and in this sense the two concepts are equivalent [47].

In what follows, in line with other expositions [41, 46, 22], we shall usually illustrate the concepts with examples from the class of tilings made from finitely many proto-tiles, but all concepts will be formulated in a way that allows to switch between such tilings and Delone sets. In particular, the central equivalence concepts will be described in that way, and it will actually make this very connection between tilings and point sets more precise.

As mentioned before, it is the aim of this contribution to summarize several rather central concepts needed for the description of quasicrystals. Since it is impossible to give a self-contained exposition of the present body of knowledge in this survey, the following text is often a pointer in nature, and further details have to be taken from the references given. Also, even though some definitions and results are high-lighted, the exposition is rather informal and contains no proofs, though we try to sketch the ideas behind them, whenever that is possible in fairly non-technical terms.

The article is organized as follows. After an introduction to non-crystallographic symmetries, we will recall some concepts from *diffraction theory*, followed by a Section on quasiperiodicity and the *projection method*, which is vital for the description of perfect quasicrystals. We then explain the issue of *minimal embedding*, illustrated with the most important examples. Then, as a first equivalence concept, *local indistinguishability* (also called *local isomorphism*) in introduced. This allows for the proper definition of (generalized) symmetry, but also calls for an effective control of the corresponding equivalence classes, which is then provided by the *torus parametrization*.

This is followed by the introduction of the important equivalence concept of *mutual local derivability*, or *local equivalence*. This paves the way for a generalization of translation and space group concepts from "ordinary" crystallography, without using Fourier transforms. Also, this setting enables a simple and unified setting for *inflation/deflation symmetries* and *perfect matching rules* as invariants of local equivalence classes. After some speculations on a possible classification scheme (which is presently still pretty incomplete), a brief introduction to the alternative approach via so-called "random tilings" is given, which is, in a way, the stochastic counterpart of the theory of perfect quasiperiodic tilings.

2. Non-crystallographic symmetries

To understand the impact of the discovery of quasicrystals, we first have to know what non-crystallographic symmetries are (to be described in this

[1] A point set S is *discrete*, if each point $x \in S$ can be surrounded by a non-empty open ball that does not contain any point from S except x. Note that uniform discreteness is a lot stronger than mere discreteness!

Section) and then why their appearance in the diffraction images of solids is, at least at first sight, astonishing (to be addressed in the next Section).

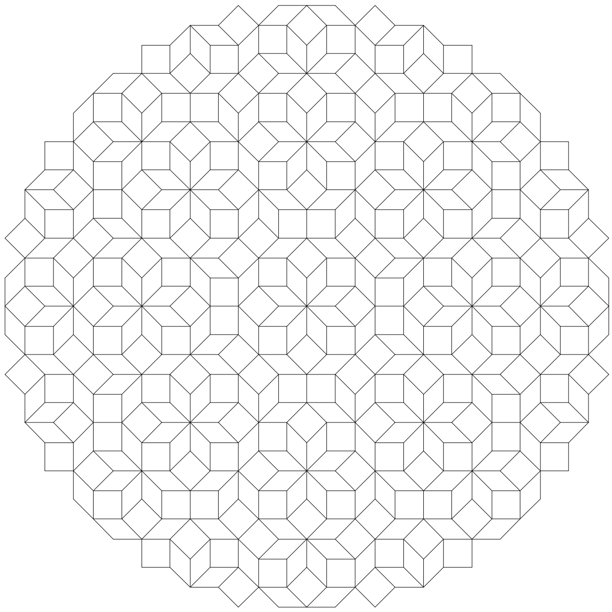

Fig. 2.1. Central patch of an exactly eightfold symmetric Ammann-Beenker tiling.

For a start, we need some crystallographic concepts, in particular that of a lattice in n-dimensional (n-D) Euclidean space. A set $\varGamma \subset \mathbb{R}^n$ is called a *lattice* if it is a discrete subgroup of $\mathbb{R}^n$ such that the factor group $\mathbb{R}^n/\varGamma$ is compact. This is equivalent to saying that there is a set of linearly independent vectors, $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$, called the *basis* of the lattice, such that

$$\varGamma = \mathbb{Z}\boldsymbol{b}_1 \oplus \cdots \oplus \mathbb{Z}\boldsymbol{b}_n \, , \tag{2.1}$$

i.e., $\varGamma$ consists of all integer linear combinations of the basis vectors. So, $\mathbb{Z}$ is a lattice in one dimension, and $\mathbb{Z}^2$ is one in the plane, but $\mathbb{Z}$ is *not* a lattice in two or more dimensions because its basis only spans a 1D ambient space.

Next, we call a set S *periodic*, if $S + \boldsymbol{t} = S$ for some $\boldsymbol{t} \neq 0$. Such a $\boldsymbol{t}$ is a *period* of S. S is called *crystallographic* if its periods form a lattice, i.e. if its periods span ambient space, compare [52]. If we now ask for the possible rotation symmetries of a crystallographic point set, we hit an obstruction, usually called the *crystallographic restriction* [68].

Theorem 2.1. *Let S be a crystallographic point set in $\mathbb{R}^n$, and R an orthogonal transformation that maps S onto itself. Then, R is of finite order,* *i.e.* $R^k = \mathbb{1}$ *for some* k. *In particular, we have* $k \in \{1, 2\}$ *for* $n = 1$, *and* $k \in \{1, 2, 3, 4, 6\}$ *both for* $n = 2$ *and* $n = 3$. *In general, the characteristic polynomial of* R *has integer coefficients only.*

The reason is that the invariance of S implies the invariance of its lattice of periods. The discreteness of the lattice forbids R to be of infinite order. Next, R must map the basis vectors onto integer linear combinations of them, so it must be similar to an integer matrix, from which the last statement follows. The case $n = 1$ is trivial, while $n = 2$ and $n = 3$ results from considering the traces of orthogonal matrices in their standard form: if φ is the rotation angle (around an axis in 3D), then $2\cos(\varphi)$ must be an integer.

As a consequence, the so-called Ammann-Beenker tiling of Figure 2.1, showing exact eightfold symmetry even in the infinite area limit where it covers the plane, cannot be crystallographic [1, 29, 4]. What is more, it cannot possess *any* period, because the symmetry would immediately complete any single period to a basis of a lattice. So, this is not a model of a 2D crystal, neither is any atomic decoration of its two cells, a square and a rhomb. The simplest such decoration, which is typical and exhaustive in a sense we will understand shortly, consists of points on all vertices of the entire tiling. If we take a diffraction image of this, with scatterers of equal strength on all points, we nevertheless obtain an image that resembles that of crystals pretty closely, see Figure 2.2. Let us thus turn our attention to diffraction for a while.

3. Diffraction

To simplify things, we will only talk about kinematic diffraction, i.e. diffraction that can be understood in terms of single scattering in the Fraunhofer picture. This is quite appropriate for X-ray and neutron diffraction, but not for electron diffraction where multiple scattering is essential, see [18] for details. Kinematic diffraction from a structure, in turn, is closely related to the Fourier transform of the corresponding potential in the sense that the observed intensities of sharp spots (Bragg peaks) are proportional to the absolute squares of the Fourier amplitudes, see [18] for a more detailed discussion and justification of this point of view.

In our idealized world, with atoms etc. replaced by point scatterers (Dirac distributions) on a set $\varLambda$, we thus consider the so-called *Dirac comb* of $\varLambda$,

$$\omega \;=\; \omega_{\varLambda} \;:=\; \sum_{x\in\varLambda} \delta_x \,, \tag{3.1}$$

where δ_x is Dirac's distribution (or measure) at point x, i.e. $(\delta_x, \psi) = \psi(x)$ for all test functions $\psi(x)$. In a second step, one defines a so-called *autocorrelation* or *Patterson function* [18] for this,

$$\gamma_{\omega} \;:=\; \lim_{r\to\infty} \frac{1}{\mathrm{vol}(B_r(0))} \sum_{x,y\in\varLambda\cap B_r(0)} \delta_{x-y} \,, \tag{3.2}$$

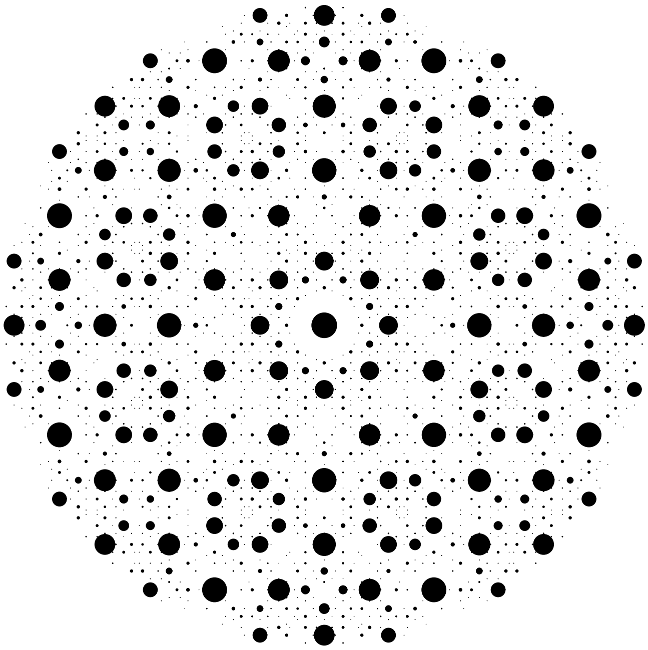

Fig. 2.2. Diffraction image of the Ammann-Beenker tiling of Figure 2.1. The area of each disc is proportional to the intensity of the corresponding peak, the cut-off is at 0.1 % of the central intensity.

where $B_r(0)$ is the (solid) ball of radius r around 0, and the limit is assumed to exist. Furthermore, we will tacitly assume that this limit stays the same if we replace the ball by any other convex region centred around the origin (with r the radius of the maximal inscribed ball) – an actually rather non-trivial feature to establish. The autocorrelation is a distribution of the form

$$\gamma_\omega \;=\; \sum_{z\in\varDelta} \nu(z)\delta_z \tag{3.3}$$

where $\varDelta = \varLambda - \varLambda$ and $\nu(z)$ is the density of points $x \in \varLambda$ such that also $x + z \in \varLambda$. In particular, the coefficient of δ_0 is $\nu(0) = d = \mathrm{dens}(\varLambda)$.

The observed intensity pattern is now given by the Fourier transform of this autocorrelation, denoted by $\hat{\gamma}_\omega$. Here, the following convention for the Fourier transforms of functions is used:

$$\hat{f}(k) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i k x} f(x)dx\;, \quad \check{g}(x) \;=\; \int_{\mathbb{R}^n} e^{2\pi i k x} g(k)dk\,. \tag{3.4}$$

With this convention, one has $\hat{\check{g}} = g$ and $\check{\hat{f}} = f$. Also, there is no need to distinguish between dual and reciprocal lattice (the factor 2π is absorbed into the argument of the exponential function) and the convolution theorem takes the nice form $\widehat{f * g} = \hat{f} \cdot \hat{g}$ where

$$(f * g)(x) \; = \; \int_{\mathbb{R}^n} f(x-y) g(y) dy \, . \tag{3.5}$$

Finally, the Fourier transform of a tempered distribution [64], T, is defined by $(\hat{T}, \psi) := (T, \hat{\psi})$, as usual. This way, the coefficient of δ_0 in $\hat{\gamma}_\omega$ is d^2.

