Limit-(quasi-)periodic point sets as quasicrystals with p-adic internal spaces

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Dedicated to Peter Kramer on the occasion of his 65th birthday

Abstract. Model sets (or cut and project sets) provide a familiar and commonly used method of constructing and studying nonperiodic point sets. Here we extend this method to situations where the internal spaces are no longer Euclidean, but instead spaces with p-adic topologies or even with mixed Euclidean/p-adic topologies. We show that a number of well known tilings precisely fit this form, including the chair tiling and the Robinson square tilings. Thus the scope of the cut and project formalism is considerably larger than is usually supposed. Applying the powerful consequences of model sets we derive the diffractive nature of these tilings.

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1. Introduction

The cut and project method of constructing nonperiodic point sets, as developed by Peter Kramer and others in the early eigthies [10, 11, 9, 3, 12], is one of the basic tools in the mathematical study of quasicrystals and aperiodic order. The intuition behind their use is that quasiperiodic point sets, such as those arising in many nonperiodic tilings and also in the diffraction patterns of physical quasicrystals, may be viewed as arising from the projection of lattices in some higher dimensional spaces. Thus the physical space is complemented by an internal space (possibly of some other dimension), a lattice is located in the combined physical-internal space pair, and the projection maps are used to create a cut and project scheme.

The same type of mathematical structure had also arisen (before the recent excitement about quasicrystals, and in a very different context) in the work of Yves Meyer [13] in which the formalism is expressed entirely in terms of locally compact Abelian groups. In [14, 19] these ideas were taken up and extended in the context of aperiodic order, with the result that a considerable amount of the mathematical theory underlying these cut and project sets (or model sets) can now be seen to hold in great generality. Up to now, however, no attempt has actually been made to see to what extent existing aperiodic structures might be explained in terms of these more general types of spaces. In this paper we address this question, showing that a number of familiar tilings and substitution systems, so far not contained under the aegis of the cut and project formalism, are in fact based on internal spaces with *non-Euclidean* topologies, namely p-adic topologies or mixed Euclidean/p-adic topologies.

There is a variety of discrete structures known that display a pure point diffraction spectrum, i.e., the Fourier transform of their autocorrelation (which is a positive measure) is pure point, compare [8, 22]. Among the known examples are model sets, but also certain inflation-generated point sets and tilings, e.g. the chair or the sphinx tiling [22]. They are limit-periodic structures with a countably, but not finitely, generated Fourier module – so, they cannot be described in the "conventional" cut and project setup where the internal space is Euclidean.

It is the aim of this contribution to start to develop a proper generalization of the projection method, in the spirit of Meyer, to include such limit-periodic and even limit-quasiperiodic sets. Here, we explain, in an illustrative fashion, how this works. A detailed approach to model sets over arbitrary internal groups, and especially to aspects of diffraction, will appear in [20].

Let us recall the notion of a cut and project scheme. By definition, this consists of a collection of spaces and mappings:

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times G & \xrightarrow{\pi_2} & G \\ & & \cup & & \\ & & L & & \end{array} \tag{1}$$

where $\mathbb{R}^d$ is a real Euclidean space and G is some locally compact Abelian group, π_1 and π_2 are the projection maps onto them, and $L \subset \mathbb{R}^d \times G$ is a lattice, i.e., a discrete subgroup such that the quotient group $(\mathbb{R}^d \times G)/L$ is compact. We assume that $\pi_1|_L$ is injective and that $\pi_2(L)$ is dense in G. We call $\mathbb{R}^d$ (resp. G) the physical (resp. internal) space.

In this definition, we have already oriented the situation to physical applications by assuming that the physical space is indeed a real Euclidean space. On the other hand, allowing G to be an arbitrary locally compact Abelian group is precisely the point at which we are going beyond the usual situation of an internal space that is also Euclidean.

Given any subset $\Omega \subset G$, we define a corresponding set $\Lambda(\Omega) \subset \mathbb{R}^d$ by

$$\Lambda(\Omega) \;=\; \{\pi_1(x) \mid x \in L,\, \pi_2(x) \in \Omega\}\,. \tag{2}$$

We call such a set Λ a *model set* (or *cut and project set*) if the following condition is fulfilled,

W1 $\Omega \;=\; \overline{\mathrm{int}(\Omega)} \;\neq\; \emptyset$ is compact.

