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The Koszul duality functor

Based on *Koszul duality patterns in representation theory*, by Beilinson, Ginzburg and Soergel

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August 11, 2015

Notation, and the Koszul complex

- k: fixed semisimple ring.
- V: k-bimodule.
- $T_k V$: k-bimodule of tensor products of copies of V.
- $R \subset V \otimes V$: subbimodule. $\leftarrow$ Tensor products and Hom-spaces without subscript are taken w.r.t. k
- $A = T_k V/(R)$: quadratic ring (which we assume is left finite, i.e. with each A_i finitely k-generated).
- $A^!$: the quadratic dual of A.
- ${}^*(A_i^!) = \bigcap_\nu V^{\otimes\nu} \otimes R \otimes V^{\otimes i-\nu-2} \subset V^{\otimes i}$.
  ${}^*(A_i^!) = \mathrm{Hom}(A_i^!, k)$.

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The Koszul complex:

$$\to A \otimes {}^*(A^!_i) \xrightarrow{d^i_K} A \otimes {}^*(A^!_{i-1}) \to \cdots \to A \otimes {}^*(A^!_2) \to A \otimes V \to A \to 0$$

$$a \otimes v_1 \otimes \cdots \otimes v_i \mapsto av_1 \otimes \cdots \otimes v_i$$

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Theorem

A is Koszul $\Leftrightarrow$ the Koszul complex is a resolution of k.

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d^i_K "moves" the leftmost degree 1 generators from the ${}^*(A^!_i)$-part of $A \otimes {}^*(A^!_i)$ to the A-part.

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Alternative description of the Koszul complex differential:

- Write $\mathrm{id}_V = \sum \check{v}_\alpha \otimes v_\alpha$, where
- $\{v_\alpha\}$: k-generators of V, and
- $V^* \ni \check{v}_\alpha = \delta_{v_\alpha} \leftrightarrow v_\alpha \in V$, where δ_{v_α} is the k-linear extension of the Kronecker delta.

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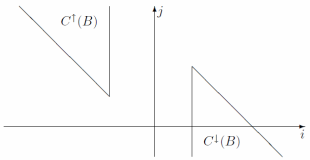
Then d_k may be written

$$A \otimes {}^*(A^!_i) = \mathrm{Hom}(A^!_i, A) \xrightarrow{d^i_K} \mathrm{Hom}(A^!_{i-1}, A) = A \otimes {}^*(A^!_{i-1})$$
$$f(\_) \mapsto \sum f(\_ \cdot \check{v}_\alpha) v_\alpha$$

Removing v_α from ${}^*(A^!_i)$ becomes multiplying $\check{v}_\alpha$ to $A^!_i$ by contravariance of *.

Theorem 2.12.1: Statement and overview



Location of nonzero M^i_j.

- $B = \bigoplus_{j \geq 0} B_j$: positively graded ring.
- $C(B)$: homotopy category of complexes in $B\text{-}\mathrm{Gr}$.
  $B\text{-}\mathrm{Gr}$ means category of graded B-modules.
- $C^{\uparrow}(B) \subset C(B)$: subcategory of objects M satisfying $M^i_j = 0$ if $i \gg 0$ or $i + j \ll 0$.
  M^i_j is the degree j part of M at position i.
- $C^{\downarrow}(B) \subset C(B)$: subcategory of objects M satisfying $M^i_j = 0$ if $i \ll 0$ or $i + j \gg 0$.

Theorem 2.12.1: Statement and overview

- $D^{\uparrow}(B)$ and $D^{\downarrow}(B)$: localizations of $C^{\uparrow}(B)$ and $C^{\downarrow}(B)$ w.r.t. quasi-isomorphisms.

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Proof outline:

1. The functors F, G

$$F : A\text{-}\mathrm{Mod} \leftrightarrow A^{!}\text{-}\mathrm{Mod} : G$$
$$M \mapsto A^{!} \otimes M$$
$$\mathrm{Hom}(A, N) \leftarrow N$$

form an adjoint pair (hom/tensor adjunction).

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- Consider $M \in C^{\downarrow}(A)$ and $N \in C^{\uparrow}(A^!)$ as modules $M = \bigoplus_i M^i \in A\text{-}\mathrm{Mod}$ and $N = \bigoplus_i N^i \in A^!\text{-}\mathrm{Mod}$.
- Endow FM (respectively GN) with bicomplex structures: differentials d' from the Koszul complex, and d'' from the complex M (respectively N).
- Take total complex to obtain $FM \in C^{\uparrow}(A^!)$ (respectively $GN \in C^{\downarrow}(A)$).