Let us take a closer look at the diffraction from a lattice Γ. Here, $\Delta = \Gamma$ and, obviously, the autocorrelation coefficients are $\nu(z) \equiv d = \mathrm{dens}(\Gamma)$. So

$$\gamma_\omega \; = \; d \sum_{x \in \Gamma} \delta_x \, . \tag{3.6}$$

It is now a direct consequence of the so-called Poisson summation formula that the diffraction image consists of Bragg peaks (or Dirac peaks) only. They are supported by the dual (or reciprocal) of the lattice of periods, and hence distributed in a discrete fashion. Poisson's summation formula reads

$$\widehat{\sum_{x \in \Gamma} \delta_x} \; = \; d \cdot \sum_{y \in \Gamma^*} \delta_y \tag{3.7}$$

with the *dual lattice* $\Gamma^* = \{ y \mid x \cdot y \in \mathbb{Z} \text{ for all } x \in \Gamma \}$. So, we obtain

$$\hat{\gamma}_\omega \; = \; d^2 \sum_{y \in \Gamma^*} \delta_y \, . \tag{3.8}$$

It is worthwhile to note that Poisson's summation formula (3.7) also provides the Fourier transform of ω itself, which is well-defined in this case. Since the autocorrelation is essentially a volume-weighted convolution ($*$) of ω with itself, the convolution theorem explains why the coefficients of $\hat{\gamma}_\omega$ are the absolute squares of those of $\hat{\omega}$. This is usually summarized in a so-called Wiener diagram:

$$\begin{array}{ccc} \omega & \xrightarrow{\;*\;} & \gamma_\omega \\ \text{FT} \downarrow & & \downarrow \text{FT} \\ \hat{\omega} & \xrightarrow{|.|^2} & \hat{\gamma}_\omega \end{array} \tag{3.9}$$

Whenever this situation applies, in the sense that all quantities exist and the diagram is commutative, things are rather simple. In particular, given the situation of a lattice, atomic profiles (extended scatterers) or more complicated decorations of a fundamental domain (multiple atoms per unit cell) can be incorporated by means of convolutions with ω and then be processed through the Wiener diagram. Formally, the same process is then always used (at least for the Bragg part), but this needs extra justification, and often hard analysis for a proof, compare [33, 70] and references therein for details.

This procedure seems perfectly well-defined and robust if applied to the diffraction from *finite* patches or samples, but the problems mentioned are then only shifted to the question in which sense larger and larger samples show diffraction images that converge – again, not at all an easy one.

What other situations give rise to well-defined diffraction? In other words: which distribution of matter diffracts? The answer to this question is far from being known. So, we shall only sketch some aspects relevant to the understanding of the diffraction of quasicrystals and related structures. There is not much freedom to obtain sharp diffraction spots that are arranged in a discrete fashion, and this is indeed essentially restricted to diffraction from crystallographic arrangements (up to deviations of density zero, because the autocorrelation of a set of positive density is not changed by adding or removing points of density 0). So, if one restricts attention to such sets, it is obvious that non-crystallographic symmetries cannot show up. This was the point of view of "classical" crystallography – and this was challenged by the discovery of quasicrystals.

Beyond any doubt, the observed diffraction spots of quasicrystals are sharp and show non-crystallographic symmetries, such as a fivefold axis or the full icosahedral group. The solution to the emerging puzzle lies in the answer to the question whether the distribution of spots is really discrete[2], i.e. whether the spots are clearly separated from one another. For a given resolution, this seems to be the case – and this is certainly the reason why quasicrystals were discovered (for a given resolution, they show up through a discrete diffraction pattern). But if one increases the sensitivity (e.g., by doubling the exposing time of the photo plate), more peaks become visible, and this process does not come to an end: the set of *all* peaks seems to lie dense, and only those of intensity beyond a given threshold result in a discrete pattern, similar to the diffraction of a crystal, but with non-crystallographic symmetry. One example was shown in Figure 2.1, another important example being the rhombic Penrose tiling [60, 9, 21] of Figure 3.1.

4. Quasiperiodicity and the projection method

It is now time to explain how such a strange diffraction behaviour can come along. The foundation of it goes back to the beginning of the century when Harald Bohr, the younger brother of Niels, developed what is now called the theory of almost periodic functions. Let us consider the example

$$f(x) \; = \; \sin(x) + \sin(\tau x) \tag{4.1}$$

[2] We only talk about diffraction from "single crystals" here, resp. its analogue for quasicrystals. A somewhat similar discussion appears for the diffraction from powders, but should be distinguished clearly, because this case has a rather different explanation, see chapter 16 of [18].

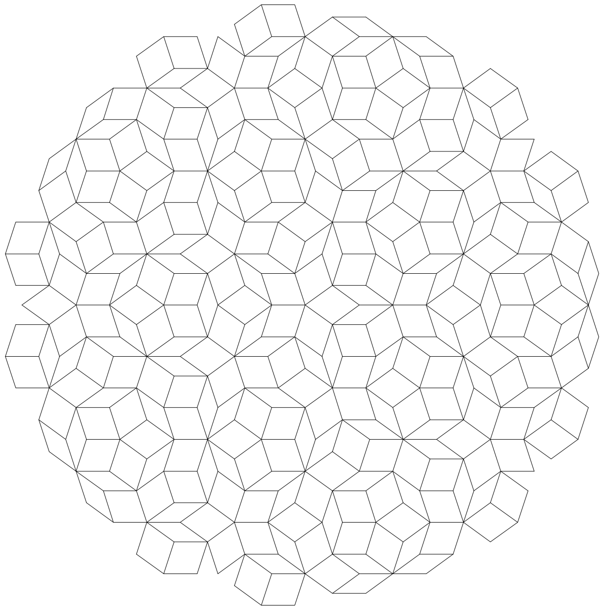

Fig. 3.1. Example of a rhombic Penrose tiling.

where $\tau = (1+\sqrt{5})/2$ is the famous *golden ratio*. This is an irrational number (in fact, as follows from its continued fraction expansion, the most irrational one), whence $f(x)$ is certainly not periodic. Nevertheless, for any given $\varepsilon > 0$, there are "almost-translations" t such that $|f(x) - f(x+t)| < \varepsilon$, for all $x \in \mathbb{R}$. Furthermore, such translations are not rare, but lie relatively dense in $\mathbb{R}$, i.e. there is a maximal distance between any two consecutive ones. The set of continuous functions with this property is closed under uniform convergence, and can be uniformly approximated by trigonometric polynomials. This results in a generalization of Fourier series which is essentially the core of Bohr's work [15]. For a more recent introduction, with additional material, see [17].

In these generalized Fourier series, pairwise incommensurate base frequencies occur (such as 1 and τ in the above example). If their number is finite, the corresponding function is called *quasiperiodic*. This subclass of functions has the property that it can be obtained as a section through a periodic function of more variables, e.g., in our example,

$$f(x) \;=\; \sin(x) + \sin(y)\big|_{y=\tau\, x} \tag{4.2}$$

This is also the essential idea to understand the diffractivity of quasicrystals, see [44, 23, 40] and various articles in [71].

Let us therefore construct non-periodic point sets by suitable sections through a crystallographic structure in higher dimension. As a first step, let us take a look at the so-called *cut and project* method, an example of which is shown in Figure 4.1. Starting with the square lattice in the plane, $\mathbb{Z}^2$, a line with irrational slope, called $\mathbb{E}$, is drawn, surrounded by a parallel strip of finite width. All lattice points inside the strip are then projected to $\mathbb{E}$. The result is a sequence of points that forms a non-periodic Delone set (due to the irrationality of the slope – otherwise it would be periodic). If the slope (as in Figure 4.1) is $1/\tau$, and if the width of the strip coincides with the projection of a fundamental square to the internal direction (which is actually perpendicular here), $\mathbb{E}_{\mathrm{int}}$, we obtain what is called the *Fibonacci chain*, the most common and best studied non-periodic 1D point set. Note that we have not given a formal definition of a quasicrystal[3], and we will not do so because the present use of the word is far from being context-free, and a really natural approach is not yet in sight. Let us add that some authors would prefer not to call the Fibonacci chain a real quasicrystal, but rather a modulated crystal. The reason is topological in nature, compare [38], but since such aspects are not important in our present context, we will suppress them.

This projection scheme, which has an obvious generalization in higher dimensions, does not seem to be an exact analogue of the section idea mentioned before, but it is equivalent to it. To see this, take the intersection of the strip with $\mathbb{E}_{\mathrm{int}}$, which is an interval here. This set, W, is called *window* or *acceptance domain*, and our Fibonacci chain F is then given by

$$F \;=\; \{P(x) \mid x \in \mathbb{Z}^2 \;\text{ and }\; P_{\mathrm{int}}(x) \in W\}\,, \tag{4.3}$$

where P and P_{int} denote the canonical projections to $\mathbb{E}$ and $\mathbb{E}_{\mathrm{int}}$, respectively[4]. The same set is obtained if we, instead of using the strip method, take an inverted copy of the window, $-W$, stitch it to each lattice point of $\mathbb{Z}^2$, and modify the rule in saying that we get a point of F whenever our cut line, $\mathbb{E}$, crosses a copy of this set. It can be considered as a *target* or a kind of *atomic hypersurface* which is point-like in the direction of the "physical" space, $\mathbb{E}$, and extended only in "internal" space, $\mathbb{E}_{\mathrm{int}}$. A third method to describe the same object goes under the name *dualization scheme* and has the advantage of directly giving cells rather than point sets, see [57, 46, 66, 9] for details.

In view of the Fourier transform, the version with the atomic hypersurfaces seems most attractive, because it is closest to the idea of describing a quasiperiodic arrangement of scatterers as a section through a crystallographic arrangement in higher dimensions. Consider now a Fibonacci chain, F, with point scatterers of equal strength on all its points, i.e. consider the Dirac comb

$$\omega_F \;=\; \sum_{x\in F} \delta_x\,. \tag{4.4}$$

[3] A reasonable working definition of a quasicrystal would include all discrete patterns which possess an autocorrelation whose Fourier transform (i.e. the diffraction) is either purely discrete, or has at least a non-trivial discrete part.

[4] For a more general and systematic formulation, we refer to [54, 67, 66].