Furthermore, we are mainly interested in the situation that the boundary of Ω does not contain any points of $\pi_2(L)$. If this is the case, we call $\Lambda(\Omega)$ *regular*. The importance of regular model sets is that they are necessarily repetitive [19]. Note that, if Ω fulfils **W1**, $\partial\Omega$ (which equals $\Omega\backslash\mathrm{int}(\Omega)$) is nowhere dense and hence a meager set. Then, it follows from the Baire category theorem that no countable union of translates of $\partial\Omega$ can cover G which is a Baire space. In particular, it is always possible to choose a shift $c \in G$ such that the boundary of $c + \Omega$ satisfies the additional regularity condition.

In the sequel, showing that a certain set is a model set actually means, more precisely, to show that there exist G, L and Ω subject to the above conditions such that $\Lambda(\Omega)$ is regular and locally isomorphic to the given set (for terminology, see [1, 19] and references therein). This way, our results are valid for entire LI-classes, even if, for simplicity, we only talk of single tilings.

2. p-adic topologies and inverse limits of finite groups

Let p be a prime number in the integers $\mathbb{Z}$. Using p we can define a metric on the rational numbers $\mathbb{Q}$, and by restriction on $\mathbb{Z}$, in the following way. For each $a \in \mathbb{Z}$, we define its p-value, $\nu_p(a)$, as the largest exponent k for which p^k divides a (with $\nu_p(0) := \infty$). This function is extended to the p-adic valuation $\nu_p : \mathbb{Q} \longrightarrow \mathbb{Z}$ by $\nu_p(a/b) := \nu_p(a) - \nu_p(b)$ for all rational numbers a/b. We now define the "distance" between two rational numbers x, y as $d(x, y) = p^{-\nu_p(y-x)}$.

It is not hard to see that this does define a metric on $\mathbb{Q}$, in which closeness to 0 is equivalent to high divisibility by the prime p. The completion of the rationals under this topology is the field of p-adic numbers $\widehat{\mathbb{Q}_p}$ and the completion of $\mathbb{Z}$ is the subring of p-adic integers, $\widehat{\mathbb{Z}_p}$. Any p-adic integer has a unique expansion (as a convergent series) in the form $\sum_{n=0}^{\infty} a_n p^n$ where the a_p are integers in the range $0 \leq a_p < p$. The topologies defined by such metrics have rather counter-intuitive properties. For example, for each non-negative integer k, the set $p^k \cdot \widehat{\mathbb{Z}_p}$, the set of elements of $\widehat{\mathbb{Z}_p}$ divisible by p^k, is the ball of radius p^{-k} and is clopen, i.e. both open and closed. $\widehat{\mathbb{Z}_p}$, seen as a topological space, is both compact and totally disconnected. In particular, $\widehat{\mathbb{Q}_p}$ and $\widehat{\mathbb{Z}_p}$ are locally compact Abelian groups under addition. Thus, we can use $\widehat{\mathbb{Z}_p}$ to construct interesting cut and project schemes for $\mathbb{R}^d$ simply by taking $G := \widehat{\mathbb{Z}_p}$ and $L = \mathbb{Z}^d$ embedded diagonally into $\mathbb{R}^d \times \widehat{\mathbb{Z}_p}$. For more on p-adic numbers and other totally disconnected groups, the reader may consult [15, 21, 7, 2].

There is another description of $\widehat{\mathbb{Z}_p}$ which is more revealing of its appearance in the context of self-similarity and generalizes what we have just done. Let

$$F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow \ldots \tag{3}$$

be an inverse system of finite Abelian groups, i.e. each F_i is a finite Abelian group (with discrete toplogy) and the arrows represent surjective group homomorphisms. Define the set of compatible sequences

$$\overleftarrow{F} := \{\tilde{x} = (x_1, x_2, \ldots) \mid x_i \in F_i,\ x_i \leftarrow\!\shortmid\ x_{i+1},\ i \in \mathbb{N}\}. \tag{4}$$

$\overleftarrow{F}$ is given the structure of a group by component-wise addition. It is structured as a topological group by the induced topology from the product $\prod_{i=1}^{\infty} F_i$. Equivalently, the subgroups

$$\overleftarrow{F}_n = \{\tilde{x} \in \overleftarrow{F} \mid x_1 = x_2 = \ldots = x_{n-1} = 0\} \tag{5}$$

form a subbase of open neighbourhoods of 0 in $\overleftarrow{F}$. Since $[F : F_n]$ is finite, the subgroups $\overleftarrow{F}_n$ are also closed. With this topology, $\overleftarrow{F}$ is a compact totally disconnected Abelian group (so, in particular, a locally compact Abelian group). Groups of this type are called profinite groups.