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   - Take total complex to obtain $FM \in C^{\uparrow}(A^!)$ (respectively $GN \in C^{\downarrow}(A)$).

2. Check that the (F, G)-adjunction is compatible with the complex structure of FM and GN.

Theorem 2.12.1: Statement and overview

3.
- Thanks to the shape of $C^{\downarrow}(A)$ and $C^{\uparrow}(A^{!})$, the spectral sequences of the bicomplexes FM and GN converge.
- Also F and G preserve mapping cones.

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It follows that

- F and G commute with quasi-isomorphisms.

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- Using that the Koszul complex is a resolution of k: The counit $FG \to \mathrm{id}_{C^{\uparrow}(A^!)}$ and unit $GF \to \mathrm{id}_{C^{\downarrow}(A)}$ from the adjunction are quasi-isomorphisms

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- Using that the Koszul complex is a resolution of k: The counit $FG \to \mathrm{id}_{C^{\uparrow}(A^!)}$ and unit $GF \to \mathrm{id}_{C^{\downarrow}(A)}$ from the adjunction are quasi-isomorphisms
- Thus F and G induce mutually inverse equivalences of triangulated categories $DF : D^{\downarrow}(A) \leftrightarrow D^{\uparrow}(A^!) : DG$.

Part 1 of proof: Construction of F and G

Proof.

- For $M \in C^{\downarrow}(A)$

$$FM = A^! \otimes M$$

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$$\bigoplus_{l,i} (FM)_{l,i} = A^! \otimes M = \bigoplus_{l,i} A^!_l \otimes M^i$$

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$$= \bigoplus_{l,i} \mathrm{Hom}({}^*(A^!_l), M^i) = \bigoplus_{l,i} \mathrm{Hom}({}^*(A^!_l), \mathrm{Hom}_A(A, M^i)) =$$

$$= \bigoplus_{l,i} \mathrm{Hom}_A(A \otimes^* (A^!_l), M^i)$$

By hom/tensor adjunction

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The Koszul complex!

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$$= \bigoplus_{l,i} \operatorname{Hom}_A(A \otimes {}^*(A^!_l), M^i)$$

Let "$d' = \pm \operatorname{Hom}_A(d_K, M^i)$".

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$$= \bigoplus_{l,i} \mathrm{Hom}_A(A \otimes {}^*(A^!_l), M^i)$$

$$d' : A^!_l \otimes M^i_j \to A^!_{l+1} \otimes M^i_{j+1} \quad (1)$$
$$a \otimes m \mapsto (-1)^{i+j} \sum a\check{v}_\alpha \otimes v_\alpha m$$
$$d'' : A^!_l \otimes M^i_j \to A^!_l \otimes M^{i+1}_j \quad (2)$$
$$a \otimes m \mapsto a \otimes \partial m$$

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Part 1 of proof: Construction of F and G

Proof cont.

$$d' : A^!_l \otimes M^i_j \to A^!_{l+1} \otimes M^i_{j+1} \tag{3}$$
$$a \otimes m \mapsto (-1)^{i+j} \sum a\check{v}_\alpha \otimes v_\alpha m$$
$$d'' : A^!_l \otimes M^i_j \to A^!_l \otimes M^{i+1}_j \tag{4}$$
$$a \otimes m \mapsto a \otimes \partial m$$

The total differential

$$d = d' + d''$$

turns FM into the complex

$$(FM)^p_q = \bigoplus_{p=i+j,q=l-j} A^!_l \otimes M^i_j.$$

Part 1 of proof: Construction of F and G

Proof cont.

