Koszulness from exceptional collections

Vincent Gelinas

Goal: Make ~~Koszulity~~ Koszulity appear out of "messing with" exceptional collections"

Set up: $k = \bar{k}$, $\mathcal{T}$ = (algebraic) triangulated, k-linear, Ext-finite cat.
($\sum \dim_k \mathrm{Hom}(X, Y[n]) < \infty$)

- $D^b(X)$, X smooth proj$/_k$
- $D^b(\text{mod-}A)$, A f.d. algebra, gldim $A < \infty$
- grmod-Λ, $\Lambda = \bigoplus_{d \geq 0} \Lambda_d$, pos. gr. Frobenius alg.

$\sigma = (E_0, E_1, \ldots, E_n)$, $E_i \in \mathcal{T}$ forms an exceptional collection if

$$\begin{cases} \mathrm{Hom}(E_j, E_i) = 0, & j > i, \\ \mathrm{Hom}(E_i, E_i) = k \cdot id_{E_i}, & \forall i \end{cases}$$

σ is full if $\langle E_0, \ldots, E_n \rangle = \mathcal{T}$

" " strong if $\bigoplus_{i=0}^{n} E_i$ is tilting

(1) $D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \ldots, \mathcal{O} \rangle = \langle \Omega^n(n), \Omega^{n-1}(n-1), \ldots, \Omega^0(0) \rangle$

$0 \to \mathcal{O}(-n) \boxtimes \Omega^n(n) \to \cdots \to \mathcal{O}_\Delta \to 0$

$$\overset{0}{\circ} \xrightarrow[\substack{x_2 \\ \vdots \\ x_n}]{x_1} \overset{1}{\circ} \xrightarrow[\substack{x_2 \\ \vdots \\ x_n}]{x_1} \overset{2}{\circ} \cdots \xrightarrow[\substack{x_2 \\ \vdots \\ x_n}]{x_1} \overset{n}{\circ}$$

Sym rel $x_i x_j = x_j x_i$

Ex: $\circ \xrightarrow[\substack{x_2 \\ \vdots \\ x_n}]{x_1} \circ \xrightarrow[\substack{x_2 \\ \vdots \\ x_n}]{x_1} \circ \cdots \xrightarrow[\substack{\vdots \\ x_n}]{x_1} \circ$

ext. rel. $x_i x_j = -x_j x_i$

Exercise

(2) Tautological

$D^b(\text{mod-}A)$, $A = kQ/I$, Q acyclic $Q_0 \sim \{0, 1 \ldots n\}$, $\exists\, i \to j$ iff $i > j$

$= \langle P(0), P(1), \ldots, P(n) \rangle$

③ $\Lambda = \bigoplus_{d=0}^{\ell} \Lambda_d$ Frob. gr. alg. ~~σ = grmod-Λ:~~ ~~$\sigma_\Lambda = (\Lambda(0)_{\le 0}, \Lambda(1)_{\le 0}$.~~

$\sigma_\Lambda = (\Lambda(0)_{\le 0}, \Lambda(1)_{\le 0}, \Lambda(2)_{\le 0}, \dots, \Lambda(d-1)_{\le 0})$ is (full) exceptional coll.

(Koto Yamara)

$$
\begin{array}{ccccc}
-2 & -1 & 0 & 1 & 2 \\
 & & \Lambda_0 & \uparrow & \\
 & \Lambda_0 & \Lambda_1 & | & \\
\Lambda_0 & \Lambda_1 & \Lambda_2 & | & \\
 & & \vdots & | & \\
\Lambda_0 \quad \dots & \dots & \Lambda_{\ell-1} & | &
\end{array}
$$

Resolve: $\dots \to P^2 \to P^1 \to \Lambda(i) \to \Lambda(i)_{\le 0} \to 0$

$$P^1 \searrow \Lambda(i)_{\ge 1} \nearrow \Lambda(i)_{\le 0}$$

$\Rightarrow$ sitting in > 0 deg

$\Rightarrow \mathcal{H}om_\Lambda(P^n, \Delta(j)_{\le 0})_0 = 0 = \mathcal{H}om_\Lambda(\Lambda(j)_{\le 0}, P^n)_0$

$\Rightarrow \underline{\mathcal{H}om}_\Lambda(\Lambda(i)_{\le 0}[n], \Lambda(j)_{\le 0}) = 0 = \underline{\mathcal{H}om}_\Lambda(\Lambda(j)_{\le 0}, \Lambda(i)_{\le 0}[n])$