As an example, for each prime number p, we can construct $\widehat{\mathbb{Z}_p}$ by the inverse system

$$\overleftarrow{\mathbb{Z}}_p : \quad \mathbb{Z}/p\mathbb{Z} \leftarrow \mathbb{Z}/p^2\mathbb{Z} \leftarrow \mathbb{Z}/p^3\mathbb{Z} \leftarrow \ldots \tag{6}$$

The relevance to the work here is this: if $\theta : L \to L$ is an injective homomorphism that is a self-similarity of Λ, then there is a clear distinction between the case that

$\theta(L) = L$ (θ is a "unit") and the case that $\theta(L) \subset L$, but $\theta(L) \neq L$. In the latter case, $[L : \theta(L)]$ is finite and we have the inverse system

$$\overleftarrow{L}(\theta) \; : \quad L/\theta(L) \leftarrow L/\theta^2(L) \leftarrow L/\theta^3(L) \leftarrow \ldots \tag{7}$$

The compact group $\overleftarrow{L}$ is invariant under the action of θ. Note that it contains a canonical copy of L itself via the mapping

$$x \;\mapsto\; ((x \mod \theta(L)) \hookleftarrow (x \mod \theta^2(L)) \hookleftarrow (x \mod \theta^3(L)) \hookleftarrow \ldots)\,. \tag{8}$$

Again, we obtain a cut and project scheme via the diagonal embedding of L in $\mathbb{R}^m \times \overleftarrow{L}$. Let us now turn to some applications.

3. A limit-periodic substitution system

Consider the primitive three-letter substitution system

$$a \to ab \qquad b \to abc \qquad c \to abcc \tag{9}$$

The standard analysis of the corresponding substitution matrix [17] shows that a proper geometric realization demands length ratios $\ell(a) : \ell(b) : \ell(c) = 1 : 2 : 3$, while all three letters finally occur with equal frequency $1/3$.

To obtain a bidirectional infinite sequence that is a fixed point, we may start with the pair $c|a$ and keep on applying the substitution rule:

$$\ldots\, ababcabccabcc|ababcababcabccababcababcabcc \,\ldots \tag{10}$$

We can imagine this as labelling the tiles of a tiling of $\mathbb{R}$ in which the tiles are of lengths $1, 2, 3$ respectively. If we identify each tile with its right-hand end point, starting with a tile of type a (length 1) at the origin, then we end up with a sequence of numbers

$$\ldots\text{-26,-24,-21,-18,-17,-15,-12,-9,-8,-6,-3,0,1,3,4,6,9,10,12,13,15,18,19}\ldots \tag{11}$$

The main property of this sequence is its invariance under the transformation $x \mapsto 3x$. This self-similarity with a *rational* scaling factor is the signal that there may be a p-adic interpretation. At the same time, as 3 is not a unit, this sequence is a candidate for a so-called limit-periodic point set, compare [5]. In fact, this sequence can be given a 3-adic interpretation. The coordinates of the tiles of the three types (resp. their right hand endpoints) can be explicitly given as follows:

- type a : $\bigcup_{k=2}^{\infty}(1 + 3 + \ldots + 3^{k-2}) \, + \, 3^k\mathbb{Z}$
- type b : $\bigcup_{k=2}^{\infty}(2 + 1 + 3 + \ldots + 3^{k-2}) \, + \, 3^k\mathbb{Z}$
- type c : $3^2\mathbb{Z} \, \cup \, \big(\bigcup_{k=3}^{\infty}(-3 - \ldots - 3^{k-2}) \, + \, 3^k\mathbb{Z}\big)$.

It is easy to see that these sets are invariant under the process of formation of the tiles (i.e., rule (9)). Furthermore, their densities in the lattice of integers $\mathbb{Z}$ are easily computed to be each equal to $1/6$, thus accounting for the entire tiling $((1+2+3)\cdot\frac{1}{6}=1)$.

Interpreting these sets p-adically, we see that they are dense subsets of the unions of open balls formed by replacing $3^k\mathbb{Z}$ by $3^k\widehat{\mathbb{Z}_3}$ in each of the sets above. In this way, we obtain three "windows", each with compact closure. Furthermore, it is not hard to see that each of them has a boundary with just finitely many points.

Using the lattice $L := \{(n,n)|n \in \mathbb{Z}\} \subseteq \mathbb{R}\times\widehat{\mathbb{Z}_3}$ we obtain a cut and project scheme and conclude that the point sets corresponding to each of the three tile types is a model set. As a consequence, the sequence is an example with pure point diffraction spectrum. This observation is consistent with the following application of Dekking's criterion [4] to a locally equivalent sequence [1], which would share pure pointness with our above example due to the invariance of this spectral property under mutual local derivability. Indeed, replacing the configuration ab by a new tile A of length 3 one obtains a sequence invariant under the substitution rule $A \to AAc$, $c \to Acc$ which has a pure point diffraction spectrum because the new substitution rule is of constant length and exhibits a so-called coincidence [4].