$$(FM)^p_q = \bigoplus_{p=i+j,q=l-j} A^!_l \otimes M^i_j.$$

The degree q "slice":

$$\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
\uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle d''} & & \\
0 & \xrightarrow{d'} & k \otimes M^{i+2}_{-q} & \xrightarrow{d'} & A^!_1 \otimes M^{i+2}_{-q+1} & \xrightarrow{d'} & A^!_2 \otimes M^{i+2}_{q+2} & \xrightarrow{d'} & \cdots \\
\uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle d''} & & \\
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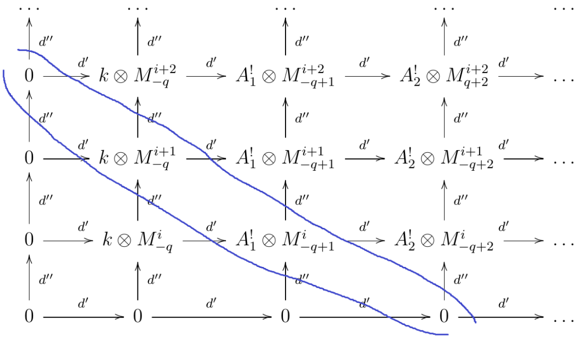
0 & \xrightarrow{d'} & 0 & \xrightarrow{d'} & 0 & \xrightarrow{d'} & 0 & \xrightarrow{d'} & \cdots
\end{array}$$

Part 1 of proof: Construction of F and G

Proof cont.



The slice lives in the 1:st quadrant $\Longrightarrow$ At each position the total complex has finitely many summands $\Longrightarrow$ The spectral sequences w.r.t. the d'- and d''-filtrations converge!

Part 1 of proof: Construction of F and G

Proof cont.

$$
\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
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0 & \xrightarrow{d'} & 0 & \xrightarrow{d'} & 0 & \xrightarrow{d'} & 0 & \xrightarrow{d'} & \cdots
\end{array}
$$

Bicomplex bounded from the left because $A^!$ is positively graded.

Part 1 of proof: Construction of F and G

Proof cont.

$$
\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
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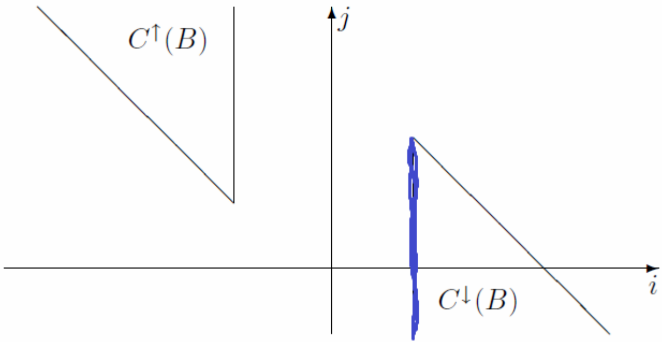
\end{array}
$$

Bicomplex bounded from below due to the shape of $C^{\downarrow}(A)$.

Part 1 of proof: Construction of F and G

Proof cont.



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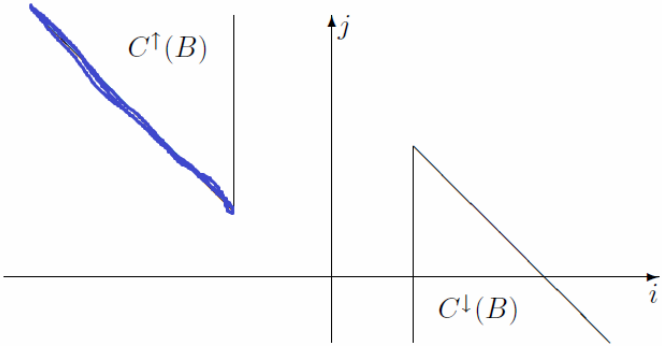

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Part 1 of proof: Construction of F and G

Proof cont.

Construction of G is similar: We get a total complex

$$(GN)^p_q = \bigoplus_{p=i+j, q=l-j} \mathrm{Hom}(A_{-l}, N^i_j) = \bigoplus_{p=i+j, q=l-j} A^*_{-l} \otimes N^i_j = $$
$$= ({}^!(A^!))^*_l \otimes N^i_j$$

Define differentials

$$d' : \mathrm{Hom}(A_{-l}, N^i_j) \to \mathrm{Hom}(A_{-(l+1)}, N^i_{j+1}) \qquad (5)$$
$$f(\_) \mapsto (-1)^i \sum \check{v}_\alpha f(v_\alpha \cdot \_)$$
$$d'' : \mathrm{Hom}(A_{-l}, N^i_j) \to \mathrm{Hom}(A_{-l}, N^{i+1}_j) \qquad (6)$$
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Part 1 of proof: Construction of F and G

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$$= ({}^!(A^!))^*_l \otimes N^i_j$$

The $A^! \otimes {}^!(A^!))^*_l$ constitute the Koszul complex of $A^!$. $\pm d'$ moves a degree 1 generator from $({}^!(A^!))^*_l$ to the $A^!$-module N. Define differentials

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Part 2 of proof: F and G are adjoint

Proof cont.