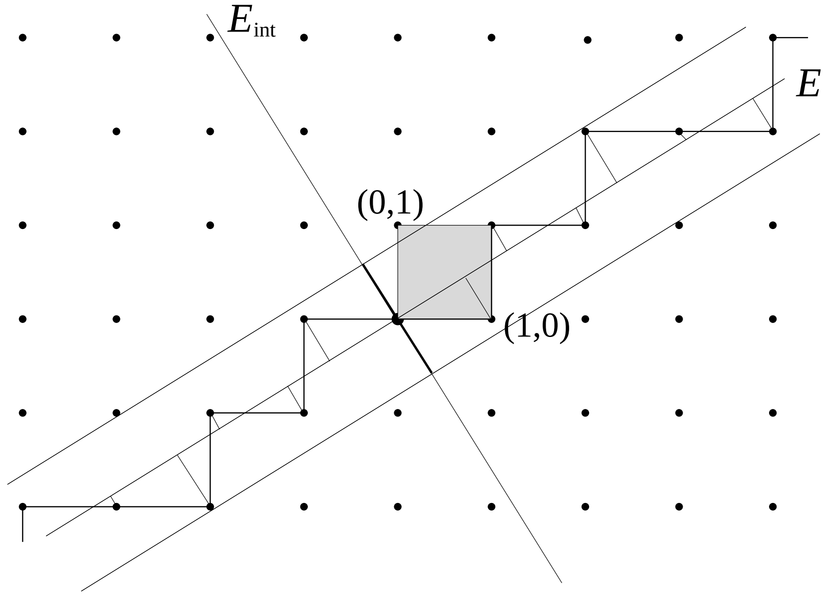

Fig. 4.1. Projection method for the Fibonacci chain and torus parametrization of its LI-class.

By a simple (formal) calculation, one finds that the Fourier transform consists of Dirac peaks on all points $P(k)$ where k is a point of the dual of the embedding lattice. Its amplitude, $a(P(k))$, is formally given by

$$a(P(k)) \;=\; \frac{d}{\mathrm{vol}(W)} \int_{-W} e^{-2\pi i\, k_{\rm int} x_{\rm int}}\, dx_{\rm int} \tag{4.5}$$

where $x_{\rm int} = P_{\rm int}(x)$ etc. and $d = \mathrm{dens}(F)$ denotes the density of F. The diffraction image is composed of Bragg peaks at the points $P(k)$ of intensity $|a(P(k))|^2$. The derivation of this can be found in many articles, but has to be taken with a grain of salt: it is purely formal, because the resulting expression is not a locally summable distribution and hence not a valid representation of a tempered distribution. That this formal way of calculating amplitudes and intensities is nevertheless correct, was proved much later by Hof, and the interested reader is referred to [33] and references therein.

In what sense does all this resolve the puzzle we started from? If we take a closer look at Eq. (4.5), we realize that, if W is an interval, the absolute squares of the amplitudes are of the form $\sin(2\pi k_{\rm int})^2/(2\pi k_{\rm int})^2$, hence bounded by $c/|k_{\rm int}|^2$ with some constant c – and this means that only *finitely* many peaks per unit volume have an intensity beyond a given threshold because $|k_{\text{int}}|$ is the distance of the (dual) lattice point k from the cut space. A cut-off for the intensities thus has an effect similar to the projection method itself! This is perhaps one of the most important observations in this context: a point set can be diffractive, and show a clear signature of this, without being crystallographic. If this is the case, there is then no longer any reason why non-crystallographic symmetries should not show up. If they do, however, we know immediately that the system cannot be crystallographic, and we would try to use the idea of a section through a lattice in higher dimensions to describe the structure.

Let me close this Section with another warning. The success of the projection method does *not* indicate that there is any need for higher-dimensional physics. It is only a convenient description of a certain class of ordered structures. Clearly, it is tempting to derive all sorts of generalizations of common properties and theorems (e.g. Bloch's theorem) by employing the embedding scheme and a chain of formal calculations. Quite frequently, this leads to wrong conclusions, and extreme care is required. For example, there is no easy analogue of Bloch's theorem. In fact, its naive generalization fails as badly as possible: the standard tight-binding model on the Fibonacci chain, in the infinite size limit, has no bands at all, and the spectrum is neither absolutely continuous nor pure point, but purely singular continuous! In other words, it is precisely of the form that was argued impossible for physical structures not too long ago. For more on this, and on the existence of a Cantor-type gap structure with topological quantum numbers, see [73, 13, 2].

5. Minimal embedding and further examples

In spite of the warnings given, the projection method, if used properly, is an extremely powerful tool, e.g. for practically indexing a diffraction pattern. But, given a diffractive system with non-crystallographic point symmetry, what is the right embedding dimension (which equals the indexing dimension) to start from? Fortunately, the answer is known:

Theorem 5.1. *The description of a planar quasiperiodic point set with n-fold symmetry by means of the projection method requires a lattice at least of dimension $\phi(n)$, where ϕ is Euler's totient function*[5]*. A 3D quasiperiodic point set with icosahedral symmetry requires an embedding lattice at least of dimension 6.*

The proof of the icosahedral case is based on the representation theory of the icosahedral group, see e.g. [22]. The statement about the planar symmetries is a direct and rather simple consequence of the structure of the so-called cyclotomic polynomials, see Appendix A of [5] for an explicit proof.

[5] $\phi(n)$ is the number of positive integers less than n which are coprime to n.

Let us note that the minimal dimension is usually sufficient (unless one wants to describe "modulated" quasicrystals, where it doubles), and using more than the minimal number only results in ambiguities of the indexing scheme – an altogether undesired feature.

Having settled the question for the correct dimension, we need to know what the "right" lattice is. It turns out that the higher-dimensional analogue of the square and cubic lattices, the hypercubic lattices, are *not* sufficient. The most common example where this becomes apparent is the Penrose tiling of Figure 3.1. It has fivefold (actually tenfold) symmetry[6], and the above Theorem then tells us that a 4D lattice is the right choice, because $\phi(5) = \phi(10) = 4$. Very often, one finds a description of the Penrose tiling based upon $\mathbb{Z}^5$ where one extra dimension has been introduced. This has the disadvantage mentioned. A simpler choice is the so-called root lattice A_4 which can be seen as the 4D lattice that is obtained by intersecting $\mathbb{Z}^5$ with the 4D hyperplane through the origin, and orthogonal to the space diagonal $(1,1,1,1,1)$. In general, root lattices provide a very nice class of simple lattices that is general enough to cover the observed cases [7] in a maximally symmetric way. For background material on root lattices, and all sorts of interesting connections to other branches of mathematics, we refer to the bible, [16].

Let us briefly mention some other planar examples. The Ammann-Beenker tiling of Figure 2.1 shows eightfold symmetry, and requires a 4D lattice ($\phi(8) = 4$). The standard choice [4] is $\mathbb{Z}^4$, but also the face-centred lattice in 4D is possible, i.e. the root lattice D_4. The latter has the advantage that, with a different choice of the cut space, also patterns with 12-fold symmetry ($\phi(12) = 4$, once more) can be obtained, see [5] for details. Sometimes, 12-fold symmetry is easier to describe with another root lattice, namely $A_2 \times A_2$.

Most prominent, in this context, are tilings made from squares and equilateral triangles, such as that shown in Figure 5.1. It is compatible with 12-fold symmetry (the triangles cover half the area), and was obtained by the projection method. Its window, however, shows a more complicated structure: it is a 12-fold symmetric region, compact, the closure of its interior, but has a fractal boundary, see Figure 5.2. It is a well-accepted conjecture that all square-triangle tilings with 12-fold symmetry, obtained by projection, require a fractally shaped window, and, in a certain sense, the one of Figure 5.1 is an example with "maximally smooth" window boundary [6]: almost everywhere, the boundary is locally smooth (a line segment, in fact), but at an uncountable set of boundary points (of vanishing Lebesgue measure) the fractal dimension is non-integral, and rather close to 2.

After these planar examples, let us briefly sketch the situation in 3D. Clearly, there are the so-called T-phases (with T for "thumbtack", to mimic their geometric structure) which are quasiperiodic in a plane and periodic along the perpendicular line. They can be modelled by 3D tilings that are stacked layers, each single layer being made from prisms (as tiles) whose base

[6] The proper use of the term symmetry will be explained in the next Section.

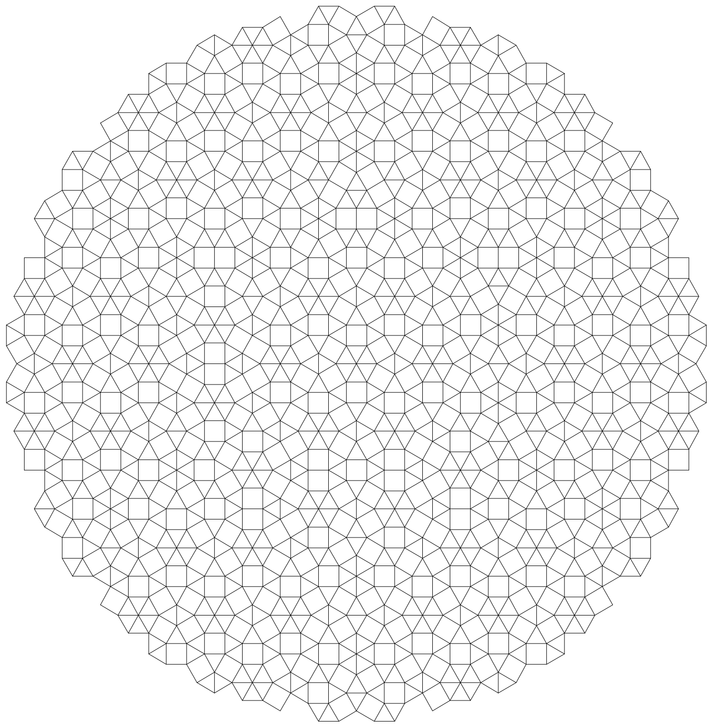

Fig. 5.1. Finite patch of a quasiperiodic square-triangle tiling.

pattern forms one of the classic planar tilings. Clearly, their Fourier image needs one extra Miller index, i.e. the Bragg peaks of standard decagonal T-phases are indexed by 5 integers, 4 being needed for the non-periodic planar degrees of freedom and one extra index for the periodic direction.

Of greatest importance probably are the tilings with icosahedral symmetry. Here, one has to distinguish three different types. All can be obtained by the projection method from hypercubic lattices in 6-space. There are three different Bravais types of them, the primitive ($\mathbb{Z}^6$), the face-centred (D_6), and the body-centred (D_6^*) one [68]. The three different icosahedral classes, see [22] for a detailed description, are then also called primitive (or P-type), face-centred (F-type) and body-centred (B-type), respectively. Since no application of the B-type models are presently known, I'll skip details of them.