4. The chair tiling

The 2D chair tiling is defined by the substitution rule in Figure 1a. This has recently been shown to display a pure point diffraction spectrum [22]. Instead of working with the chair tiling directly, we introduce a convenient modification by substituting each "chair" by three decorated squares as in Figure 1b. Since this transformation is local in the sense of [1] and can be locally inverted by the rule given in Figure 1c, it follows that showing that the chair tiling is a cut and project tiling is equivalent to showing that the modified tiling is. With the help of the transformation rules, one sees immediately that the modified tiling fulfils the substitution rule in Figure 1d.

A particular tiling $\mathcal{T}_0$ in the LI class under consideration can be defined in the following way. Starting with a decorated square S_0 of side length 1 centered at the origin of a fixed coordinate system where the arrow points towards the upper right corner $(1/2,1/2)$, one performs successively the following two steps:

(i) perform the affine transformation

$$T: \quad x \mapsto Tx := 2Rx + \frac{1}{2}(\vec{e}_1+\vec{e}_2), \tag{12}$$

where R denotes rotation by $\pi/2$ and $\vec{e}_i$ are the canonical unit vectors;

(ii) apply the (appropriately rotated) substitution rule of Figure 1d.

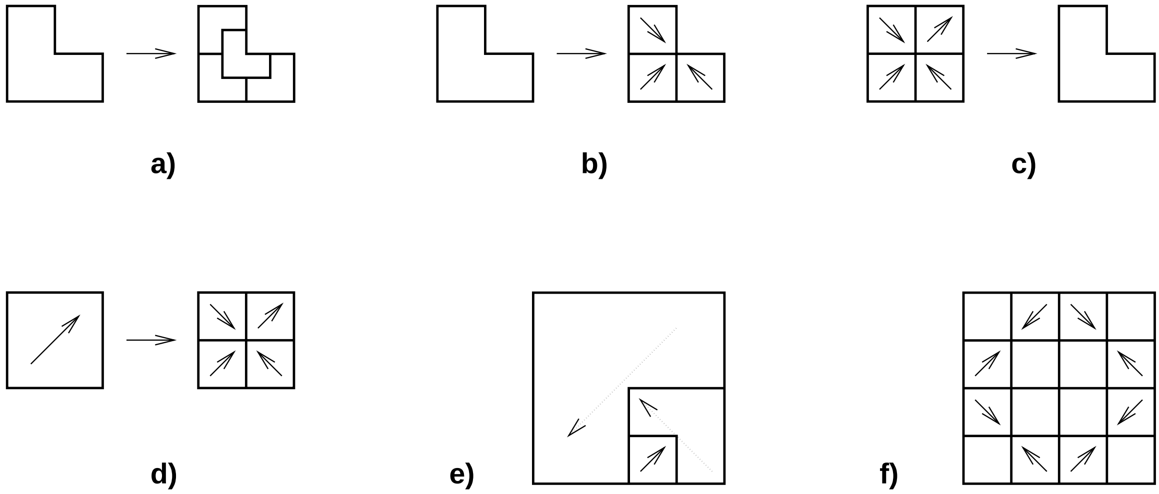

Figure 1. Geometric realization of the chair tiling.

This way, larger and larger portions of a unique member of the LI class are obtained (see Figure 1e).

As a consequence of the construction, the centers of the decorated squares form the lattice $\mathbb{Z}^2$. Let P_k be the subset of the centers of squares oriented as $R^k S_0$, $k = 0, 1, 2, 3$, respectively. Clearly, the tiling is completely determined by these subsets.

We will show that each P_k is a model set with internal group $G = \widehat{\mathbb{Z}_2} \times \widehat{\mathbb{Z}_2}$. As in our previous one-dimensional example, this will be done by writing P_k as a union of cosets of certain sublattices of $\mathbb{Z}^2$. Because of the partial symmetry of the decoration of the second substitution step (see Figure 1f), the tiles which are decorated in this Figure must repeat with period $4\mathbb{Z}^2$. If C is the undecorated square underlying S_0, viewed as a subset of $\mathbb{R}^2$, we must have, as a consequence of the self-similarity involved in the definition of $\mathcal{T}_0$, that

$$(P_k \cap T^i C) + 2^i \cdot 4\mathbb{Z}^2 \subseteq P_k \tag{13}$$

for all $i \in \mathbb{N}$, $k \in \{0, 1, 2, 3\}$. Therefore, if we set $P_{k,i} := P_k \cap T^i C$, we get

$$P_k = \bigcup_{i \in \mathbb{N}} \bigcup_{t \in P_{k,i}} (t + 2^i \cdot 4\mathbb{Z}^2) \,, \tag{14}$$

which is the desired decomposition of P_k.

