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SYMMETRIES AND REVERSING SYMMETRIES OF TRACE MAPS

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ABSTRACT

A (discrete) dynamical system may have various symmetries and reversing symmetries, which together form its so-called reversing symmetry group. We study the set of 3D trace maps (obtained from two-letter substitution rules) which preserve the Fricke-Vogt invariant $I(x,y,z)$. This set of dynamical systems forms a group $\mathcal{G}$ isomorphic with the projective linear (or modular) group $PGl(2,\mathbb{Z})$. For such trace maps, we give a complete characterization of the reversing symmetry group as a subgroup of the group $\mathcal{A}$ of all polynomial mappings that preserve $I(x,y,z)$.

1. Introduction

Trace maps of two-letter substitution rules proved useful for the determination of physical properties of various 1D systems[1] on non-periodic self-similar structures such as the Fibonacci chain. These mappings found a systematic algebraic description in terms of the free group with two generators and its automorphism (resp. homomorphism) group[2,3]; for a summary of theory and applications, see Ref. 3.

Here, we are interested in trace maps as 3D dynamical systems, in particular in those of the Nielsen class which preserve the Fricke-Vogt invariant[3,4],

$$I(x,y,z) \;=\; x^2+y^2+z^2-2xyz-1. \tag{1}$$

The set $\mathcal{G}$ of *Nielsen trace maps* is a group[3,4] isomorphic with the projective linear (or modular) group $PGl(2,\mathbb{Z}) \simeq Gl(2,\mathbb{Z})/\{\pm\mathbb{1}\}$. In the standard representation[2,3], every $F \in \mathcal{G}$ is an invertible polynomial mapping of 3-space with integer coefficients. The group $\mathcal{G}$ can be characterized as

$$\mathcal{G} \;=\; \{F \in \mathbb{Z}[x,y,z]^3 \;\mid\; I(F(x,y,z)) = I(x,y,z) \text{ and } F(1,1,1) = (1,1,1)\}\,, \tag{2}$$

with composition of mappings as group multiplication and $\mathcal{G} \simeq PGl(2,\mathbb{Z})$.

2. Symmetries and reversing symmetries

A trace map F may possess certain *symmetries* (i.e., there may be invertible transformations S with $F \circ S = S \circ F$) which form a group, $\mathcal{S}(F)$. Typically, one specifies a subclass $\mathcal{B}$ of mappings in which to look for possible symmetries, which we indicate here by the notation $\mathcal{S}_{\mathcal{B}}(F)$. The choice of this subclass is guided by key properties of the original transformations, i.e., in our case, the trace maps. We shall give a systematic description of $\mathcal{S}_{\mathcal{A}}(F)$, the group of symmetries of Nielsen trace maps within the set $\mathcal{A}$ of polynomial mappings that leave $I(x,y,z)$ invariant,

$$\mathcal{A} \;=\; \{A \in \mathbb{C}[x,y,z]^3 \;\mid\; I(A(x,y,z)) = I(x,y,z)\}. \tag{3}$$

It turns out that $\mathcal{A}$ is a *group* (hence, $A \in \mathcal{A}$ implies A invertible) which contains $\mathcal{G}$ as a subgroup via the semidirect product

$$\mathcal{A} \;=\; \Sigma \otimes_s \mathcal{G} \;\simeq\; \Sigma \otimes_s PGl(2,\mathbb{Z})\,, \tag{4}$$

where $\Sigma = \{Id, \sigma_1, \sigma_2, \sigma_3\} \simeq C_2 \otimes C_2$ (Klein's 4-group) is the normal subgroup[2,4]. (Here and in what follows, we use C_n for the cyclic group of order n.) The involutions σ_i are given by $\sigma_1(x,y,z) = (x,-y,-z)$ etc. The set $\mathcal{A}$ is quite remarkable: for any mapping $A \in \mathcal{A}$, all polynomial coefficients are integers[2,4], wherefore we automatically have $A \in \mathbb{Z}[x,y,z]^3$.