The standard P-type tiling is made from two rhombohedra, an acute and an obtuse one. It has, in various degrees of completeness, a long history [72], and was first described by means of the projection technique in [44]. It will be denoted by KN. The diffraction shows icosahedral symmetry and a clear scaling with inflation multiplier τ^3. This is characteristic of P-type

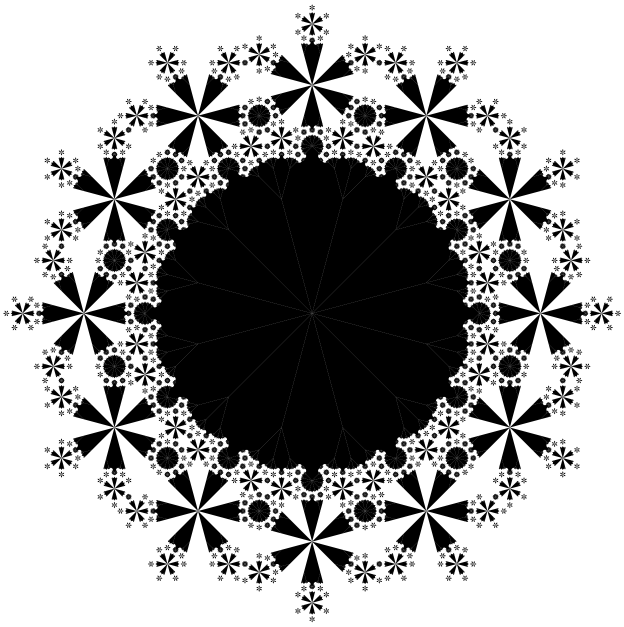

Fig. 5.2. Window for the square-triangle tiling of Figure 5.1.

structures, and makes the distinction from F-type rather simple as the latter displays scaling with an inflation multiplier[7] τ (the same would be true of the B-type).

The more important class (in terms of applications) is that of F-type tilings. One of the earliest examples is the zonohedral tiling by Socolar and Steinhardt [69], abbreviated as SS. It is built from four proto-tiles, namely the acute rhombohedron met above, the rhombic dodecahedron, the rhombic icosahedron and the famous triacontahedron, also known as Kepler's body [29]. Another example was found by Danzer [19], which is very closely related (i.e. locally equivalent) as we shall see later. Danzer's tiling (called DT) is built from 4 tetrahedra. Finally, based on the projection technique, several other F-type tilings have been investigated, see [8, 45] and references therein for details. The most important of those (called $\mathcal{T}^{*(2F)}$) is built from six tetrahedra and is again very closely related to the two tilings mentioned before (SS and DT), although it contains more local information – a concept to be made more precise in a shortwhile.

Up to this point, no further mathematical details or concepts were needed to get a first impression (see [59] for a general construction scheme). But for a better understanding of the structures, their symmetries and some of the

[7] The meaning of this will become clear in the Section on inflation symmetry.

new features, we now have to dive a little deeper into the world of discrete geometry. In particular, we definitely need some good tools to handle the zoo of possibilities. Later, we shall see that the number of "known" examples, al least those with "nice" properties, is actually rather small, and can be handled with little more difficulty than needed for the crystallographic patterns.

6. LI-Classes and Symmetry

One basic concept for the general analysis of global order properties of discrete structures is the equivalence concept of *local indistinguishability*, also known as local isomorphism[8] [49]. Since the infinite (mathematical) structure is considered as an approximation to large but finite physical objects, it is natural to identify those structures which are locally indistinguishable on arbitrarily large but finite scales. Such structures are called locally indistinguishable, or locally isomorphic.

We will use this term frequently in the sequel, so for a precise definition we introduce some notation. The mathematical objects we deal with are, most generally, discrete structures in Euclidean space, i.e., sets of (possibly decorated) bounded subsets of the space which are locally finite in the sense that each ball of finite radius meets only finitely many structure elements. If A is such a discrete structure, then we call an *r-patch* of A each subset of A which is completely contained in a ball of radius r. Now, two structures, A and B, are *locally indistinguishable* (or locally isomorphic) if each r-patch of A is, up to a translation, also an r-patch of B and *vice versa* (a moment's reflection reveals that, in the general case, this 'vice versa' is necessary to get a proper equivalence relation). The corresponding equivalence class of a structure is called its *local indistinguishability class*, or LI-class, for short.

It should be emphasized that this formal definition does not quite reflect the intuitive description of the first paragraph, because we have insisted on identity of r-patches up to *translations* only, rather than up to more general Euclidean motions. This more restrictive relation will prove useful for other concepts to be introduced in the next Section, and especially for the definition of generalized point symmetries.

One of the most outstanding properties of the experimentally observed aperiodically ordered structures like quasicrystals is the occurence of crystallographically forbidden symmetries in their diffraction spectra, e.g., fivefold axes. On the other hand, it is clear that, e.g., a 3D discrete structure can possess at most one axis of exact fivefold point symmetry in a given direction, because otherwise there would be a dense set of such axes, which is impossible for a locally finite structure. Therefore, to take into consideration

[8] Since this term is occupied with a different meaning in other (connected) areas, I suggest to replace the word isomorphism by indistinguishability here, which also avoids the introduction of a new abbreviation.

also spatially homogeneous structures with non-crystallographic symmetry properties, one has to enlarge the symmetry concept slightly. This is easily done with the help of LI as defined above: we say that the isometric linear transformation T is a *generalized point symmetry element* of the structure A if, and only if, A and $T(A)$ belong to the same LI-class.

What do these concepts mean in the crystallographic case? Firstly, one sees at once that a structure A is locally indistinguishable from a crystallographic structure B if and only if there is a translation vector t which translates A to B: $B = t + A$. As a consequence, in the crystallographic case, the generalized point symmetry of a structure coincides with the conventional point symmetry, as it should do.

For aperiodic patterns, the generalized symmetry is a proper extension of exact symmetry. For example, the well-known Penrose tiling of the plane may have exact fivefold symmetry, i.e., there are precisely four representatives (up to translations) in its LI-class which have one point of exact (global) fivefold symmetry each; most of the members of the Penrose LI-class, abbreviated by LI(PT) from now on, have no exact symmetry at all. The generalized point symmetry group of each elements of this LI class coincides, however, with the symmetry group of the regular decagon, i.e., it is the dihedral group[9] D_{10}.

7. Parametrization of LI-classes

Having given the definition of an LI-class is not quite the same as understanding its structure. The latter is, in fact, more complicated than one might expect. To see this, let us first consider the case of a crystallographic pattern $\mathcal{P}$: its LI-class consists of all its translates, and can thus be parametrized by the points of a fundamental domain of the corresponding lattice, Γ, of translations (e.g., its Voronoi or Wigner-Seitz cell, to be specific) because $\mathcal{P} = \mathcal{P} + t$ for all $t \in \Gamma$. In particular, the LI-class $LI(\mathcal{P})$ simply consists of one translation class of patterns.

The correspondence between patterns in LI($\mathcal{P}$) and points of a fundamental domain is called the *torus parametrization* of LI($\mathcal{P}$) because such a domain, upon identifying Γ-equivalent boundary points, becomes a *torus* of the dimension of the lattice. Clearly, the answer cannot be this simple for non-crystallographic patterns. Here, LI($\mathcal{P}$) does not only contain all translates of $\mathcal{P}$, but also all other patterns that can be obtained as limits of these (w.r.t. the obvious topology of patch-wise comparison) – and "most" members are of the latter type. In fact, for repetitive[10] aperiodic patterns, the LI-class

[9] The symbol D_n appears in two different meanings, once for the corresponding root lattice and once for the dihedral group of order $2n$. Since both are standard in the literature, and misunderstandings unlikely, we stick to this convention.

[10] The term repetitive means the following: for each radius r, there is another radius, $R = R(r)$, such that each $\mathcal{P}$-patch of radius r can be found in every $\mathcal{P}$-patch of radius R.

contains uncountably many ($2^{\aleph_0}$) translation (even congruence) classes [66] – so, things are a lot more complicated, and a parametrization would be handy.

It is not known how to achieve this in general, but, for certain patterns, it is indeed possible. Among them are those point sets and tilings that can be constructed by the projection method introduced in Section 3. Let us take another look at Figure 4.1. It shows, in addition to the ingredients needed to visualize the projection mechanism, a shaded square that represents a fundamental domain of the lattice $\mathbb{Z}^2$. It becomes a torus on identifying opposite faces in the usual way. Now, let us mark a special point of the cut space $\mathbb{E}$ by a handle (this can be thought of as a reference point for the pattern). If we move $\mathbb{E}$ around, and with it the strip, we obtain different cut and project sets for each position of the handle in the fundamental domain, but we do not get anything new beyond it, due to the periodicity of the embedding lattice. So, the points on the 2-torus parametrize different Fibonacci chains, and it is well known that one actually exhausts the entire LI-class this way [66]. This is the *torus parametrization* of quasiperiodic LI-classes [3].

There is one subtlety which we have suppressed so far. The projection method is unique as long as no lattice point lies on the boundary of the strip – in which case the corresponding projected object is called *regular* (or *generic*). Such regular members form the majority of the LI-class. Situations where lattice points fall on the boundary of the strip, in turn, correspond to the union of several *singular* patterns, each of which can be seen as a limit of regular patterns. The singular patterns also belong to the LI-class. In this new light, the torus parametrization is one-to-one for regular members of the LI-class, but multiple-to-one for singular members. This point will become important in any potential classification of LI-classes beyond the torus parametrization. We will briefly come back to this in the next Section.

In our Fibonacci example, the parametrization of singular members is two-to-one. An interesting question is how different from one another two such singular chains are. The answer relies on the structure of the window. As long as its boundary (the intersection of the strip with $\mathbb{E}_{\rm int}$) is of vanishing Lebesgue measure, the different singular members attached to the same torus parameter differ from one another only at places of zero density – hence, their identification is physically reasonable[11]. Vanishing Lebesgue measure of the boundary of the window, in turn, does not seem to be too restrictive. In particular, the example of Figure 5.2 is still included, as are all other compact sets in $\mathbb{R}^n$ with boundary of Hausdorff (or fractal) dimension $< n$.

The great advantage of the torus parametrization is its universality in the sense that we can use the same torus for all projection structures attached to the same embedding lattice and the same choice of the cut space $\mathbb{E}$. Some individual properties of the LI-classes are then encoded in the precise way

[11] For certain applications, it is advantageous to distinguish regular and singular patterns and to adopt a topological point of view, e.g. for questions such as the spectra of Schrödinger operators, see [13] for details.

the singular members behave, but they are usually less important or even irrelevant for considerations such as symmetry, inflation invariance etc. In particular, the torus parametrization allows us to find all members of an LI-class showing *exact* invariance under a given symmetry operation, including new types of symmetry such as inflation/deflation symmetry to be discussed later. This is based on lifting the symmetry operation under consideration to a mapping on the torus. Then, the number of fixed points can be determined by calculating certain determinants. For details, together with explicit examples and a full treatment of the physically relevant symmetries, see [3, 31].

An extension of this analysis to groups of transformations (rather than single operations) is possible, and it is instructive to look at subgroups of the icosahedral group and their action on the three possible types of LI-classes, see Table 7.1. In each case, there are precisely 64 inversion symmetric members of the LI-class, and they distribute in a very peculiar way on the subgroups of Y_h, the full icosahedral group. To be more specific, there are two rhombohedral tilings in LI(KN) with full Y_h symmetry, one being regular and one singular, while there are 4 such members in LI(DT), say, three regular and one singular. This shows at least one reason why F-type structures are more frequent than the other possibilities: as a consequence of this analysis, and using the implications of local indistinguishability, it must be concluded that F-type tilings or Delone sets have a denser distribution of clusters with exact (or almost exact) icosahedral symmetry – an idea pretty close to the concept of a Frank-Kasper phase. This would suggest that F-type icosahedral quasicrystals should be more frequent than P-type ones, as is indeed the case.

group	B-type	P-type	F-type
Y_h	1	2	4
D_{5d}	18	12	0
D_{3d}	30	20	0
D_{2h}	15	30	60

Table 7.1. Point symmetries of the 64 inversion symmetric tilings for the three different types of icosahedral LI-classes.