Using the substitution rule, the finite sets $P_{k,i} := P_k \cap T^i C$ can actually be calculated by recursion:

$$P_{0,0} = \{0\} \,, \;\; P_{k,0} = \emptyset \,, \;\; k \in \{1, 2, 3\}, \tag{15}$$

$$P_{k,i+1} = \bigcup_{l=0}^{3} T^i M_l T^{-i} (P_{(k-n_l)_4, i}) \tag{16}$$

for the integers $n_0 := 0$, $n_1 := 1$, $n_2 := 2$, $n_3 := 1$ and the affine transformations M_l given by $M_0 x := x$, $M_1 x := Rx + \vec{\boldsymbol{e}}_1$, $M_2 x := R^2 x + \vec{\boldsymbol{e}}_1 + \vec{\boldsymbol{e}}_2$, $M_3 x := Rx + \vec{\boldsymbol{e}}_2$.

In much the same way as in the one-dimensional example, the decomposition (14) leads to a description of P_k as model sets. The lattice $\mathbb{Z}^2$ is embedded in G in the canonical fashion. In the embedding space $\mathbb{R}^2 \times G$ we choose the lattice $L := \{(\vec{n}, \vec{n}) | \vec{n} \in \mathbb{Z}^2\}$. For each $i \in \mathbb{N}$, the closure with respect to the 2-adic topology of the sublattice $2^i \cdot 4\mathbb{Z}^2$ is an open and compact subgroup of G, actually equal to $2^i \cdot 4G$. Replacing $\mathbb{Z}^2$ by G in (14) and taking the 2-adic closure gives the description of windows $\Omega_k \subseteq G$ in G. It is easily seen that P_k is the model set using G as internal space, L as lattice and Ω_k as window.

Finally, we show that the boundary of each open set Ω_k has Haar measure 0. Let μ be the Haar measure on G and assume it is normalized to $\mu(G) = 1$. Then the measure of any coset $t + 2^i \cdot 4 \cdot G$ is $1/[G : (2^i \cdot 4 \cdot G)]$. On the other hand, the proportion of points of $\mathbb{Z}^2$ lying in the coset $t + 2^i \cdot 4\mathbb{Z}^2$ is $1/[\mathbb{Z}^2 : (2^i \cdot 4\mathbb{Z}^2)]$ which is exactly the same number. Then the proportion of points of $\mathbb{Z}^2$ occupied by the cosets belonging to P_k is exactly the same as $\mu(\Omega_k)$. Since $\bigcup_{k=0}^{3} P_k = \mathbb{Z}^2$ we obtain $\sum_{k=0}^{3} \mu(\Omega_k) = 1$. From $\overline{\Omega}_0 \cap (\Omega_1 \cup \Omega_2 \cup \Omega_3) = \emptyset$ we have $1 \geq \mu(\overline{\Omega}_0) + \sum_{k=1}^{3} \mu(\Omega_k) \geq \sum_{k=0}^{3} \mu(\Omega_k) = 1$ and so $\mu(\overline{\Omega}_0) = \mu(\Omega_0)$. Similarly, one gets $\mu(\partial\Omega_k) = 0$ for $k = 0, 1, 2, 3$ as required.

5. A limit-quasiperiodic example

The substitution matrix of the primitive two letter substitution system

$$a \to aab \qquad b \to abab \tag{17}$$

has the Perron-Frobenius eigenvalue $\lambda := 2 + \sqrt{2}$ which is a Pisot-Vijayaraghavan number but not a unit. Therefore, any possible description as a model set of the resulting substitution sequence will have to use more complicated groups than $\mathbb{R}^n$ as embedding space.

A geometric representation of the substitution system is obtained by replacing symbols a and b by intervals of length $\ell(a) = 1$ and $\ell(b) = \sqrt{2}$. If we denote the sets of left endpoints of the a and b intervals by Λ_a and Λ_b, the substitution rule leads to the following system of recursion relations:

$$\Lambda_a = (\lambda\Lambda_a) \cup (\ell(a) + \lambda\Lambda_a) \cup (\lambda\Lambda_b) \cup (\ell(a) + \ell(b) + \lambda\Lambda_b) \tag{18}$$

$$\Lambda_b = (2\ell(a) + \lambda\Lambda_a) \cup (\ell(a) + \lambda\Lambda_b) \cup (2\ell(a) + \ell(b) + \lambda\Lambda_b) \tag{19}$$