$n > 0$

<u>Ex</u>: $\Lambda = \Lambda^* V$, calculate the quiver

<u>Mutating collections</u>: Fix (E, F)

Left mutation: $L_E(F) \to R\mathcal{H}om(E,F) \otimes_k E \xrightarrow{ev} F \to L_E(F)[-1]$

$E \to R\mathcal{H}om(E,F)^* \otimes F \to R_F(E)$

<u>Prop</u>: $(L_E(F), E)$ and $(F, R_F(E))$ are both exceptional pairs

$L_i(E_0, \dots, E_n) = (E_0, \dots, L_{E_i}(E_{i-1}), E_i, E_{i+2}, \dots, E_n)$

R_i similar

<u>Prop</u>:
- R_i, L_i are inverses
- $\{L_i\}_i, \{R_i\}_i$ satisfies the braid relations
- R_i, L_i preserve <u>full</u> collection

• Mutation does not preserve "strong"

$X = \mathbb{P}(V)$, Euler seq: $0 \to \mathcal{O}_{\mathbb{P}(V)} \xrightarrow{\varepsilon} V \otimes_k \mathcal{O}(1) \to T_{\mathbb{P}(V)} \to 0$

$$V = \mathrm{Sym}^1(V^*)^*$$

Check: $\cancel{0 \to}\ \mathcal{O}_{\mathbb{P}(V)} \to V \otimes \mathcal{O}(1) \to R_{\mathcal{O}(1)}(\mathcal{O}) \to \mathcal{O}_{\mathbb{P}(V)}[1]$

$$R_{\mathcal{O}(1)}(\mathcal{O}) \cong T_{\mathbb{P}(V)}$$

① $\sigma = (E_0, E_1, \dots, E_{n-1}, E_n)$

$(E_0, \dots \quad L_{E_{n-1}}(E_n), E_{n-1})$

$(L^n E_n, E_0, \dots, E_{n-3}, E_{n-2}, E_{n-1})$

$L^n = L_{E_0} \cdot \ldots \cdot L_{E_{n-2}} \cdot L_{E_{n-1}}$

$(L^n E, L^{n-1} E_{n-1}, E_0, \dots, E_{n-2})$

$\rightsquigarrow (L^n E_n, L^{n-1} E_{n-1}, \dots, L^1 E_1, E_0)$ This braid element $\Delta^{1/2}$

$D^b(\text{mod-}A)$: $\sigma = (P(0), P(1), \dots, P(n))$ $\quad A = \mathrm{End}_A A$

<u>Prop (Bondal)</u>: $\Delta^{1/2} \sigma = (S(n)[-n], S(n-1)[-n+1], \dots, S(1)[-1], S(0))$

(might not be strong)

<u>Corollary</u>:

$$\begin{array}{ccc} & \sigma & \\ \swarrow\!\!\swarrow \sim & & \searrow \wr \\ D^b(\text{mod-}A) & & D^{\mathrm{perf}}(R\mathrm{End}(\bigoplus_{i=0}^n S(i)[-i])) \end{array}$$

What is $(\Delta^{1/2})^2 \stackrel{?}{=:} \Delta$

$(E_0, E_1, \dots, E_n)$

$(L^n E_n, E_0, \dots, E_{n-1})$

$(L^n E_{n-1}, L^n E_n, E_0, \dots, E_{n-2})$

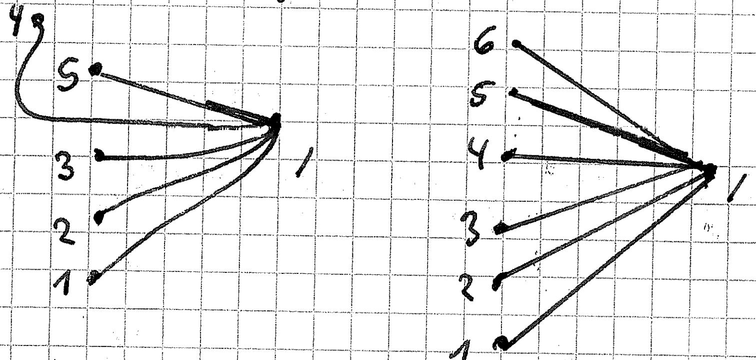
$\vdots$

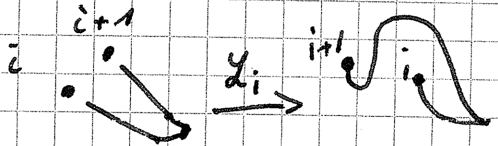
$\to (L^n E_0, L^n E_1, L^n E_2, \dots, L^n E_n)$ $\quad \Delta \in Br_n$

