Benjamin Briggs

Summer school on Koszul-duality Bad-Driburg. 11/8 '15

Talk 7: BGG-correspondence.

Short nice looking version:

$$\mathcal{D}(\mathbb{P}^n_k) \cong \underline{\mathrm{mod}}\text{-}\Lambda$$

Derived cat of coh. sheaves on $\mathbb{P}^n$

stable module cat (graded right) over Λ = exterior alg/k on $n+1$ var

$$\partial : \mathcal{F} \longrightarrow \mathcal{F}(-1)$$

$\mathcal{O}_{\mathbb{P}}$-linear

<u>Notation</u>: k field. A graded alg $A_0 = k$ usually.

mod A = graded right A-modules.

DG-mod A = DG modules over A.

<u>homological gradings</u>: $|\partial| = -1$.

Let W be a graded k-vsp concentrated in <u>negative</u> <u>even</u> degrees.

$S = \mathrm{Sym}_k W$ $\quad W \subseteq S$ ← linear fcts.

DG-module F is called <u>linear</u> if it is equipped with a generating set (graded k-subspace) $\bar{F} \subseteq F$ s.t. $\bar{F} \otimes S \longrightarrow F^{\#}$ an iso. and $\partial(\bar{F}) \subseteq \bar{F}W$.

$F^{\#}$: underlying graded module.

Construction of the Koszul complex:

The minimal resolution $\mathbb{K}_S$ of k_S begins $\cdots \to W \otimes S[1] \to S$

so inside $\operatorname{Ext}^{\cdot}_S(k,k) = H^{\cdot}\operatorname{Hom}_S(\mathbb{K}_S, k) =$

$$= \operatorname{Hom}_S(\mathbb{K}_S, k) \supseteq \operatorname{Hom}_S(W \otimes S[1], k) = W^*[-1].$$

$V = W^*[-1]$ sits in <u>pos</u> odd, so

$$\Lambda = \operatorname{Sym}_k V = \text{exterior algebra on } V.$$

It is "well known" that $\operatorname{Ext}_S(k,k) \cong \Lambda$.

[if $x \in W$ is nonzero divisor in M_S.

$M \xrightarrow{x} M \to M/x \rightsquigarrow \operatorname{Ext}(M,k) \rightsquigarrow \operatorname{Ext}(M/x, k)$

adjoins an exterior var.]

Diff? Anytime I have $V \subset X$ of free module,

I can draw

$(w) = m$

$$\begin{tikzcd}
\cdots \arrow[r] & F_i \arrow[r] \arrow[dr, "\partial"] & F_{i-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & F_i M \arrow[r] \arrow[u, hook] & F_{i-1} M \arrow[r] \arrow[u, hook] & \cdots
\end{tikzcd}$$

$\partial : F \to F_m$

tensor with k:

$m/m^2 = W$

$$\begin{tikzcd}
\cdots \arrow[r] & F_i \otimes k \arrow[r, "0"] \arrow[dr, "\partial_{\mathrm{lin}}"] & F_{i-1} \otimes k \arrow[r] & \cdots \\
\cdots \arrow[r] & F_i \otimes m/m^2 \arrow[r, "0"] \arrow[u, "0"] & F_{i-1} \otimes m/m^2 \arrow[r] \arrow[u, "0"'] & \cdots
\end{tikzcd}$$

In this situation, one can fully recover F as
$F \cong ((F \otimes_S k) \otimes S, \partial_{lin})$ when F is linear.

OBS: (F semi-free) this ∂_{lin} is the connecting hom
$\partial_{lin}: Tor^S(F,k) \to Tor(F, \frac{m}{m^2})$ associated to the s.e.s.

$$0 \to \frac{m}{m^2} \to \frac{S}{m^2} \to k \to 0.$$

Moreover, this is the action of the class
$\delta \in Ext^1_S(k, m/m^2) : Tor(F,k) \xrightarrow{\partial_{lin}} Tor(F, m/m^2)$

In fact, $\delta \in V \otimes W \subseteq Ext^0_S(k,k) \otimes m/m^2 = Ext^0(k, m/m^2)$
is none other than trace element $\in V \otimes W = W^* \otimes W[-1] \cong Hom(W,W)[1]$