8. Local derivability and MLD-classes

Local indistinguishability (or local isomorphism), as defined above, is certainly a useful tool for the study of various properties of individual tilings. However, for the description of more general aspects of their order (regardless of local details that might be more accidental), one has to extend the equivalence concept under consideration [10]. Let us make this vague statement a bit more concrete by considering a crystallographic structure. Its order is, up to a scaling factor, fully described by its space group, without caring about how the fundamental domain is actually decorated. Also, it does not matter which representative of the fundamental domain we actually choose – we can certainly transfer the detailed description from one choice to another in a strictly local way. Now, in the aperiodic case, the (naively defined) space group is almost always trivial and therefore *cannot* serve as a classification tool. One way out is the consideration of the diffraction intensities of the structure, as has been done in [62, 53]. Though this (non-rigorous!) approach works in a large class of structure types, it has a number of shortcomings. First, it is confined to the case of structures which show Bragg diffraction, i.e., are essentially almost periodic. Next, the method is a little bit indirect, namely working in k-space, and offers therefore no good intuition for the things going on in real space. Finally, such an approach might lead to a rather coarse picture, not distinguishing between locally inequivalent structures. This does not happen in the crystallographic case, but it does so in the extension of this approach to quasicrystals.

To describe an appropriate alternative, let us consider, as an example, two tilings which certainly are to be considered equivalent: the well known rhombic Penrose tiling (PT), and its Robinson decomposition (RD) into isosceles golden triangles, see [9] and references therein for details. By definition, there is a clear-cut rule to transform a PT into its corresponding RD, and it is not hard to see that, for a given RD, the underlying PT can be reconstructed, at each place, by observing just a few triangles in the vicinity, see Figures 3.1, 8.1 and 9.1 (right half). In particular, this means that we can transfer *any* decoration of RD into one of PT, and vice versa – eventually for the price that we have to distinguish several congruent copies of the proto-tiles from one another, depending on their (uniformly!) local neighbourhood in the tiling.

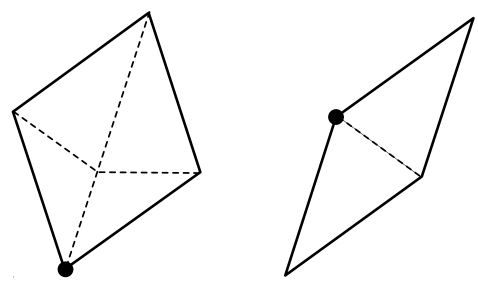

Fig. 8.1. Robinson's decomposition of the Penrose tiling.

Let us put this type of equivalence in more formal terms. It is clear that the details of the transformation process described above are not important, e.g., we will certainly not care about tiles being properly dissected or composed, or in fact about tiles at all: quite frequently a representative discrete point set [42] is what one really needs or wants! The essential feature which allows the abstraction from local details while keeping track of the global order is the *uniform locality* of the transformation rule. It is easy to see that such a uniformly local rule for the transformation of some structure A into a structure B exists precisely under the following condition: There is a fixed finite radius r such that if the r-patches of A around two points, p, q, are equal up to the translation $t = p - q$, then the structure B at the points p and q is the same, again up to the translation t.

If this condition is fulfilled, then we call B *locally derivable* [11] from A. If it is also fulfilled with the roles of A and B interchanged (with a possibly different radius r') then we call A and B *mutually locally derivable* from one another, or *locally equivalent*.

It is clear that this equivalence relation can be extended to entire LI-classes. That is, if A and B are locally equivalent, then, for any A' in the LI-class of A, some B' can be found in a canonical fashion (just using "the same rule") such that A' and B' are locally equivalent, thereby defining a one-to-one correspondence between the two LI-classes. Therefore, we can combine these equivalences defining the *MLD-class* of a structure A to be the set of all structures which are locally indistinguishable from some structure locally equivalent to A. Needless to say that PT and RD in the example above belong to the same MLD-class in this sense. It is sometimes more useful to view MLD-classes, which are defined as *unions* of LI-classes, directly as *sets* of LI-classes, but we will identify these two points of view for simplicity.

Let A and B be locally equivalent structures. Obviously, if A is invariant under a certain translation t, then B must be invariant under t as well, by the very definition of local equivalence. A little further reflection shows that, on the other hand, if A and B are crystallographic with the same translation lattice, Γ, then they are locally equivalent, by "transformation rules" involving only a couple of fundamental domains. Therefore, local equivalence is a generalization of "having the same translation lattice" in the periodic case:

Theorem 8.1. *Two crystallographic patterns, A and B, are locally equivalent if and only if they share the same translation lattice Γ.*

This result also explains why we restricted our definition of local indistinguishability (or patch-equivalence) to translations only, rather than using a version involving congruence. Below, we will refine the MLD concept in order to achieve a generalization of the space group classification.

9. Local equivalence and limit translation module

In general, it may be a hard problem to decide whether two given structures are locally equivalent or not. As one tool for this task, we introduce an object associated to each discrete structure. It also generalizes the translation lattice of a periodic structure in a certain sense, but contains less information[12].

Let A be a discrete structure. For each radius r, one can collect all translation vectors t which "move patches inside A", that is, for which an r-patch P exists in A such that the translate $t + P$ is also an r-patch of A. These translation vectors for fixed r generate a $\mathbb{Z}$-module, Λ_r, which simply consists of all integer linear combinations of the translations found. This module gets smaller if r becomes larger: $r \leq r'$ implies $\Lambda_r \supseteq \Lambda_{r'}$. This property allows us to define a limit,

$$\Lambda := \bigcap_{r>0} \Lambda_r \tag{9.1}$$

which is again a $\mathbb{Z}$-module. We call Λ the *limit translation module* (LTM) of the structure A. It is obviously an invariant of LI-classes.

In the crystallographic case, the LTM is just the translation lattice itself; on the other hand, in the general case, Λ may turn out to be trivial, and this might even be the "typical" situation. However, there is a large class of structures where this LTM is nontrivial and provides important information on the order of the structure, among them being point sets or tilings obtained by the projection method. Anyway, the LTM is an invariant of MLD-classes: if A and B are locally equivalent, then their LTM must coincide; furthermore, if B' is locally indistinguishable from B, then its LTM is the same as well. On the other hand, if B is derivable from A, one can only conclude that LTM(A) $\subseteq$ LTM(B). Therefore, the determination of the limit translation module is a natural first step for proving or disproving the local equivalence of two given structures.

That the LTM does not specify the MLD class completely, even if it is non-degenerate, can also be studied in the special case of cut and project patterns as introduced above. If A is such a quasiperiodic structure, minimally embedded [66] into the periodic structure S (this is important!), then the limit translation module of A turns out to be just the projection image of the LTM of S (i.e. its lattice of translations, which then serves as the embedding lattice) into the subspace containing A, i.e. into $\mathbb{E}$. This also explains the connection of LTM(A) to the Fourier module of A in such a situation: the latter is the (generally dense) set of points in k-space where we have to expect Bragg peaks. It is obtained by projecting the *dual* of the embedding lattice into $\mathbb{E}$, as explained earlier. But, given the higher-dimensional lattice and the cut space, the quasiperiodic structures obtained by cut-and-project are, by no means, all in the same MLD-class.

In fact, one has the following simple necessary and sufficient criterion for two cut-and-project structures to share the same MLD-class: it must be possible to reconstruct the acceptance domains of A from those of A' by finitely many union, intersection and set complement operations, and vice versa (for a more detailed account of the relationship between the projection

[12] This slightly more difficult Section may be skipped on first reading.

formalism and local derivability, see [11]). This way, one can actually prove that PT and TTT (the so-called Tübingen triangle tiling, another decagonal tiling built from the golden triangles, see Figure 9.1 and Ref. [9]) are *not* in the same MLD-class (although it is possible to rescale PT such that it becomes locally derivable from TTT). These two LI-classes actually differ in the distribution of singular tilings, and the transformation rule from TTT to PT maps certain sets of singular tilings of LI(TTT) onto single, but regular members of LI(PT) – something that clearly cannot be inverted. This is remarkable as TTT and PT certainly have the same space group according to [62], i.e. they *cannot* be distinguished on the basis of the symmetry properties of their Fourier transforms.

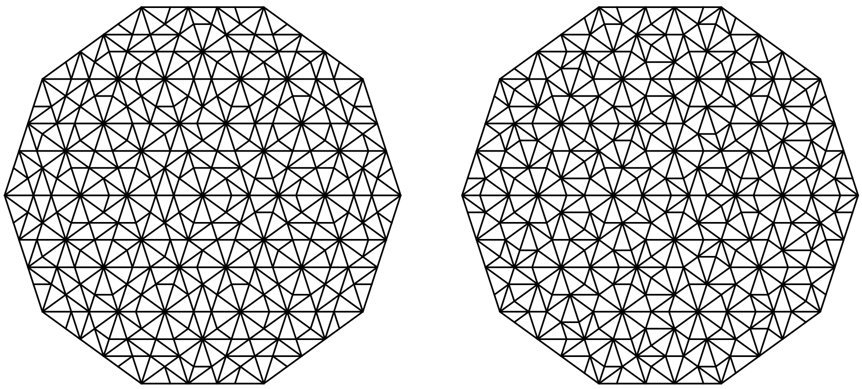

Fig. 9.1. Decagonal patch of the Tübingen triangle tiling TTT (left) versus Robinson's decomposition (RD) of the rhombic Penrose tiling (right).

10. Local derivability and symmetry preservation

The careful reader will have noticed that we did not include rotations etc. in the definition of local derivability – for very good reasons. Nevertheless, symmetry is important, in particular for many physical applications, and it is obvious how to include it by one further step of refinement [10]. Let us say that a certain local derivation rule preserves the symmetry of a structure if derivation rule and symmetry operation commute. This then defines S-MLD-classes (for symmetry preserving MLD) which obviously form (pairwise disjoint) subclasses of MLD-classes.

As in the case of MLD-classes, the specialization of the S-MLD concept to the periodic case fits well into classical crystallography: Two crystallographic structures belong to the same S-MLD-class if and only if their space groups are identical (for a true generalization of the crystallographic classification scheme of periodic structures one has to broaden the S-MLD-classes by allowing for global similarity transformations). Simultaneously, this fact shows that S-MLD is a proper refinement of MLD, i.e., there are MLD-classes which contain several S-MLD-classes. An aperiodic planar example for such a behaviour is given at the end of the following Section.

In 3-space, Danzer's tiling (DT) and that of Socolar and Steinhardt (SS) are in the same S-MLD-class, see [63, 20] for a proof. So, in this sense, they really describe the same class of structures. One of the D_6 based tilings allows the local derivation of Danzer's tiling from it, but there is no local rule to go back – a 3D analogue of the situation previously met with TTT versus PT.