where the right hand sides represent disjoint unions. Both Λ_a and Λ_b are subsets of the group $\mathbb{Z}[\sqrt{2}]$ which can be mapped onto $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ by sending $\ell(a)$ to $\vec{\boldsymbol{e}}_1$ and $\ell(b)$ to $\vec{\boldsymbol{e}}_2$. The transformation $t \mapsto \lambda t$ in $\mathbb{Z}[\sqrt{2}]$ induces the linear transformation $\phi : \vec{\boldsymbol{e}}_1 \mapsto 2\vec{\boldsymbol{e}}_1 + \vec{\boldsymbol{e}}_2,\ \vec{\boldsymbol{e}}_2 \mapsto 2\vec{\boldsymbol{e}}_1 + 2\vec{\boldsymbol{e}}_2$ of $\mathbb{R}^2$. Notice that $\phi(\mathbb{Z}^2) \subset \mathbb{Z}^2$, but $\phi(\mathbb{Z}^2) \neq \mathbb{Z}^2$.

The transformation ϕ has the two eigenvalues $2\pm\sqrt{2}$; we may identify the "physical" space $\mathbb{R}$ with the subspace V of $\mathbb{R}^2$ corresponding to the eigenvalue $2+\sqrt{2}$. Then, all points Λ_a and Λ_b of a substitution sequence according to (17) are images of uniquely determined points of $\mathbb{Z}^2$ under the projection onto the physical space V along the second invariant subspace of ϕ. (Note that, as the transformation matrix of ϕ is not normal, this projection is not orthogonal with respect to the canonical metric of $\mathbb{R}^2$; see Figure 2.) Because the second eigenvalue is smaller than 1, the preimages must lie in a bounded strip parallel to V, and it is easily calculated that, for the sequence generated from ba, where the middle vertex is the origin, the preimages lie in the strip $V+\{t(0,-1-\sqrt{2})\,|\,0\le t\le 1\}$ (see Figure 2).

The problem is that not all points of $\mathbb{Z}^2$ which are in the strip are preimages of points in the sequence, therefore the embedding so far does not exhibit the sequence as a model set. However, there is an open substrip whose model set is a subset of the sequence. This can be seen by observing that the preimages of the sequence can be connected by a path which only passes along horizontal and vertical bonds in the square lattice (cf. Figure 2). If one would omit any point of $\mathbb{Z}^2$ in the strip

$$V+\{(0,-\sqrt{2}/2)+t(0,-1-\sqrt{2}/2)\,|\,0\le t\le 1\}\,, \tag{20}$$

then no such connected path would be possible any more. This observation is the analogue of finding periodic subsets in the limit periodic examples; all further steps are more or less completely determined.

We extend $\mathbb{R}^2$ by the inverse limit $G:=\overleftarrow{\mathbb{Z}}^2(\phi)$ (see Eq. (7)) and embed $\mathbb{Z}^2$ in $\mathbb{R}^2\times G$ in the canonical fashion as the lattice L. The homomorphism ϕ can be uniquely extended to $\mathbb{R}^2\times G$. From the above considerations, if we take as internal group G' the product of the second eigenspace of ϕ with G, we find an open window in G' such that the corresponding model set is a subset of the substitution sequence $\Lambda:=\Lambda_a\cup\Lambda_b$. The recursion relations (18) and (19) can be transfered to the internal group G'. This gives an iterated function system for two windows Ω_a and Ω_b related to type a and b vertices. This system has a unique pair of compact sets (Ω_a,Ω_b) as its attractor.

Obviously, $\Lambda\subseteq\Lambda(\Omega)$ for $\Omega:=\Omega_a\cup\Omega_b$. Due to the recursion relation (18), $\lambda\Lambda$ is a subset of Λ_a. From the observation (20) we can find an open subset of G inside Ω_a. Then, using the recursion relation both for $\Lambda_{a,b}$ and $\Omega_{a,b}$ we can find an open set $U\subset\Omega$ which has Ω as its closure such that $\Lambda(U)\subseteq\Lambda\subseteq\Lambda(\Omega)$. Since $\Omega\backslash U$ has no interior, $\Lambda(U)$ and $\Lambda(\Omega)$ differ only on a set of points that is *not* relatively dense, i.e. on a set (in $\mathbb{R}$) that has gaps of arbitrary length.

By the argument at the end of the Introduction, we can find a $c\in G'$ so that $\Lambda(c+\Omega)$ is regular, and hence repetitive, as is the original substitution sequence Λ. Now let us show that Λ and $\Lambda(c+\Omega)$ are locally isomorphic, thus establishing Λ to be a model set as defined in the Introduction.