Furthermore, $F \in \mathcal{G}$ can be *(weakly) reversible* (i.e., there is a reversing symmetry G with $G \circ F \circ G^{-1} = F^{-1}$) or *(strictly) reversible* (if $G^2 = Id$, the identity mapping), see Ref. 4. We will drop the historical attributes "weakly" and "strictly", but distinguish trivial reversibility (where $F^2 = Id$) from *true* reversibility (where F is not an involution). Reversibility is an important tool for the analysis of dynamical systems[5,6]. The symmetries and reversing symmetries together form the so-called reversing symmetry group[6], $\mathcal{R}(F)$. If F with $F^2 \neq Id$ is reversible, $\mathcal{R}(F)$ is a *group extension* of the symmetry group $\mathcal{S}(F)$ of index 2. Alternatively, it can be seen as a group with a natural $\mathbb{Z}_2$-grading[7]. Here, we shall describe $\mathcal{R}_{\mathcal{A}}(F)$. This is not really a restriction – see Ref. 4 for a first discussion of extensions, in particular to *arbitrary* polynomial mappings of finite order.

3. Results

Let us now formulate our results. We do this in two steps, giving the structure of the (reversing) symmetry group first within $\mathcal{G} \simeq PGl(2,\mathbb{Z})$ and then within $\mathcal{A}$.

Theorem 1 *The symmetry group $\mathcal{S}_{\mathcal{G}}(F)$ for $F \in \mathcal{G}$ is:*
(1) $\mathcal{G}$ if and only if $F = Id$,
(2) $C_2 \otimes C_2$ if and only if $F^2 = Id \neq F$,
(3) C_3 if and only if $F^3 = Id \neq F$,
(4) C_∞ in all remaining cases, i.e., whenever F is not of finite order.

The symbol C_∞ is used for the infinite cyclic group generated from one generator, i.e. $C_\infty \simeq \mathbb{Z}$ as groups. Unfortunately, we can only give a hint of the proof here, for details see Ref. 7. One has to use $\mathcal{G} \simeq PGl(2,\mathbb{Z})$ and to distinguish the elements of

finite order (possible is order 1, 2, or 3, where the proof is almost straightforward) from those of infinite order. In the latter case, $\mathcal{S}_{\mathcal{G}}(F)$ does certainly contain the subgroup $\{F^n \,|\, n \in \mathbb{Z}\}$, but often more: F may have roots within $\mathcal{G}$ which still give rise to the group C_∞. To show that nothing else can happen, one uses the Dirichlet unit theorem of algebraic number theory for quadratic fields. This gives the answer for all truly hyperbolic or elliptic 2×2-matrices M_F, to be taken mod $\pm\mathbb{1}$. The remaining case, $\mathrm{tr}(M_F) = 2$ with $\det(M_F) = 1$, can be checked explicitly.

Theorem 2 *The reversing symmetry group $\mathcal{R}_{\mathcal{G}}(F)$ of $F \in \mathcal{G}$ is:*
(1) C_∞ if and only if F is not reversible,
(2) $\mathcal{G}$ if and only if $F = Id$,
(3) $D_2 \simeq C_2 \otimes C_2$ if and only if $F^2 = Id \neq F$,
(4) D_3 if and only if $F^3 = Id \neq F$, or
(5) D_∞, if F is truly reversible but not of finite order.

Here, $D_n \simeq C_n \otimes_s C_2$ is the dihedral group, with D_∞ being the extension to $n = \infty$. This Theorem shows that all elements of finite order are reversible[4], which is trivial only for order 1 and 2. If F is an involution, we just have $\mathcal{S}_{\mathcal{G}}(F) = \mathcal{R}_{\mathcal{G}}(F)$. The question whether a given element of infinite order actually is reversible involves the representation of ± 1 by integer quadratic forms and is, for any given F, decidable in finitely many steps[7].

Now, the extension of $\mathcal{S}_{\mathcal{G}}(F)$ ($\mathcal{R}_{\mathcal{G}}(F)$) to $\mathcal{S}_{\mathcal{A}}(F)$ ($\mathcal{R}_{\mathcal{A}}(F)$), respectively, requires Eq. (4) and the precise knowledge of how $\mathcal{G}$ acts on Σ. With $\Sigma = \{Id, \sigma_1, \sigma_2, \sigma_3\}$, $F \in \mathcal{G}$ acts – via conjugation – on Σ as a permutation $\pi_F \in S_3$:

$$F \circ \sigma_i \circ F^{-1} \;=\; \sigma_{\pi_F(i)}. \tag{5}$$

The permutation π_F is uniquely determined by F and easy to calculate[4,7]. Such a permutation can only be of order 1, 2, or 3, so $(\pi_F)^6 = id$ (the identity in S_3) for *all* $F \in \mathcal{G}$ and

$$\mathcal{S}_{\mathcal{A}}(F) \;\subset\; \mathcal{S}_{\mathcal{A}}(F^6) \;=\; \Sigma \otimes_s \mathcal{S}_{\mathcal{G}}(F^6)\,. \tag{6}$$