11. Inflation symmetries and matching rules

Let us now investigate how the MLD-concept works in the context of two outstanding properties which important aperiodic structures exhibit: *inflation/deflation symmetry* and *perfect matching rules* (compare [39, 48, 50, 52, 41] for commonly used definitions).

As there are various concepts of inflation/deflation in the literature, we have to make precise what we mean by it, thereby taking the opportunity to put the MLD concept into operation. Usually, an inflation of a structure consists in a certain rule for a local transformation of structure elements into patches of a new structure which turns out to be of the same type as the original one, but on a smaller scale. For example, the dissection of the golden triangles depicted in Figure 11.1 gives the inflation rules both for the triangles of TTT and of RD. So far, this does not seem to be too interesting, as one can do this sort of procedure with the periodic tiling of the plane by squares. However, in certain cases, such an operation does not result in *any* loss of information on the original structure, i.e., it is possible to recover it by an inverse transformation, also in a local fashion.

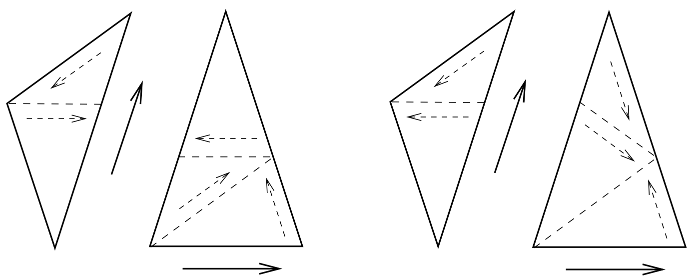

Fig. 11.1. Inflation rule for TTT (left) and RD (right).

Rereading the definitions in the last Sections, one sees that the above description is precisely what is meant by the following formal definition: a structure A has an *inflation/deflation symmetry* related to a similarity transformation T, if $T(A)$ is in the same MLD-class as A (i.e., if $T(A)$ is locally indistinguishable from a structure which, in turn, is locally equivalent to A; it is necessary to phrase it this way, because the situation where already $T(A)$ is locally equivalent to A is too special).

From the last Section, we may conclude that no periodic structure can have *any* in-/deflation symmetry related to a nontrivial T, i.e., where T is not just a rigid motion. The reason can be glimpsed from the following example: subdividing the square cells of the lattice $\mathbb{Z}^2$ into smaller squares of half the edge length, say, is obviously a local rule. But the converse, re-grouping four adjacent squares into a bigger square, is not – it requires the knowledge where the process was started to guarantee fault-free operation, and this means it is not possible by a local rule. On the other hand, the existence of nontrivial in-/deflation seems to be a very common feature among the interesting aperiodic structures [52]. In many cases, T is just a rescaling, but there are important examples where rotation-dilations are needed, as in certain 2D tilings with 12-fold symmetry (cf. Refs. [55, 5]). We have

Theorem 11.1. *The existence of an inflation/deflation symmetry is a property of an entire MLD-class: either all members share it, or none has it.*

This fact may serve into two directions. Firstly, having established the existence of in-/deflation for a single structure, one already has in-/deflation for its entire MLD-class. Secondly, simultaneous (non-)existence of in-/deflation provides a necessary criterion for two structures to be locally equivalent.

The study of perfect matching rules is another subject where the concept of local equivalence proves fruitful. We say that a structure A possesses *perfect matching rules* (essentially in the sense of Ref. [50]) if its LI-class is determined by the set of its r-patches for some *finite* radius r (which we call, if chosen minimally, the matching rule radius of A), i.e., if every other structure which contains, up to translations, only r-patches which also occur in A necessarily belongs to the LI-class of A. In the case of LI(PT), a very simple version in terms of tiles with oriented edges can be given, see [27] for an illustration. For obvious reasons, this property of a structure is particularly interesting in the case that this structure is supposed to describe the global order of physically realized structures as quasicrystals [39, 41]. One may think of a Hamiltonian that favours the patches of the atlas, this way restricting the groundstate to a member of the LI-class.

It is almost immediate that every structure which is locally equivalent to one with perfect matching rules must itself possess perfect matching rules:

Theorem 11.2. *The existence of perfect matching rules is a property of an entire MLD-class: either all LI-classes contained in this MLD-class possess perfect matching rules, or none of the LI-classes can have them.*

Note, however, that the matching rule radius is *not* an invariant of MLD-classes. This is of some relevance in the physical context, if one tries to relate the matching rules to the local interaction of some suitable Hamiltonian, i.e. if one searches for a Hamiltonian whose ground states form a specific LI-class with perfect matching rules. This is due to the fact that the information contained in a structure may be delocalized (gradually) by the local derivation of another structure. An estimate for the matching rule radius of a structure A which is locally equivalent to a structure B with perfect matching rules is given in [26]. It involves the matching rule radius of B and the relevant radii for the transition from B to A and vice versa. It is an interesting question what the infimum of all matching rule radii of the LI-classes inside one MLD-class is. It has been conjectured that it might actually be zero under some extra condition. This is rather plausible for systems with inflation-deflation symmetry, as shrunk-down representatives exist on arbitrarily small scales.

It should be noted that there are certain tilings, such as the Ammann-Beenker octagonal tiling [1] or Gähler's dodecagonal "shield" tiling [25], which do *not* possess perfect matching rules if one considers only undecorated tiles, but can be transformed into structures with matching rules by convenient decorations. In these cases, the introduction of the decoration cannot be achieved in a local fashion (see Ref. [25]), the naked and decorated tilings form different MLD-classes and should therefore be distinguished clearly [43].

One might ask for the number of different possibilities to construct tilings of a given symmetry with perfect matching rules (for a study of the 8-, 10-, and 12-fold symmetrical cases see Ref. [51]). Here, one is not interested in the (infinite) variety of representatives of one and the same MLD-class, but in *different* MLD-classes with perfect matching rules, such as those defined by LI(PT) and LI(TTT). At the moment, the only candidate of an infinite family of different tilings with perfect matching rules is provided by the generalized Penrose patterns with parameter $\gamma = m + n\tau$, see [48] for details. A closer inspection, compare also Ref. [34, 48], shows that these tilings belong to only one MLD-class, or to two S-MLD-classes – one with fivefold and one with tenfold symmetry (which contains LI(PT)). This analysis has been extended in [48] to rational values of γ, which results in an entire tower of LI-classes that allow a local derivation down the tower, but not upwards. On the bottom of this tower, we find a well-known friend: the rhombic Penrose tiling, PT. Similar towers certainly exist for other examples, e.g. in 3-space, but, to our knowledge, have not yet been analyzed in detail.

12. Summary of the perfect world

So far, it has been outlined, in rather elementary terms, how equivalence concepts such as that of local indistinguishability (LI) and that of mutual local derivability (MLD) are helpful in sorting out local properties of locally finite tilings or other discrete patterns. Furthermore, we believe that concepts along the lines presented above are needed to continue a sound classification program of aperiodically ordered structures, and for various aspects it is advantageous not to depend on Fourier transforms.

Let us continue with a speculation. We have seen that there were serious connections between different examples of patterns with perfect matching rules. An interesting question is how serious these connections are. From the past fifteen years of research on quasicrystals and aperiodic order, we have the feeling that the variety of S-MLD-classes with *all* magic properties is limited, if organized properly. To this end, one has to form towers of them, in which two consecutive members allow a local derivation down the tower, but not up – as in the case of MLD(TTT) being on top of MLD(PT). With this, we tend to the following conjecture: the number of towers of quasiperiodic S-MLD-classes with fixed symmetry, limit translation module of minimal rank[13], local inflation/deflation symmetry (with fixed inflation multiplier) and local perfect matching rules is finite.

This statement is rather fragile: removing essentially any of its conditions, it is wrong. So, a further exploration of this question (and a proof or disproof) would be a logical next step in the classification of aperiodic structures – complementing and perhaps even completing the existing classification of Fourier modules [36, 53] (mainly based on symmetry alone).

Although a good classification of order, even along these lines, is not in sight (and it might actually be very far away), one should not close ones eyes in front of other possibilities. In fact, even if we had the answer to the above question, it would not be sufficient physically: so far, we have totally, and deliberately, ignored any *stochastic* aspect of point sets or tilings. This is not really tolerable, and the last Section is now devoted to a very brief and sketchy introduction to a totally different (though, fortunately, not totally disconnected) universe ...

13. Alternatives: disorder and random tilings

The first indication on incompleteness of the above approach comes from the observation that also quasicrystals will show defects (in fact, probably more than ordinary crystals), and one would like to know the possible scenarios. As a first step, one can investigate so-called defective vertex configurations within the geometric setting of a tiling, compare [12] and references therein for a survey. Although this provides rather interesting insight, and even allows for some simplistic, but not too unrealistic, models, we shall focus here on an alternative to perfect tilings or Delone sets that starts from a rather different, if not antipodal, point of view.

[13] Minimal w.r.t. the symmetry, e.g. rank 4 for tenfold symmetry in the plane or rank 6 for icosahedral symmetry in 3-space.

The above discussion implicitly concentrated on discrete structures that could be seen as idealizations of *energetically* stabilized structures, and LI-classes with perfect matching rules may be interpreted in this direction: in principle, there exists a Hamiltonian whose ground state is necessarily an element of the LI-class defined by the perfect matching rules. Although this idea seems attractive, it has a number of pitfalls. On the one hand, such a Hamiltonian will be rather artificial and unrealistic, and, on the other hand, such matching rules do not lead to local growth rules [27, 58] – unless the structure is crystallographic. So, even if matching rules can provide a toy model for energetic stabilization, they cannot explain why quasicrystals form and how they grow. A similar problem is also present in a slightly relaxed scenario, known as “maxing rules”, see [28, 37].

An alternative, or complementary, idea is provided by statistical physics. Recall that stability is related to minimizing the free energy which is given as $F = U - TS$, with U the internal energy, S the entropy and T the temperature. Therefore, stabilization can also have a significant *entropic* contribution, and we will now briefly sketch some ideas that have been developed along that route. To this end, we assume that the internal energy, U, is essentially degenerate, and minimizing F thus means maximizing S. So, we are talking about an idealization (first pointed out by Elser [24]) which corresponds to a high-temperature phase – a picture that is not at all absurd in view of the experimental observations!

Let us go back to the Fibonacci chain for a moment. It showed two different atomic distances between neighbouring points, cells a and b, say. Their arrangement is entirely deterministic, with frequencies $1/\tau$ and $1/\tau^2$, respectively. Furthermore, one can show that precisely $n+1$ different words (in a and b) of length n occur in an infinite Fibonacci chain – no matter which representative of its LI-class we take. So, its (combinatorial) entropy density per letter vanishes, as can easily be calculated:

$$s \;=\; \lim_{n\to\infty} \frac{\log(n+1)}{n} \;=\; 0\,. \qquad (13.1)$$

The same phenomenon happens for the entropy density of all other deterministically ordered structures obtained by the projection method or by an inflation rule [14]. This is caused by the inherent long-range repetitive order of those structures, which is also the reason for a well-defined diffraction spectrum[14]. So, if we want to introduce some entropy, we have to go beyond such structures, but we certainly want a scheme that does not totally destroy the long-range (orientational) order.