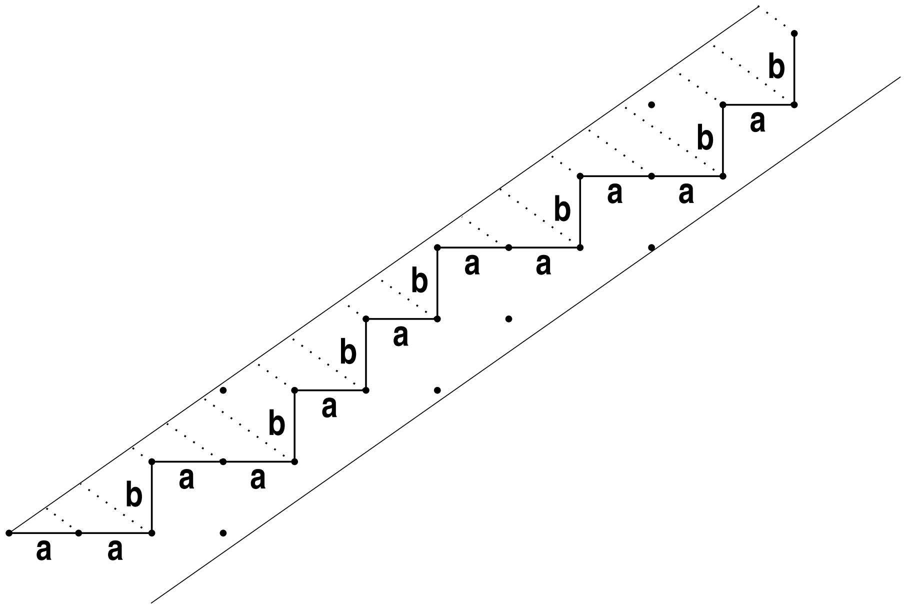

Figure 2. The limit quasiperiodic example.

From the above argument, we know that there are arbitrarily long intervals where $\Lambda(U)$, Λ and $\Lambda(\Omega)$ coincide. Let us select such an interval of length R. Then, we also know that there exists a relatively dense set of translations t such that $t + \Lambda(c + \Omega)$ coincides both with $\Lambda(U)$ and $\Lambda(\Omega)$ on that interval, too. Since Λ is repetitive, this establishes that Λ and $\Lambda(c + \Omega)$ are locally isomorphic.

This finally reveals the points Λ_a and Λ_b as model sets based on the mixed internal space $\mathbb{R} \times \overleftarrow{\mathbb{Z}}^2(\phi)$.

6. Diffraction

The diffraction of a generalized model set can be calculated in much the same way as for model sets in the conventional framework [20]. Given a model set Λ in $\mathbb{R}^d$, one can show that its characteristic Dirac comb, i.e. the measure

$$\omega \;=\; \omega_\Lambda \;:=\; \sum_{t\in\Lambda} \delta_t\,, \tag{21}$$

has a unique autocorrelation,

$$\begin{aligned} \gamma \;=\; \gamma_\omega &:= \lim_{r\to\infty} \frac{1}{\mathrm{vol}(B_r(0))} (\omega_{\Lambda_r} * \tilde{\omega}_{\Lambda_r}) \\ &= \lim_{r\to\infty} \frac{1}{\mathrm{vol}(B_r(0))} \sum_{s,t\in\Lambda_r} \delta_{t-s}\,, \end{aligned} \tag{22}$$

where $B_r(0)$ is the ball of radius r around 0, $\Lambda_r := \Lambda \cap B_r(0)$ and $\tilde{\omega}$ denotes the measure defined by $(\tilde{\omega}, \phi) = \overline{(\omega, \tilde{\phi})}$ with $\tilde{\phi}(x) := \overline{\phi(-x)}$.

A good theory of diffraction exists for model sets under the additional assumption

W2 The boundary of Ω has measure 0 (measure being the Haar measure of G).

If internal space is Euclidean, this is tantamount to saying that the window is a Riemann measurable set.

If one interprets the measure ω as a set of point scatterers at the sites of Λ, then the corresponding diffraction pattern is the Fourier transform $\hat{\gamma}$ of the autocorrelation γ. This Fourier transform $\hat{\gamma}$ is itself a positive measure and has, for general model sets Λ with property **W2**, only a point component, i.e., can be written in the form

$$\hat{\gamma} \;=\; \sum_{k\in F} C(k)\delta_k \tag{23}$$

with non-negative coefficients $C(k)$. The set F in (23) is the projection into $\mathbb{R}^d$ of the dual lattice of L in the dual of $\mathbb{R}^d \times G$.