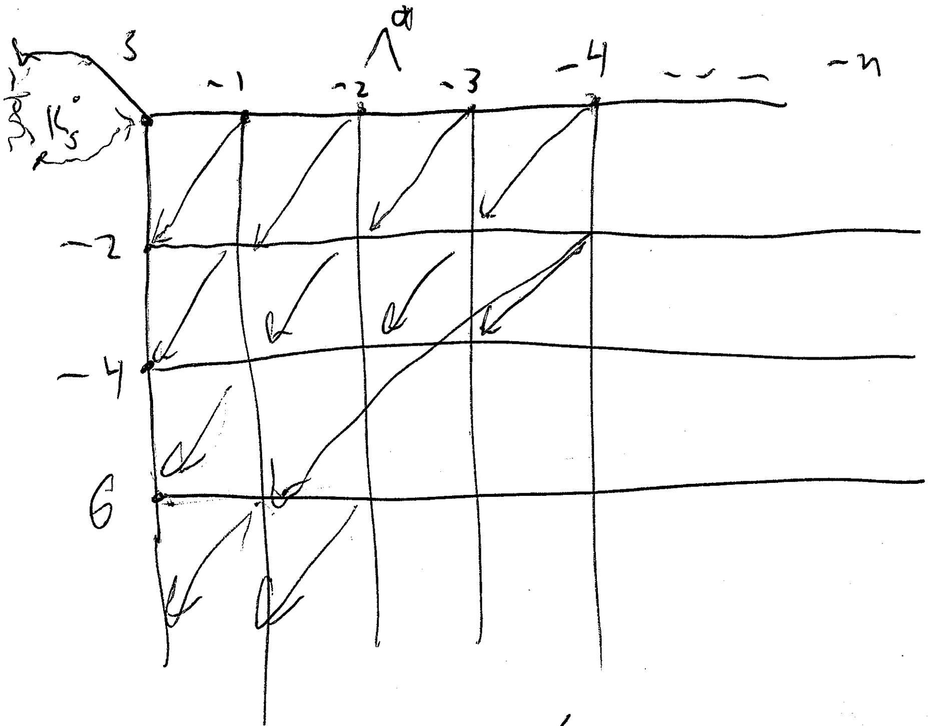
think: $\delta \in (\Lambda \otimes S^{op})_{-1}$ element of this alg.
graded complex $\to \delta^2 = 0$.

know: $K_S^\# = (\Lambda^*_\Lambda \otimes_S S) \in S-\Lambda$ bimodule
$\to$ mult by $\delta: K_S^\# \to K_S^\#[1]$ is $\partial_{lin} = \partial$

Check $\mathbb{K}_S := (\Lambda^* \otimes S)^\delta$ is exact $\to$ Koszul.

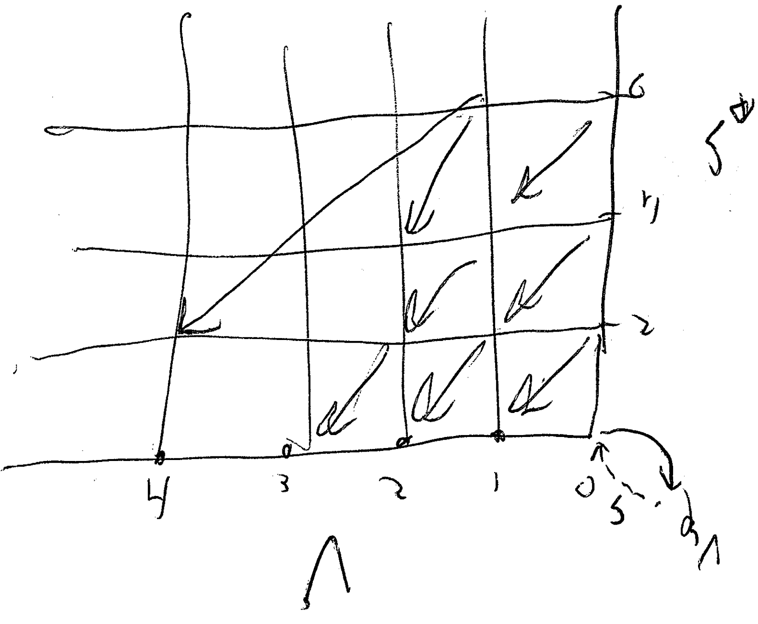
Rk: ${}_S M_\Lambda \rightsquigarrow M^\delta$ as a deformation of the cat. of $S-\Lambda$ DG-modules. $\partial_M \mapsto \partial_M + \delta$

The Koszul complex: $\mathbb{K}_S = (\Lambda^*_\Lambda \otimes_S S)^\delta$



Hilbert: These strands are exact.

Dualize $\rightsquigarrow$ $\mathbb{K}_\Lambda = (S^*_S \otimes_\Lambda \Lambda)^\delta$



Formality: $\mathrm{End}_{D(S)}(k_S) = R\mathrm{Hom}_S(k,k) = \mathrm{Hom}_S(\mathbb{K}_S, k)$

$$\overset{\text{qis}}{=} \mathrm{Hom}_S((\Lambda^* \otimes S)^\delta, k)$$

$$= \mathrm{Hom}_S(\Lambda^* \otimes S, k) = \Lambda$$

Check: This is com. alg. map.