The exponent 6 can of course be replaced by the number $n = \mathrm{ord}(\pi_F)$. Now, we have two possibilities: either $\mathcal{S}_{\mathcal{G}}(F^6) = \mathcal{G}$ (if F is of finite order) or $\mathcal{S}_{\mathcal{G}}(F^6) = \mathcal{S}_{\mathcal{G}}(F) \simeq C_\infty$ (if F is not of finite order). The latter statement is a consequence of the Cayley-Hamilton theorem applied to unimodular matrices. If C_∞ is generated by F' (which could be a root of F in $\mathcal{G}$) and if $\pi_{F'} = id$, the product in Eq. (6) is direct.

Similarly, in case of *true* reversibility (i.e., $[\mathcal{R}_{\mathcal{A}}(F) : \mathcal{S}_{\mathcal{A}}(F)] = 2$), one obtains

$$\mathcal{R}_{\mathcal{A}}(F) \;\subset\; \mathcal{R}_{\mathcal{A}}(F^6) \;=\; \Sigma \otimes_s \mathcal{R}_{\mathcal{G}}(F^6), \tag{7}$$

where we can again simplify the right hand side through the relation $\mathcal{R}_{\mathcal{G}}(F^6) = \mathcal{R}_{\mathcal{G}}(F)$ if F is not of finite order[4,7], and through $\mathcal{R}_{\mathcal{G}}(F^6) = \mathcal{G}$ if $F^6 = Id$.

However, one might not only be interested in the (reversing) symmetry group of some power of F, cf. Ref. 6, but in that of F itself. To this end, we have to define

$$K_\Sigma(F) \;:=\; \{g \in \Sigma \mid [g, F] = 0\} \tag{8}$$

which is a subgroup of Σ, hence one of the groups $\{Id\}$, $\{Id, \sigma_1\}$, $\{Id, \sigma_2\}$, $\{Id, \sigma_3\}$, or Σ. From the factorization property[4] of the semidirect product (4), we then obtain

Theorem 3 *Let F be a Nielsen trace map. The symmetry group within $\mathcal{A}$ is*

$$\mathcal{S}_{\mathcal{A}}(F) = K_{\Sigma}(F) \otimes_s \mathcal{S}_{\mathcal{G}}(F)$$

while – in case of true reversibility – the reversing symmetry group is

$$\mathcal{R}_{\mathcal{A}}(F) = K_{\Sigma}(F) \otimes_s \mathcal{R}_{\mathcal{G}}(F).$$

The products can be direct if the elements of $K_{\Sigma}(F)$ commute with all elements of the symmetry group $\mathcal{S}_{\mathcal{G}}(F)$ resp. the reversing symmetry group $\mathcal{R}_{\mathcal{G}}(F)$ (which includes the case $K_{\Sigma}(F) = \{Id\}$). A comparison of Thm. 3 with Eqs. (6) and (7) illustrates how the semidirect product structure of $\mathcal{A}$ can lead to an element of $\mathcal{G}$ that has fewer symmetries and reversing symmetries than certain of its iterates. One example is provided by the Fibonacci trace map[4] where F^3 commutes with every element of Σ but F does not.

From the three Theorems, the structure of the (reversing) symmetry group within $\mathcal{G}$ as well as within $\mathcal{A}$ is completely classified. Additionally, due to several examples, we tend to believe that any F which is not reversible within $\mathcal{A}$ is not reversible at all[4,7].

4. Concluding remarks

Let us briefly comment on general trace maps of two-letter substitution rules. If they do not belong to the Nielsen class discussed above, they cannot be of finite order and, even more, they are not globally invertible. This also implies that non-Nielsen trace maps are never reversible in the above sense. However, certain (generalized) symmetries and reversing symmetries may show up in subspaces. This calls for a suitable generalization of the (reversing) symmetry concept, in particular for the consideration of non-invertible mappings, e.g., projectors in combination with an invertible transformation. Though this gives a variety of interesting possibilities, we presently do not see anything close to a classification like that given above for the Nielsen class.

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