BIREP Summer School on Koszul duality - BAD DRIBURG

I[st] talk

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Koszul rings & their duals.

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I. Preliminaries on graded rings.

- Ring : associative unital ring.
- A ring R is $\mathbb{Z}$-graded if :

$$R = \bigoplus_{n\in\mathbb{Z}} R_n \quad w/ \quad R_n R_m \subset R_{n+m}$$

- An element r is homogeneous of degree n if $r \in R_n$.
- Idempotents of R are of degree 0.

[BGS]: A ring R is positively graded if $R_n = 0 \quad \forall n < 0$.

- An R-module M is graded over the graded ring R if:

$$M = \bigoplus_{n\in\mathbb{Z}} M_n \quad \text{s.t.} \quad R_m M_n \subset M_{m+n}.$$

- A graded submodule $N \subset M$ is s.t. $N_n = N \cap M_n$.

$\rightsquigarrow$ graded quotients $(M/N)_n = {M_n + N}/{N}$.

- $\exists$ a shift (functor): $M\langle n\rangle$: $(M\langle n\rangle)_m = M_{n+m}$

(In [BGS], $m-n$ instead of $n+m$).

- A map $f: M \to N$ is degree preserving if $f(M_n) \subset N_n$.

Im(f) and Ker(f) are also graded.

- R-Gr: category of all graded left R-modules.
- R-gr: category of f.g. graded left R-modules

  (where maps are degree-preserving maps)

- Similarly, gr-R, Gr-R, mod-R, Mod-R, etc.

- Let $M, N \in R\text{-gr}$.

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{R\text{-Gr}}(M, N\langle n \rangle) \cong \mathrm{Hom}_R(\tilde{M}, \tilde{N})$$

where $\tilde{M}$ means forget grading on M.

- Let A be positively gr. ring.

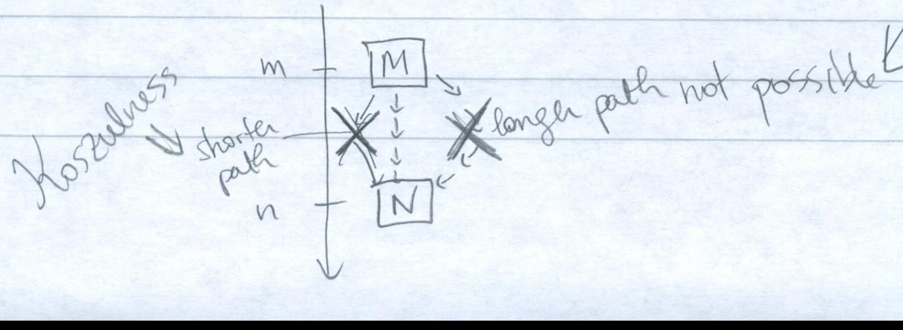
Lemma: If A_0 is semisimple (as an A-module), then:

① simple objects in R-gr are concentrated in a single degree : $\exists n \in \mathbb{Z}$, $M = M_n$.

② Conversely, $M = M_n \Rightarrow M$ is semi-simple in A-gr.

③ M, N concentrated in single degrees m, n resp.
$\Rightarrow \mathrm{ext}^i_A(M,N) := \mathrm{Ext}^i_{A\text{-gr}}(M,N)$

$$= 0 \quad \text{if} \quad i > n - m.$$



"Motivation"

↳ Refine ③ so that shorter path is not possible.

Definition. A is Koszul if

① A_0 is semi-simple

② gr. projective resolution of A_0 is linear.

$$\cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow A_0 \longrightarrow 0$$

s.t. $P^i = A(P^i)_i$

(i.e. i-th term of proj. res. is gen. by elts in dg. i)

Example. * $k[x_1, \ldots, x_n] \cong S^{\cdot}(V)$ - symmetric alg. of V,
V v.s. of dim n.

* $\Lambda^{\cdot}(V^*)$ - exterior algebra

Exercise. Find the grading s.t. these algebras are Koszul.

Proposition. A pos. graded, w/ A_0 semisimple. TFAE

① A Koszul

② M concentrated in deg m, N concentrated in deg n $\Rightarrow \mathrm{ext}^i_A(M,N) = 0 \quad \forall i \neq n-m$

③ $\mathrm{ext}^i_A(A_0, A_0\langle n\rangle) = 0, \quad \forall i \neq n$

Sketch of the proof.