Prop: $(\Delta^{1/2})^2 = \Delta$

Proof: (by pictures) (Hurwitz diagrams):

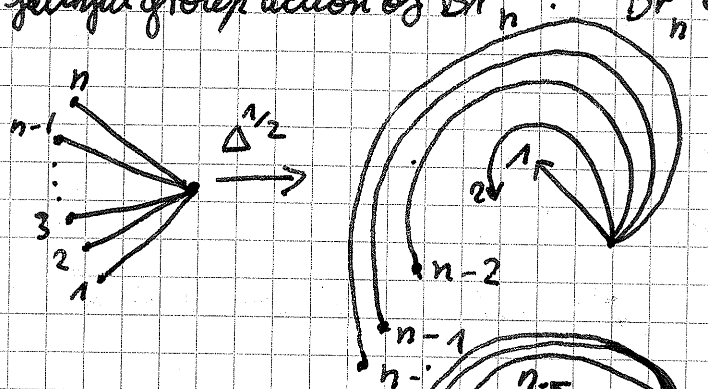
H. D:



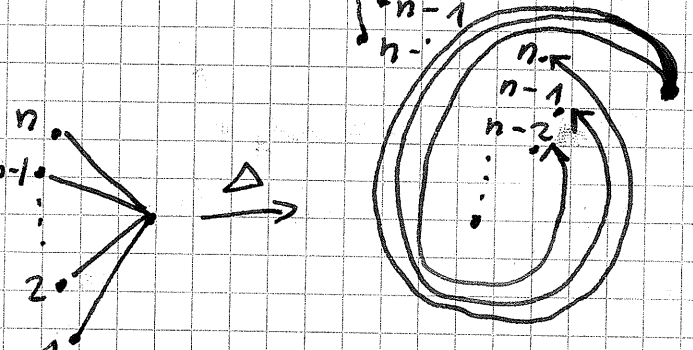


faithful group action of Br_n : $Br_n \hookrightarrow Aut(F_{n+1})$

Ex:



"half-twist"



"full turn"

$(\Delta^{1/2})^2 = \Delta$

Thm (Bondal, Kapranov (?)):

(Assume that $\mathcal{T}$ has a Serre functor)

$\sigma = (E_0, \ldots, E_n)$

$\Delta\sigma = (S(E_0)[-n], S(E_1)[-n], \ldots, S(E_n)[-n])$

Corollary: Let $\sigma = (E_0, \ldots, E_n)$, $A(\sigma) = End_{dg}(\bigoplus_{i=0}^{n} E_i)$, $\Delta^{1/2}\sigma =: \sigma_L^!$

then $A((\sigma_L^!)_L^!) = End_{dg}(\bigoplus_{i=0}^{n} S(E_i)[-n]) \xleftarrow[\text{qis}]{\sim} End_{dg}(\bigoplus_{i=0}^{n} E_i)$

Q: σ is strong $\overset{?}{\Longrightarrow}$ $\sigma_i^!$ is strong $\left(A(\sigma_i^!) \underset{q.is.}{\cong} H^0(\mathcal{A}(\sigma_i^!)) \right.$

$\Rightarrow \mathcal{A}(\sigma) \cong H^0(A(\sigma))$ $\qquad\qquad\qquad = B^!$

$= B$

honest assoc. alg.

Ans: (Bondal): When B is Koszul (graded path alg)

$\sigma = D^b(\text{mod-}A) = \langle P(0), \dots, P(n) \rangle = \langle S(n)[-n], S(n-1)[-n-1], \dots, S(0) \rangle$
$(\sigma^!)$

$\sigma^!$ is strong $\Leftrightarrow \mathcal{H}om_{D^b(A)}(S(i)[-i], S(j)[-j][n]) = 0 \quad \forall n \neq 0$

$\Leftrightarrow \mathrm{Ext}^{n+i-j}(S(i), S(j)) = 0, \; n \neq 0$

Pf: Restrictive assumptions:

$\exists i \to j \Rightarrow j = i-1$ $\quad \overset{n}{\circ} \Rightarrow \overset{n-1}{\circ} \Rightarrow \overset{n-2}{\circ} \cdots \Rightarrow \circ$

$e_i K: \qquad \dots \otimes P(i-2) \to e_i B^*_{i-1} e_{i-1} \otimes P(i-1) \to P(i) \to S(i) \to 0$

$A, B := A^!$ quadratic dual

gr. alg

$K_\bullet$ resol. $\Leftrightarrow \mathcal{A}$ is Koszul