Want to show that the adjointness

$$\mathrm{Hom}_{A^!}(A^! \otimes M, N) \cong \mathrm{Hom}(M,N) \quad \cong \mathrm{Hom}_A(M, \mathrm{Hom}(A,N))$$
$$a^! \otimes am \overset{\tilde{f}}{\mapsto} n \leftrightarrow am \overset{f}{\mapsto} a^! n \quad \leftrightarrow m \overset{\hat{f}}{\mapsto} (a \mapsto a^! n)$$

is compatible with the total complex structure. We need to check two things:

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is compatible with the total complex structure. We need to check two things:

(i)

$$\tilde{f}((FM)^i_j) \subset N^i_j \text{ for all } i, j \iff \hat{f}(M^p_q) \subset (GN)^p_q \text{ for all } p, q.$$

Pick $a, a^!, m$ and n corresponding to $\tilde{f}$ and $\hat{f}$ and check bidegrees . Omitted.

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(ii)

$$\partial \tilde{f}(1 \otimes m) = \tilde{f}(d(1 \otimes m)) \iff d\hat{f}(m)(1) = \hat{f}(\partial m)(1).$$

Since it is sufficient to check at values where $a^! = 1 = a$.

Part 2 of proof: F and G are adjoint

Proof cont.

$$\partial\tilde{f}(1\otimes m)=\tilde{f}(d(1\otimes m)) \iff d\hat{f}(m)(1)=\hat{f}(\partial m)(1).$$

(ii)

$$\begin{aligned}
&\tilde{f}(d(1\otimes m))-\partial\tilde{f}(1\otimes m)\\
&=\tilde{f}(1\otimes(\partial m)+(-1)^{i+j}\sum \check{v}_\alpha\otimes v_\alpha m)-\partial f(m)\\
&=f(\partial m)+(-1)^{i+j}\sum \check{v}_\alpha f(v_\alpha m)-\partial f(m)\\
&=\hat{f}(\partial m)(1)-d\hat{f}(m)(1).
\end{aligned}$$

This part is similar, see paper.

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F preserves mapping cones

Recall: For $X, Y \in C^{\downarrow}(A)$ and a morphism $f : X \to Y$, the mapping cone of f is the complex

$$X[1] \oplus Y$$

$[1]$ means the complex is shifted 1 position to the left, and the differential multiplied by -1

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Now

- F is additive and clearly commutes with $[1]$. Hence $F(X[1] \oplus Y) = F(X)[1] \oplus F(Y)$.

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- F commutes with f (since F is a functor).

It follows that F preserves mapping cones.

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Part 3 of proof: F and G induce inverse equivalences $D^{\downarrow}(A) \cong D^{\uparrow}(A^!)$

Proof cont.

This part relies on the theory of spectral sequences.

(i)

- We saw that the bicomplex $(FM)_q$ lives in the 1:st quadrant, so by Theorem 2.15 of (2), there exist spectral sequences with first terms

$$H^{\bullet}(FM, d'') \text{ and } H^{\bullet}(F(GN), d')$$

respectively that converge to

$$H^{\bullet}(FM, d) \text{ and } H^{\bullet}(F(GN), d)$$

respectively.

Part 3 of proof: F and G induce inverse equivalences $D^{\downarrow}(A) \cong D^{\uparrow}(A^{!})$

Proof cont.

- The cone, M, of a quasi-isomorphism, f, is acyclic. Also:

$$H^{\bullet}(M, \partial) = 0 \Longrightarrow H^{\bullet}(FM, d'') = 0 \overset{\text{Rem 1 of (3)}}{\Longrightarrow} H^{\bullet}(FM, d) = 0,$$

  so the cone FM of $F(f)$ is acyclic too, so $F(f)$ is a quasi-isomorphism. Hence F preserves quasi-isomorphisms, and induces a functor

$$DF : D^{\downarrow}(A) \to D^{\uparrow}(A^{!}).$$

- That we get a functor $DG : D^{\uparrow}(A^{!}) \to D^{\downarrow}(A)$ is analogous.