$\longrightarrow$ $\mathrm{End}_{D(S)}(k)$ is formal

Similarly: $\mathrm{End}_{D(\Lambda)}(k_\Lambda) \simeq S$ as DG-algs.

~~DG~~ <u>BGG</u>

$$
\begin{tikzcd}
D(\mathrm{End}_{D(S)}(k)) \cong D(\Lambda) \arrow[r, shift left, "{\mathrm{RHom}(k_\Lambda,-) \,=\, \mathrm{Hom}_\Lambda((S^*\otimes\Lambda)^\delta,-)}"] & D(S) \cong D(\mathrm{End}_{D(\Lambda)}(k_\Lambda)) \arrow[l, shift left, "- \otimes^L_S k"] \\
\mathrm{thick}(k) = D^f(\Lambda) \arrow[u] \arrow[r, shift left] & D^f(S) = \mathrm{thick}(S) \arrow[u] \arrow[l, shift left] \\
\mathrm{thick}(\Lambda) = D_{perf}(\Lambda) \arrow[r, shift left, "\simeq"] & D_{f.g}(S) = \mathrm{thick}(k) \arrow[u] \arrow[l, shift left]
\end{tikzcd}
$$

BGG: These restrict to an equivalence here

Moreover equiv. here

Consequently, take Verdier quotients:

$$D(\mathrm{coh}\,\mathbb{P}^{n-1}) \cong \frac{D^f(S)}{\mathrm{thick}(k)} \simeq \frac{D^f(\Lambda)}{\mathrm{perf}} \simeq \underline{\mathrm{mod}}-\Lambda \quad \text{(Rickard)}$$

(Serre)

<u>proof</u>: ~~But~~ Just need to check unit and counit are isos of $k_\Lambda, k_S, S_S, \Lambda_\Lambda$

$$
\begin{tikzcd}
K_\Lambda \arrow[r, squiggly] \arrow[rrr, bend right, "q\text{-is}"'] & S_S \arrow[r, squiggly] & S \otimes^L_S k \simeq & k_\Lambda
\end{tikzcd}
$$

check this q-is is the counit on k. □

Let $f_1 \cdots f_c \in S$ be a regular sequence in m^2.

Tate resolution: $k_R \simeq$

$$\mathbb{K}_k^{\#} = \Lambda^* \otimes_k \Gamma_k(df_1, \dots, df_c), \text{ where } \Gamma_k = \text{divided ? alg}$$
$$= \mathrm{Sym}_k(\partial_{f_1}, \dots, \partial_{f_c})^*.$$

On $(\Lambda^* \otimes R) = \mathbb{K}_S \otimes R$ has usual diff $\partial \otimes 1$

on Γ_R diff is given by $\partial(df_i) = \sum \frac{\partial f_i}{\partial w^*} w \in R \otimes W$
$\subseteq R \otimes \Lambda^*$

extended to be a DG-algebra.

Assume f_i are quadrics, i.e. $E \subseteq W^2 \subseteq S$

then this is linear $\longrightarrow$ Koszul

Get $E = \mathrm{Ext}_R(k,k)$ Koszul dual.

General picture

$$\begin{tikzcd}
 & D(R) \arrow[r, shift left, "- \otimes^L_R k"] & D(E) \arrow[l, shift left, "R\mathrm{Hom}_E(k,-)"] & \\
 & D^f(R) \arrow[u] \arrow[r, shift left, "\cong"] & D^f(E) \arrow[u] \arrow[l, shift left] & \\
D_{perf}(R) \arrow[ur, hook] & D_{fd}(R) \arrow[u, hook] \arrow[rr, bend right, "\cong"] & D_{fd}(E) \arrow[u] \arrow[ll, bend left, "\cong"'] & D_{perf}(E) \arrow[ul]
\end{tikzcd}$$

Consequently:

$$D(\mathrm{coh}\ \mathrm{Proj}\, R) \overset{\text{Buchweitz}}{\cong} \frac{D^f(E)}{\mathrm{perf}} \cong \underline{MCM}-E$$

Serre:

$$\frac{D^f(E)}{D_{fd.}(E)} =: D(\mathrm{coh}\ \mathrm{proj}_{\mathrm{nc}} E) \cong \underline{MCM}-R.$$

$$R\mathrm{Hom}_S(k,M) = \mathrm{Hom}_S(\mathbb{K}_S, M) = \mathrm{Hom}_S(\Lambda^{\#} \otimes S, M)$$
$$= (\Lambda \otimes M)^{\delta}.$$