The simplest idea is to relax the possible configurations of cells. If we allow any sequence of a's and b's subject to the sole restriction that their frequencies ν_a and ν_b match those of the Fibonacci chain, we obtain a Bernoulli-type ensemble of infinite chains, this time with positive entropy density

[14] Inflation generated tilings need not have pure point diffraction spectrum. This depends on the nature of the inflation multiplier, see [70] for a detailed account.

$$s = -\nu_a \log(\nu_a) - \nu_b \log(\nu_b) = \frac{\tau+2}{\tau+1}\log(\tau) \simeq 0.665 . \tag{13.2}$$

This (essentially unrestricted) ensemble has two disadvantages: first, the maximal entropy would occur for $\nu_a = \nu_b = 1/2$ as $\log(2) \simeq 0.693$, and not for the Fibonacci frequencies. Second, the new ensemble is so random that the typical ensemble member does no longer show sharp diffraction peaks. Consequently, this model is not suitable to explain Fibonacci-type structures.

To overcome these difficulties, at least in dimensions two and higher, the idea of a *random tiling* was put forward by the Cornell group, see [30] and references therein. Starting from the same proto-tiles as in a perfect quasiperiodic tiling, one allows all gap-less space-fillings that are face-to-face and overlap-free, and eventually subject to further restrictions. It is this set of local, geometric constraints which transforms the unrestricted Bernoulli-type ensemble into a more interesting and realistic Markov-type ensemble. E.g., one could start from the two rhombi of the Penrose tiling of the plane but relax the matching rules. This way, one obtains a random tiling ensemble with unique entropy maximum at the tile frequencies of the perfect tiling, and this maximum also corresponds to the unique point (in parameter space) of maximum symmetry, D_{10}. So far, the situation seems considerably improved. However, the diffraction side is not yet fully satisfactory: such planar tiling ensembles display sharp peaks, but they are, in general, no Bragg peaks.

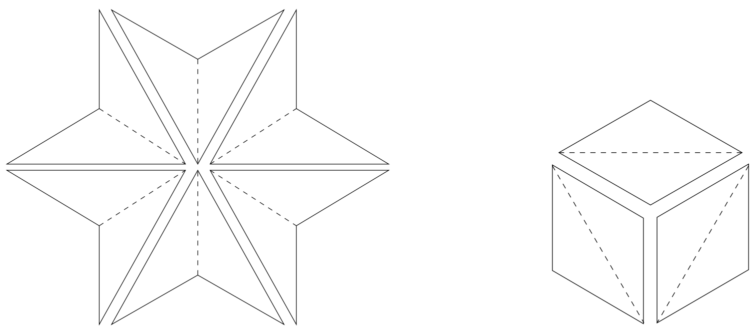

Fig. 13.1. Prototiles of the dart-rhombus random tiling.

Since we live in three dimensions, we should ultimately be concerned only with that situation. But then, doing the analogous exercise with the tilings of 3-space by the two rhombohedra mentioned earlier, we find again the entropy maximum at the point in parameter space that shows icosahedral symmetry. This time, because statistical fluctuations of a 3D ensemble are bounded, the diffraction would still show Bragg reflexes, plus a structured diffuse background. Furthermore, the entropy around the maximum, as a function of the parameters (e.g., the densities of different proto-tiles), is a strictly concave function with the maximum being locally quadratic. This allows to develop a kind of entropic elasticity theory around the maximum, and the corresponding symmetry adapted invariant forms are characterized by what are called the *elastic constants* [30].

We have met the basic ingredients. The collection of all tilings of n-space from a given set of proto-tiles (face-to-face, gap-less and overlap-free, as usual, plus eventually some extra constraints) is called a *random tiling ensemble* if it has positive entropy density in the sense that the number of possibilities to cover a ball of radius r grows exponentially in r^n. Such ensembles, if well-defined statistically, have some rather general features, two of which are often stated as kind of axioms. They are the so-called random tiling hypotheses [30] which, in our language, read

RTH1 The point of maximum entropy is automatically a point of maximum symmetry.

RTH2 The entropy, around the maximum, is a locally quadratic function of the parameters.

It might be interesting to note that RTH1 is actually a theorem and follows by some group theoretic arguments from considerably simpler assumptions. The status of RTH2, however, is more difficult: this really is an assumption that has to be verified, as explicit counterexamples are known, see [61] for details.

To give an impression, let us, at this point, present one example of a planar random tiling, made from two proto-tiles, a rhombus and a dart, see Figure 13.1. Both proto-tiles are built from two copies of an isosceles triangle, and thus share the same area. Rhombi occur in 3 orientations, darts in 6. In addition to the usual rules to put them together, we also demand that the rhombi observe an alternation rule, i.e., no two rhombi that are mere translates of one another are allowed to share an edge. Also, two neighbouring darts must not share a short edge. This is then one of the very few non-trivial examples that can be solved completely, in an exact and even rigorous way, by methods from statistical mechanics. Although it is only compatible with sixfold symmetry (and could thus be called a crystallographic random tiling ensemble), it shows many features that are also typical of “quasicrystalline” random tilings (though the diffraction is different). In particular, it displays a unique entropy maximum of quadratic nature [RTH2] at the point of maximal symmetry [RTH1]. Furthermore, somewhere else in the phase diagram, it shows an order-disorder phase transition (of Onsager type) which can alternatively be interpreted as a percolation phase transition, compare [32, 61] for details. In Figure 13.2 we show a typical snapshot from the ensemble at its entropy maximum, which turns out to be

$$s \;=\; \frac{1}{3}\log(2) \;\simeq\; 0.231\,. \tag{13.3}$$

This result is so simple that it must be possible to give a direct derivation of it, without referring to the somewhat elaborate exact solution. This is indeed possible, see [61] or chapter 3.4 of [32].

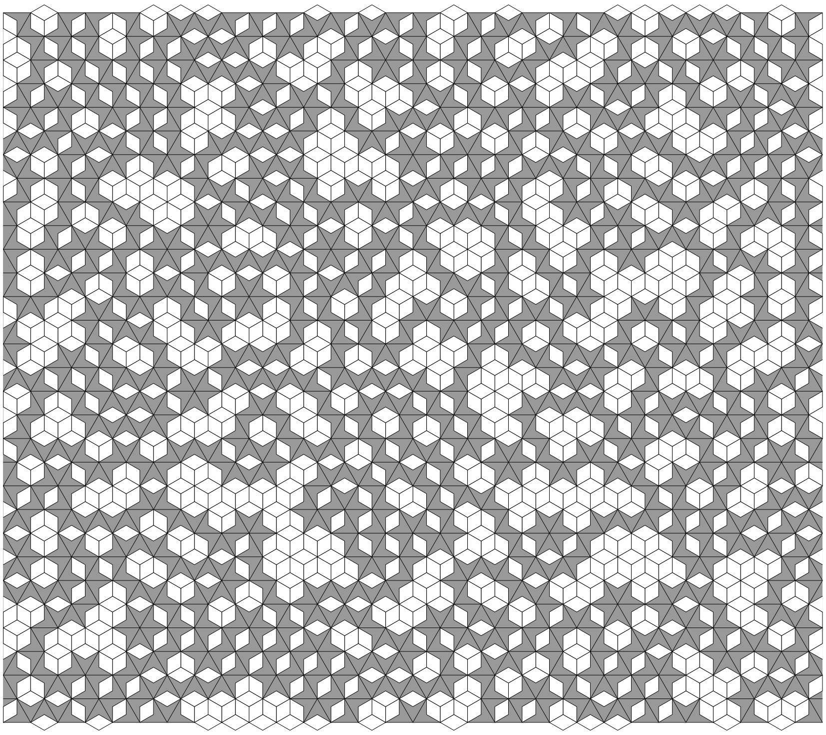

Fig. 13.2. Snapshot of a dart-rhombus random tiling at maximum entropy.

14. Concluding remarks

After this little excursion into the world of aperiodic order, one should realize that the displayed material was not only a short (and possibly insufficient) glimpse, it was also highly biased. This resulted not so much in a strange view at the things discussed, I hope, but in the omission of many other aspects, often fascinating, sometimes frustratingly complicated, occasionally in the t2c2e-category[15], or just unnoticed by the ignorant author. But, even fifteen

[15] Things too complicated to explain, at least on a few pages.

years after the field took off, it is still in its infancy, and any other statement bares the risk of delusive security. It is amazing how many things seem to be known, or almost known, and how few of them are really established, or fully known. And for many aspects, we have just scratched the surface.

In view of this, I should try to mention some of the points not discussed above. However, even that would be biased again, and it is probably better to give the advice to read on in this volume, and to consult the introductory articles from other summer or winter schools, some of which are given in the references below. A more complete list of books and proceedings can be found in a short extra chapter of this book, together with some guiding comments on what one can find in the titles listed. Anyway, one "take-home" message should be to stay open-minded to new aspects, to challenge "common knowledge", to follow interesting paths, and – last not least – to have fun!

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References

1. R. Ammann, B. Grünbaum and G. C. Shephard, "Aperiodic Tiles", *Discr. Comput. Geom.* **8** (1992) 1–25.
2. M. Baake, U. Grimm and D. Joseph, "Trace maps, invariants, and some of their applications", *Int. J. Mod. Phys.* **B7** (1993) 1527–50.
3. M. Baake, J. Hermisson and P. A. B. Pleasants, "The torus parametrization of quasiperiodic LI-classes", *J. Phys.* **A 30** (1997) 3029–56.
4. M. Baake and D. Joseph, "Ideal and defective vertex configurations in the planar octagonal quasilattice", *Phys. Rev.* **B42** (1990) 8091–102.
5. M. Baake, D. Joseph and M. Schlottmann, "The root lattice D_4 and planar quasilattices with octagonal and dodecagonal symmetry", *Int. J. Mod. Phys.* **B5** (1991) 1927–53.
6. M. Baake, R. Klitzing and M. Schlottmann, "Fractally shaped acceptance domains of quasiperiodic square-triangle tilings with dodecagonal symmetry", *Physica* **A191** (1992) 554–8.