① ⇒ ②: Assume $m=0$, $M \in \mathrm{add}(A_0)$.

$$\begin{tikzcd} A P^{i+1}_{i+1} \arrow[r] & A P^i_i \arrow[r] \arrow[d, "\exists"] & \cdots \arrow[r] & A P^0_0 \arrow[r] & M \arrow[r] & 0 \\ & N & & & & \end{tikzcd}$$

(think carefully)

② ⇒ ③: straightforward.

③ ⇒ ①: Construct gr. proj. res.^{n} inductively:

$$0 \to \Omega^{i+1} \to \underbrace{P^i \xrightarrow{\pi^i} \cdots \to P^0}_{\text{linear w/ } P^i \in \mathrm{proj}} \to A_0 \to 0$$

where $\Omega^{i+1} = \mathrm{Ker}\, \pi^i$.

$\mathrm{ext}^{i+1}_A(A_0, N) = \mathrm{hom}_A(\Omega^{i+1}, N)$ (where $\mathrm{hom}_A = \mathrm{Hom}_{A\text{-gr}}$)

use ③ $\Rightarrow$ $\Omega^{i+1} = A\,\Omega^{i+1}_{i+1}$ $\Rightarrow$ Put $P^{i+1} := A \otimes_{A_0} \Omega^{i+1}$

$\square$

II. "A_0-linear" dual.

- $M \in A\text{-gr} \implies M \in A_0\text{-Mod}$.

$M^* := \mathrm{Hom}_{A_0}(M, A_0)$

${}^*M := \mathrm{Hom}_{-A_0}(M, A_0)$

$\rightsquigarrow$ M as right A_0-module.

$\rightsquigarrow \quad M^{\circledast} \underset{\text{abelian gpr}}{:=} \mathrm{Hom}_{A_0}(M, A_0)$

with grading : $(M^{\circledast})_i := (M_{-i})^*$

similarly for ${}^{\circledast}M$.

Proposition. A Koszul $\implies A^{op}$ (opposite ring) is Koszul.

Idea: linear proj. res^n $\rightsquigarrow$ A_0-linear dual linear injective res^n over opposite ring.

III. Quadratic ring.

Definition. A is quadratic if:

① pos. gr.

② A_0 semi-simple

③ A is generated by A_1 over A_0 s.t relations are in deg. 2 — I generated by homogeneous elts of deg. 2.

$$\iff A = T_{A_0}(A_1)/I \quad \text{for some ideal } I \subset A_{>0}.$$

$$\iff \mathrm{ext}^1_A(A_0, A_0\langle -n\rangle) = 0 \quad \forall n \neq 1.$$

Theorem A generated by A_1 over A_0 w/ A_0 semisimple.

If $\left[\mathrm{ext}^2_A(A_0, A_0\langle -n\rangle) \neq 0 \implies +n = 2\right]$,

Then A is quadratic.

Corollary: A Koszul $\implies$ quadratic.

Question. When is a quadratic ring Koszul ?

→ **Theorem** [BGS] A quadratic, say $A = T_{A_0}(A_1)/(R)$
w/ R - set of homogeneous elts of deg 2.
then A is Koszul $\iff$ $\exists$ quasi-isomorphism between the Koszul complex and A_0.

- If your ring is not very "nice", then this is the Koszul complex $K^{\bullet}$.

$$K^i := A \otimes_{A_0} \widetilde{K(i)}$$

where $\widetilde{K(i)} := \prod_j A_1^{\otimes j} \otimes (R) \otimes A_1^{\otimes i-j-2} \subset A_1^{\otimes i}$

and differential comes from restricting

$$\begin{aligned} A_1^{\otimes i+1} &\longrightarrow A^{\otimes i} \\ a_0 \otimes \cdots \otimes a_i &\longmapsto a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_i \end{aligned}$$

- In particular, starting terms are:

$$\begin{tikzcd} \cdots \arrow[r] & A \otimes_{A_0} (R) \arrow[r] & A \otimes_{A_0} A_1 \arrow[r] & A \otimes_{A_0} A_0 \arrow[r] \arrow[d, phantom, "\cong"] & A_0 \arrow[r] & 0. \\ & & & A & & \end{tikzcd}$$

From now on, we look at <u>left finite</u> (resp. <u>right finite</u>) rings.

$A_i \in A_0\text{-mod}$ $\qquad$ $A_i \in \text{mod-}A_0$.