Part 3 of proof: F and G induce inverse equivalences $D^{\downarrow}(A) \cong D^{\uparrow}(A^!)$

Proof cont.

(ii) $F \circ G$ is quasi-isomorphic to $\mathrm{id}_{C^{\uparrow}(A^!)}$:

- We show that the counit map

$$F(GN) \twoheadrightarrow N$$

from the adjunction is a quasi-isomorphism. Over k, the map splits, and we get the splitting map

$$\varphi : N = k \otimes \mathrm{Hom}(k, N) \hookrightarrow \bigoplus A_p^! \otimes \mathrm{Hom}(A_l, N_j^i)$$

$$= \bigoplus_l \mathrm{Hom}(A_l \otimes {}^*(A_p^!), N_j^i) = F(GN),$$

with $F(GN)$ having the bicomplex structure from F, where

$$d' : f \mapsto (-1)^{i+j}(f \circ d_K).$$

Part 3 of proof: F and G induce inverse equivalences $D^{\downarrow}(A) \cong D^{\uparrow}(A^!)$

Proof cont.

$$F(GN) = \bigoplus_l \mathrm{Hom}(A_l \otimes {}^*(A^!_p), N^i_j)$$

$$d' : f \mapsto (-1)^{i+j}(f \circ d_K).$$

- Because the Koszul complex is a resolution of k (Here we use Koszulity of A) we get:

$$H^{\bullet}(F(GN), d') = N.$$

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Part 3 of proof: F and G induce inverse equivalences $D^{\downarrow}(A) \cong D^{\uparrow}(A^!)$

Proof cont.

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$$d' : f \mapsto (-1)^{i+j}(f \circ d_K).$$

- Because the Koszul complex is a resolution of k (Here we use Koszulity of A) we get:

$$H^\bullet(F(GN), d') = N.$$

- We may consider the same bicomplex structure on N (via φ), and check that also

$$H^\bullet(N, d') = N.$$

Part 3 of proof: F and G induce inverse equivalences $D^{\downarrow}(A) \cong D^{\uparrow}(A^{!})$

Proof cont.

- Thus there are spectral sequences with common first term

$$H^{\bullet}(N, d') = N = H^{\bullet}(F(GN), d')$$

that converge to

$$H^{\bullet}(N, d = d' + d'') = H^{\bullet}(N, \partial) \text{ and } H^{\bullet}(F(GN), d)$$

respectively.

Part 3 of proof: F and G induce inverse equivalences $D^{\downarrow}(A) \cong D^{\uparrow}(A^{!})$

Proof cont.

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respectively.

- In fact, the spectral sequences must be the same, since all terms are determined by the first terms and the boundary maps, which are the same for both sequences.
- Thus $H^{\bullet}(N, \partial) = H^{\bullet}(F(GN), d)$, so φ is a quasi-isomorphism.

Part 3 of proof: F and G induce inverse equivalences $D^{\downarrow}(A) \cong D^{\uparrow}(A^!)$

Proof cont.

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respectively.

- In fact, the spectral sequences must be the same, since all terms are determined by the first terms and the boundary maps, which are the same for both sequences.
- Thus $H^\bullet(N, \partial) = H^\bullet(F(GN), d)$, so φ is a quasi-isomorphism.
- That also $G \circ F$ is quasi-isomorphic to $\mathrm{id}_{C^{\downarrow}(A)}$ is similar.

□

Theorem 2.12.5: Statement

We call $K := DF$ *the Koszul duality functor*.

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Theorem

Let A be a left finite Koszul ring over k.

(i) *The functor $K : D^{\downarrow}(A) \to D^{\uparrow}(A^!)$ together with the obvious canonical isomorphism $K(M[1]) \cong (KM)[1]$ is an equivalence of triangulated categories.*

$[1]$ *means the complex is shifted 1 position to the left, and the differential multiplied by -1*
We saw in Theorem 2.12.1 that DF is an equivalence and preserves mapping cones.

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Let A be a left finite Koszul ring over k.

(i) *The functor $K : D^{\downarrow}(A) \to D^{\uparrow}(A^!)$ together with the obvious canonical isomorphism $K(M[1]) \cong (KM)[1]$ is an equivalence of triangulated categories.*

(ii) *We have $K(M\langle n\rangle) \cong (KM)[-n]\langle -n\rangle$, canonically.*

$\langle n\rangle$ means the degrees have been shifted so as to increase by n.