7. M. Baake, P. Kramer, D. Joseph and M. Schlottmann, "Root lattices and quasicrystals", *J. Phys.* **A23** (1990) L1037–41.
8. M. Baake, P. Kramer, Z. Papadopolos and D. Zeidler, "Icosahedral dissectable tilings from the root lattice D_6" in: *Group Theoretical Methods in Physics*, eds. V. V. Dodonov and V. I. Man'ko, Springer, Berlin (1991), p. 543–47.
9. M. Baake, P. Kramer, M. Schlottmann and D. Zeidler, "Planar patterns with fivefold symmetry as sections of periodic structures in 4-space", *Int. J. Mod. Phys.* **B4** (1990) 2217–68.
10. M. Baake and M. Schlottmann, "Geometric aspects of tilings and equivalence concepts", in: *Quasicrystals*, eds. C. Janot and R. Mosseri, World Scientific, Singapore (1995), p. 15–21.
11. M. Baake, M. Schlottmann and P. D. Jarvis, "Quasiperiodic tilings with tenfold symmetry and equivalence with respect to local derivability", *J. Phys.* **A24** (1991) 4637–54.
12. S. I. Ben-Abraham, "Defective vertex configurations in quasicrystalline structures", *Int. J. Mod. Phys.* **B7** (1993) 1415–26.
13. J. Bellissard, A. Bovier, and J. M. Ghez, "Gap labelling theorems for one-dimensional discrete Schrödinger operators". *Rev. Math. Phys.* **4** (1992) 1–37.
14. V. Berthé, "Entropy in deterministic and random systems", in: *Beyond Quasicrystals*, eds. F. Axel and D. Gratias, Springer, Berlin (1995), p. 441–63.
15. H. Bohr, *Fastperiodische Funktionen*, Springer, Berlin (1932), reprint 1974.
16. J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 2nd ed., Springer, New York (1993).
17. C. Corduneanu, *Almost Periodic Functions*, 2nd ed., Chelsea, New York (1989).
18. J. M. Cowley, *Diffraction Physics*, 3rd ed., North Holland, Amsterdam (1995).
19. L. Danzer, "Three-dimensional analogues of the planar Penrose tilings and quasicrystals", *Discr. Mathem.* **76** (1989) 1–7.
20. L. Danzer, Z. Papadopolos and A. Talis, "Full equivalence between Socolar's tilings and the (A,B,C,K)-tilings leading to a rather natural decoration", *Int. J. Mod. Phys.* **7** (1993) 1379–86.
21. N. G. de Bruijn, "Algebraic theory of Penrose's non-periodic tilings of the plane", part I: *Math. Proc.* **A84** (1981) 39–52, and part II: *Math. Proc.* **A84** (1981) 53–66, reprinted in [71].
22. M. Duneau, "N-dimensional crystallography and the icosahedral group", in: *Lectures on Quasicrystals*, eds. F. Hippert and D. Gratias, Les Editions de Physique, Les Ulis (1994), p. 153–86.
23. M. Duneau and A. Katz, "Quasiperiodic patterns", *Phys. Rev. Lett.* **54** (1985) 2688–91;
24. V. Elser, Comment on "Quasicrystals: a new class of ordered structures", *Phys. Rev. Lett.* **54** (1985) 1730, reprinted in [69].
25. F. Gähler, "Matching rules for quasicrystals: the composition-decomposition method", *J. Non-Cryst. Solids* **153&154** (1993) 160–4.
26. F. Gähler, M. Baake and M. Schlottmann, "Binary tiling quasicrystals and matching rules", *Phys. Rev.* **B50** (1994) 12458–67.
27. U. Grimm and D. Joseph, "Modelling quasicrystal growth", this volume.
28. P. Gummelt, "Penrose tilings as coverings of congruent decagons", *Geom. Dedicata* **62** (1996) 1-17.
29. B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, Freeman, New York (1987).
30. C. L. Henley, "Random tiling models", in: *Quasicrystals – The State of the Art*, eds. D. P. DiVincenzo and P. J. Steinhardt, World Scientific, Singapore (1991), p. 429–524.

31. J. Hermisson, C. Richard and M. Baake, "A guide to the symmetry structure of quasiperiodic tiling classes", *J. Phys. I (France)* **7** (1997) 1003–18.
32. M. Höffe, *Zufallsparkettierungen und Dimermodelle*, Diplomarbeit, Universität Tübingen (1997).
33. A. Hof, "Diffraction by aperiodic structures", in: *The Mathematics of Long-Range Aperiodic Order*, ed. R. V. Moody, NATO ASI Series C 489, Kluwer, Dordrecht (1997), p. 239–68.
34. K. Ingersent, "Matching rules for quasicrystalline tilings", in: *Quasicrystals: The State of the Art*, eds. D. P. DiVincenzo and P. J. Steinhardt, World Scientific, Singapore (1991), p. 185–212.
35. T. Ishimasa, H.-U. Nissen and Y. Fukano, "New ordered state between crystalline and amorphous in Ni-Cr particles", *Phys. Rev. Lett.* **55** (1985) 511–3.
36. T. Janssen, "Aperiodic crystals: a contradictio in terminis?", *Phys. Rep.* **168** (1988) 55-113.
37. H.-C. Jeong and P. J. Steinhardt, "Constructing Penrose-like tilings from a single prototile and the implications on quasicrystals", *Phys. Rev.* **B55** (1997) 3520–32.
38. A. Katz, "On the distinction between quasicrystals and modulated crystals", in: *Quasicrystals*, eds. M. V. Jarić and S. Lundqvist, World Scientific, Singapore (1990), p. 200–17.
39. A. Katz, "Matching rules and quasiperiodicity: the octagonal tilings", in: *Beyond Quasicrystals*, eds. F. Axel and D. Gratias, Springer, Berlin (1995), p. 141–89.
40. A. Katz and M. Duneau, "Quasiperiodic patterns and icosahedral symmetry", *J. Physique* **47** (1986) 181–96, reprinted in [71].
41. A. Katz and D. Gratias, "Tilings and quasicrystals", in: *Lectures on Quasicrystals*, eds. F. Hippert and D. Gratias, Les Editions de Physique, Les Ulis (1994), p. 187–264.
42. R. Klitzing and M. Baake, "Representation of certain self-similar quasiperiodic tilings with perfect matching rules by discrete point sets", *J. Phys. I (France)* **4** (1994) 893–904.
43. R. Klitzing, M. Schlottmann and M. Baake, "Perfect matching rules for undecorated triangular tilings with 10-, 12-, and 8-fold symmetry", *Int. J. Mod. Phys.* **B7** (1993) 1455–73.
44. P. Kramer and R. Neri, "On periodic and non-periodic space fillings of $\mathbb{E}^m$ obtained by projection", *Acta Cryst.* **A40** (1984) 580–7, and *Acta Cryst.* **A41** (1985) 619 (Erratum), reprinted in [71].
45. P. Kramer and Z. Papadopolos, "Symmetry concepts for quasicrystals and noncommutative crystallography", in: *The Mathematics of Long-Range Aperiodic Order*, ed. R. V. Moody, NATO ASI Series C 489, Kluwer, Dordrecht (1997), p. 307–30.
46. P. Kramer and M. Schlottmann, "Dualization of Voronoi domains and klotz construction: a general method for the generation of proper space fillings", *J. Phys.* **A22** (1989) L1097–102.
47. J. C. Lagarias, "Geometric Models for Quasicrystals I. Delone Sets of Finite Type", and: "Geometric Models for Quasicrystals II. Local Rules Under Isometries", preprint (1997), to appear in *Discr. Comput. Geometry.*
48. T. Q. T. Le, "Local rules for quasiperiodic tilings", in: *The Mathematics of Long-Range Aperiodic Order*, ed. R. V. Moody, NATO ASI Series C 489, Kluwer, Dordrecht (1997), p. 331–66.
49. D. Levine and P. J. Steinhardt, "Quasicrystals. I. Definition and structure", *Phys. Rev.* **B34** (1986) 596–616, reprinted in [71].

50. L. S. Levitov, "Local rules for quasicrystals", *Commun. Math. Phys.* **119** (1988) 627–66.
51. R. Lück, "Basic ideas of Ammann bar grids", *Int. J. Mod. Phys.* **B7** (1993) 1437–53.
52. W. F. Lunnon and P. A. B. Pleasants, "Quasicrystallographic tilings", *J. Math. pures et appl.* **66** (1987) 217–63.
53. N. D. Mermin, "(Quasi) crystallography is better in Fourier space", in: *Quasicrystals: The State of the Art*, eds. D. P. DiVincenzo and P. J. Steinhardt, World Scientific, Singapore (1991), 133–83.
54. R. V. Moody (ed.), *The Mathematics of Long-Range Aperiodic Order*, NATO ASI Series C 489, Kluwer, Dordrecht (1997).
55. K. Niizeki, "Self-similarity of quasilattices in two dimensions: II. The non-Bravais-type n-gonal quasilattice", *J. Phys.* **A22** (1989) 205–18.
56. H.-U. Nissen and C. Beeli, "Electron microscopy of quasicrystals and the validity of the tiling approach", *Int. J. Mod. Phys.* **B7** (1993) 1387–413.
57. C.Oguey, M. Duneau and A. Katz, "A geometrical approach of quasiperiodic tilings", *Commun. Math. Phys.* **118** (1988) 99–118.
58. G. van Ophuysen, M. Weber and L. Danzer, "Strictly local growth of Penrose patterns", *J. Phys.* **A28** (1995) 281–90.
59. P. A. B. Pleasants, "The construction of quasicrystals with arbitrary symmetry group", in: *Quasicrystals*, eds. C. Janot and R. Mosseri, World Scientific, Singapore (1995), p. 22–30.
60. R. Penrose, "The rôle of aesthetics in pure and applied mathematical research", *Bull. Inst. Math. Appl.* **10** (1974) 266–71, reprinted in [71].
61. C. Richard, M. Hoeffe, J. Hermisson and M. Baake, "Random tilings: concepts and examples", *J. Phys.* **A31** (1998) 6385–408.
62. D. S. Rokhsar, N. D. Mermin and D. C. Wright, "The two-dimensional quasicrystallographic space groups with rotational symmetry less than 23-fold", *Acta cryst.* **A44** (1988) 197–211.
63. J. Roth, "The equivalence of two face-centred icosahedral tilings with respect to local derivability", *J. Phys.* **A26** (1993) 1455–61.
64. W. Rudin, *Functional Analysis*, 2nd ed., McGraw Hill, New York (1991).
65. D. Shechtman, I. Blech, D. Gratias and J. Cahn, "Metallic phase with long-range orientational order and no translational symmetry", *Phys. Rev. Lett.* **53** (1984) 1951–3, reprinted in [71].
66. M. Schlottmann, "Geometrische Eigenschaften quasiperiodischer Strukturen", Dissertation, Univ. Tübingen (1993); and: "Periodic and quasi-periodic Laguerre tilings", *Int. J. Mod. Phys.* **B7** (1993) 1351–63.
67. M. Schlottmann, "Projection formalism for locally compact Abelian groups", in: *Quasicrystals and Discrete Geometry*, ed. J. Patera, Fields Institute Monographs, vol. 10, AMS, Rhode Island (1998), pp. 247–64.
68. R. L. E. Schwarzenberger, *N-dimensional Crystallography*, Pitman, San Francisco (1980).
69. J. E. S. Socolar and P. J. Steinhardt, "Quasicrystals. II. Uni-cell configurations", *Phys. Rev.* **B34** 1986 617–47, reprinted in [71].
70. B. Solomyak, "Dynamics of self-similar tilings", *Ergod. Th. & Dynam. Syst.* **17** (1997) 695–738.
71. P. J. Steinhardt and S. Ostlund (eds.), *The Physics of Quasicrystals*, World Scientific, Singapore (1987).
72. J. B. Suck, "Prehistory of quasicrystals", this volume.
73. A. Sütő, "Schrödinger difference equation with deterministic ergodic potentials", in: *Beyond Quasicrystals*, eds. F. Axel and D. Gratias, Springer, Berlin (1995), p. 481–549.