<u>Definition</u>. $A = T_{A_0}(A_1)/(R)$ - quadratic ring, left finite

Then define its quadratic dual:

$$A^! := T_{A_0}(A_1^*)/(R^{\perp}),$$

where $R^{\perp} := \{ f \in \overset{A_2^*}{\cancel{(A_1 \otimes A_1)^*}} \mid f(R) = 0 \}$

? question mark ?

see equiv. def next page.

Equiv. definition

$I := (R)$

$$0 \longrightarrow I \longrightarrow A_1 \otimes_{A_0} A_1 \longrightarrow C \longrightarrow 0$$

Apply $\mathrm{Hom}_{A_0}(-, A_0)$:

$$0 \longleftarrow I^* \longleftarrow (A_1 \otimes_{A_0} A_1)^* \longleftarrow C^* \longleftarrow 0$$

$$(A_1 \otimes_{A_0} A_1)^* \cong A_1^* \otimes_{A_0} A_1^*$$

and $(R^\perp) := C^*$.

Exercise. $S^\cdot(V)^! \cong \Lambda^\cdot(V^*)$.

Fact: $\underbrace{{}^!(A^!) \cong A}_{A \text{ left finite}}$, $\underbrace{({}^!A)^! \cong A}_{A \text{ right finite}}$

(hence the name "quadratic DUAL".)

Theorem. A Koszul, left finite (resp. right finite)
$\Longrightarrow$ $A^!$ Koszul, right finite.
(resp. ${}^!A$ Koszul, left finite)

Theorem A left finite Koszul

$\Longrightarrow$ $E(A) := \mathrm{Ext}^\cdot_{A-}(A_0, A_0)$ (ring)
$\cong$
$(A^!)^{op}$

Theorem A left finite Koszul

$E(E(A)) \cong A.$

Exercise. Compute $E(S^\bullet(V))$ directly to verify previous theorems.

If A is left finite, then can use this Koszul complex:

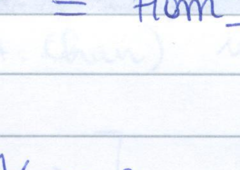
$K^\bullet$ where $K^i := A \underset{A_0}{\otimes} {}^*(A^!_i)$

(Reason why K^i is well-def. for left finite rings:
$W \underset{A_0}{\otimes} {}^*V \cong \mathrm{Hom}_{-A_0}(V, W)$)

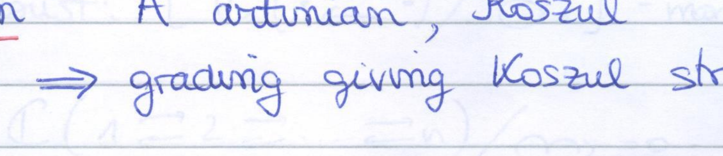
Theorem A artinian, Koszul
$\Rightarrow$ grading giving Koszul structure on A is unique.

- (Comm) Ring / Geometer, Combinatorics $\Big\}$ $S^\bullet(V)$, $\Lambda^\bullet(V^*)$
  $k\langle x, y\rangle / (x^2, y^2, xy - yx)$

- Quiver Algebraist ①.



/ $i \to 0 \to j$
$i \circlearrowleft 0 - 0 \circlearrowleft j$, $i \neq j$
$i, j \in \{1, 2, 3, 4\}$



quadratic dual of preprojective algebra assoc. to extended Dynkin graph $D_4^{(1)}$

②. KQ/J $J = \langle$all paths of length 2$\rangle$.
(radical-square zero algebras)

Lie theorist: $[\mathbb{C}(\cdot \rightleftarrows \cdot)/\circlearrowleft_{\cdot}]$-mod $\simeq \mathcal{O}_0(\mathfrak{sl}_2)$

$$\mathbb{C}(1 \rightleftarrows 2 \rightleftarrows \cdots \rightleftarrows n)\Big/ \begin{array}{ll} \curvearrowright\curvearrowright = 0 & \circlearrowleft_i = \circlearrowright_i \quad i \in \{2, \dots, n\} \\ \curvearrowleft\curvearrowleft = 0 & \circlearrowleft_n = 0 \end{array}$$

$$\mathcal{O}^{\mathfrak{gl}_{n-1}\times \mathfrak{gl}_1}(\mathfrak{gl}_n)$$

Ex $k = \overline{\mathbb{F}_p}$ $\quad B_0(S(p,p))$

$S(p,p)$: Schur alg.

B_0: princ. block.