Without going into the detailed calculation, let us give the result in the case of the limit-periodic substitution sequence (9) for which we can easily verify condition **W2** – indeed, the boundary of the window is a finite point set. We denote the right-hand end points of the intervals a, b and c by Λ_a, Λ_b and Λ_c, and consider them as the positions of point scatterers of strengths h_a, h_b and h_c, respectively. The Fourier transform of the autocorrelation γ_ω of the measure

$$\omega \;=\; h_a \sum_{t\in\Lambda_a} \delta_t + h_b \sum_{t\in\Lambda_b} \delta_t + h_c \sum_{t\in\Lambda_c} \delta_t \tag{24}$$

is then given by

$$\hat{\gamma} \;=\; \sum_{k\in F} |h_a A_a(k) + h_b A_b(k) + h_c A_c(k)|^2 \delta_k \tag{25}$$

with the "amplitudes"

$$\begin{aligned}
A_a(k) &= \frac{1}{3^n} e^{\pi i m/3^n} \left(e^{-\pi i m/3} + \frac{(-1)^m}{2} \right) , \\
A_b(k) &= \frac{1}{3^n} e^{-\pi i m/3^{n-1}} \left(e^{-\pi i m/3} + \frac{(-1)^m}{2} \right) , \\
A_b(k) &= \frac{1}{3^n} e^{-\pi i m/3^{n-1}} \left(e^{\pi i m/3} + \frac{(-1)^m}{2} \right) .
\end{aligned} \tag{26}$$

The sum in (25) runs over the Fourier module F,

$$F \;:=\; \left\{ k = \frac{m}{3^n} \;\mid\; (n=2, m\in\mathbb{Z}) \text{ or } (n\geq 3, m \not\equiv 0 \mod(3)) \right\} , \tag{27}$$

namely the set of all rational numbers k whose denominators are, at worst, powers of 3. Each such number k is uniquely expressible in the form indicated in (27). It is this one-to-one parameterization that appears in (26).

It is easy to see that F is indeed the projection into $\mathbb{R}$ of the dual of $\mathbb{Z}$ in the dual of $\mathbb{R} \times \widehat{\mathbb{Z}_3}$.

In the case of the chair tiling, we already established **W2**, and the diffraction can be calculated along similar lines to the previous example. The limit-quasiperiodic substitution system of Section 5 is a lot more complicated, and we have not even been able to verify **W2** so far.

7. Comments

The formalism of model sets has been shown to encompass situations not hitherto considered within its scope by using internal spaces with non-Euclidean topologies. The situations in which such topologies occur are signalled by the presence of chains of decreasing sublattices of ever increasing scale.

Let us point out a few more examples. The period doubling substitution rule $a \to ba$, $b \to aa$, which is known to have pure point spectrum from Dekking's criterion [4], gives rise to a 2-adic model set. One of the oldest aperiodic tilings is the Robinson tiling [18], which is based on a set of six decorated squares. This tiling has an interpretation in terms lattices of overlapping squares (see [6] for a picture) which clearly shows its 2-adic nature (something already realized by Robinson). In fact, the centers of the tiles of each type form a 2-adic model set. In the course of working out these examples, we discovered that the chair tilings and the Robinson square tilings are actually closely related. Suitably decorated, the chair tiling can be transformed into a Robinson tiling and in the reverse direction, suitably undecorated, the Robinson tiling can be transformed into a chair tiling.

The sphinx tiling as well as the new hexagonal tiling of Penrose [16] undoubtedly also admit 2-adic interpretations.

The substitution tilings described above have so far been considered as belonging to the classes of aperiodic tilings called limit-periodic and limit-quasiperiodic tilings, compare [5]. Potential limit-(quasi-)periodic tilings can be recognized by displaying an inflation/deflation symmetry in the sense of [1] with an inflation multiplier that is an algebraic integer larger than 1, but not a unit. Not all of them will display a pure point diffraction spectrum, as can be seen from the Thue-Morse chain (defined by $a \to ab$, $b \to ba$) or a variant of our system (9) (defined by $a \to ab$, $b \to abc$, $c \to ccab$). These cases cannot be model sets. Our analysis shows nevertheless that a unified description of at least some of the cases with pure point spectrum is possible if one slightly generalizes the class of internal spaces which are admitted.

This generalization turns out to be a very natural one. Many properties of conventional model sets can be proved to hold also in the more general case. Among these are the uniform densities of general model sets (see [19]) and, if also **W2** is fulfilled, the pure point character of the diffraction spectrum (see [20]). It would be nice to find an exhaustive criterion for those cases with pure point diffraction spectrum.

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