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(i) *The functor $K : D^{\downarrow}(A) \to D^{\uparrow}(A^!)$ together with the obvious canonical isomorphism $K(M[1]) \cong (KM)[1]$ is an equivalence of triangulated categories.*

(ii) *We have $K(M\langle n \rangle) \cong (KM)[-n]\langle -n \rangle$, canonically.*

(iii) *For any $p \in k$ we have $K(A_0 p) = A^! p$ and $K(A^{\circledast} p) = A_0^! p$.*

Where $A^{\circledast} = \bigoplus_l (A^{\circledast})_l \in A\text{-}\mathrm{Gr}$, with $(A^{\circledast})_l = (A_{-l})^*$, is the injective hull of k. For the statement to make sense, we view the modules $A_0 p, A^! p, A^{\circledast}$ and $A_0^! p$ as complexes concentrated in position zero.

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Proof of part (ii)

(ii) We have $K(M\langle n\rangle) \cong (KM)[-n]\langle -n\rangle$, canonically.

Proof.

The position/degree components are the same:

$$\begin{aligned}
(K(M\langle n\rangle))_q^p &= \bigoplus_{p=i+j,q=l-j} A_l^! \otimes (M\langle n\rangle)_j^i \\
&= \bigoplus_{p=i+j,q=l-j} A_l^! \otimes M_{j-n}^i \\
&= \bigoplus_{p=i+j+n,q=l-j-n} A_l^! \otimes M_j^i \\
&= (KM)_{q+n}^{p-n} \\
&= ((KM)[-n]\langle -n\rangle)_q^p.
\end{aligned}$$

Proof of part (ii)

(ii) We have $K(M\langle n\rangle) \cong (KM)[-n]\langle -n\rangle$, canonically.

Proof cont.

Recall that

$$\begin{aligned} d' : A^!_l \otimes M^i_j &\to A^!_{l+1} \otimes M^i_{j+1} \\ a \otimes m &\mapsto (-1)^{i+j} \sum a\check{v}_\alpha \otimes v_\alpha m \\ d'' : A^!_l \otimes M^i_j &\to A^!_l \otimes M^{i+1}_j \\ a \otimes m &\mapsto a \otimes \partial m. \end{aligned}$$

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Effect on differentials:

- $K(M\langle n\rangle)$ multiplies d' by $(-1)^n$.
- $(KM)[-n]\langle -n\rangle$ multiplies $d = d' + d''$ by $(-1)^n$.

To compensate for this discrepancy, we need to multiply every second position i of M by $(-1)^n$, i.e. take as isomorphism

$$(-1)^{in} : K(M\langle n\rangle) \xrightarrow{\sim} (KM)[-n]\langle -n\rangle.$$

□

Proof of part (iii)

(iii) For any $p \in k$ we have $K(A_0 p) = A^! p$ and $K(A^{\circledast} p) = A_0^! p$.

Proof.

- $$K(A_0 p) = A^! \otimes (A_0 p) = A^! \otimes p = A^! p,$$

  with differential $d = d' + d'' = 0$, so again a module.

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Proof of part (iii)

(iii) For any $p \in k$ we have $K(A_0 p) = A^! p$ and $K(A^\circledast p) = A^!_0 p$.

Proof.

- $$K(A_0 p) = A^! \otimes (A_0 p) = A^! \otimes p = A^! p,$$

  with differential $d = d' + d'' = 0$, so again a module.

- $$K(A^\circledast p) = A^! \otimes A^\circledast p = \bigoplus_l A^! \otimes (A_l)^* p,$$

  with differential $d = d' + d'' = d'$. This is the Koszul complex (up to sign of the differential) of $A^!$, times p. This is a resolution of $A^!_0 = k$, times p, hence quasi-isomorphic to $A^!_0 p$.

□

Theorem 2.12.6.

Theorem

Let A be a Koszul ring over k. Suppose A is a finitely generated generated k-module both from the left and from the right, so that $A_i = 0$ for $i \gg 0$. Suppose in addition that $A^!$ is left noetherian. Then Koszul duality induces an equivalence of triangulated categories

$$K : D^b(A\text{-}\mathrm{gr}) \to D^b(A^!\text{-}\mathrm{gr}).$$

D^b means the bounded derived category. $A\text{-}\mathrm{gr}$ means the category of finitely generated graded A-modules.

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Thank you!