Examples from combinatorics 10/08/2015

Speaker: Geoffrey Janssens.

EXAMPLES...

1) $A = K\langle X_1, -, X_n\rangle$ has the minimal resolution

$$0 \to A^n \xrightarrow{\phi} A \to K \to 0, \ \phi = (X_1, -, X_n)$$

Question: For which $I \triangleleft K\langle X_1, -, X_n\rangle$ is

$$K\langle X_1, -, X_n\rangle / I \quad \text{Koszul?}$$

Notation: $A = \bigoplus_{n\in\mathbb{N}} A^n, \quad A_0 = K,$

$$h_A(z) = \sum_{n\in\mathbb{N}} \dim_K (A^n) z^n. \text{ the Hilbert series.}$$

NON-EXAMPLE: 2) $A = \dfrac{K[X_1, -, X_4]}{(X_1^2, X_2^2, X_3^2, X_4^2, X_1X_2 + X_3X_4)}$

If it were Koszul, we would have

$$h_A(z)\, h_{A^!}(-z) = 1$$

Here $h_A(z) = 1 + 4z + 5z^2$

Together, $h_{A^!}(z) = \dfrac{1}{h_A(-z)}$

$$= 1 + 4z + 11z^2 + 24z^3 + 41z^4$$
$$+ 44z^5 - 29z^6 + \dots$$

Negative coefficients don't make sense! So this is a contradiction.

Def^n: Let $<$ be a monomial order on $A := K\langle x_1, \_, x_n \rangle$.

Fix $I \lhd K\langle x_1, \_, x_n \rangle$

i) $in_<(\psi_{\in A}) :=$ monomial in ψ of highest order

ii) $in_<(I) := (in_<(\psi) : \psi \in I)$

Proposition: (Fröberg, Kempf)

'If there exists a monomial order such that $in_<(I)$ is quadratic, then $K\langle x_1, \_, x_n \rangle / I$ is Koszul'.

Note: A monomial order $<$ is a total order for which $u < v \Rightarrow uw < vw$ $(\forall u, v, w \in k\langle X_1, \dots, X_n\rangle)$

§ 1 : Koszulness preserving operations

Def^{n}: i) $A \sqcap B$; $(A \sqcap B)_0 := k$ & $(A \sqcap B)_i = A_i \oplus B_i$ (*)

ii) $A \sqcup B$: the free product, i.e

$$A \sqcup B = \bigoplus_{i \in \mathbb{N}} \left(\bigoplus_{\varepsilon_1, \varepsilon_2 \in \{0,1\}} A_+^{\varepsilon_1} \otimes (B_+ \otimes A_+)^{\otimes i} \otimes B_+^{\varepsilon_2} \right)$$

*

where $A_+ = \bigoplus_{\substack{i \in \mathbb{N} \\ i > 0}} A_i$ & B_+ similar.

(*) Stipulate $A_+ B_+ := 0 =: B_+ A_+$

Proposition: $h_{A \sqcap B}(z) = h_A(z) + h_B(z) - 1$,

$$h_{A \sqcup B}(z) = \left(h_A^{-1}(z) + h_B^{-1}(z) - 1\right)^{-1}$$

$$h_{A \otimes_k B}(z) = h_A(z) . h_B(z)$$

where also we define (for *)

iii) $(A \otimes_k B)_i := \bigoplus_{\substack{j, e : \\ j + e = i}} A_j \otimes_k B_e$

Proposition: $\mathrm{Ext}^\bullet_{A\sqcap B}(k,k) \cong \mathrm{Ext}^\bullet_{A}(k,k) \sqcap \mathrm{Ext}^\bullet_{B}(k,k)$

Corollary: A & B are both Koszul iff $A \sqcup B$ is

— " — $\mathrm{Ext}^\bullet_{A\sqcup B}$ — " — $\sqcup$ — " —

— " — A & B — " — $A \sqcup B$ is

— " — $\mathrm{Ext}^\bullet_{A\oplus_k B}$ — " — $\oplus_k$ — " —

— " — A & B — " — $A \oplus_k B$ is

§1.2 Veronese sub-algebra & Segre products

Def^n: For A, B, the Segre product is

$$A \circ B = \bigoplus_{i\in\mathbb{N}} (A_i \otimes_k B_i)$$

Proposition: If A & B are Koszul, then so is $A \circ B$

Def^n: For A, & $d \geq 2$, then the d^{th} Veronese subalgebra is $A^{(d)} := \bigoplus_{i\in\mathbb{N}} A_{id}$

Proposition: If A is Koszul then so is $A^{(d)}$ $\forall d \geq 2$

Theorem: If A is commutative & generated in degree 1, then for large enough d, $A^{(d)}$ is Koszul.

PAGE 5

__Remark:__ $\text{rate}(A) := \sup \left\{ \frac{j-1}{i-1} \,\middle|\, \text{Tor}^A_{i,j}(k,k) \neq 0 \right\}$

gives that, if $\text{rate}(A) \leq d$, $A^{(d)}$ is Koszul

Consider also the map

$$S(A_1) \twoheadrightarrow A \quad \& \text{ define}$$

$$\text{rate}(S(A_1), A) := \sup_{p \in \mathbb{N}} \left(\frac{\text{hg}_{S(A_1),A}(p)}{p} \right)$$

with $\text{hg}_{S(A_1),A}(p) = \min\left(i \,\middle|\, \text{Tor}^{S(A_1)}_{pi}(A,B) \neq 0\right)$

In the proof one bound rate(A) using that rate(S(A), A) can be proved to be finite.

__Remark:__ $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ with $(x,y) \mapsto (x^2, xy, y^2)$ is called a Veronese embedding. By embedding in a high enough $\mathbb{P}^d$ we see that "projective varieties are Koszul."

§1 Affine semi-groupoids

Let Λ be a finitely generated submonoid of $(\mathbb{N}^d, +)$, & $k[\Lambda]$ its semigroup algebra

$k[\Lambda]$ is called an affine semigroup ring

Let $A := \{\alpha_1, \dots, \alpha_n\}$ be a minimal set of generators and $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,d})$

PAGE 6

__Remark__: $k[\Lambda] \cong k[t^{\alpha_i} \mid \alpha_i \in \mathcal{A}] \leq k[t_1, \ldots, t_d]$

where t^{α_i} denotes $t_1^{\alpha_{i,1}} \ldots t_d^{\alpha_{i,d}}$. Now define

$$k[x_1, \ldots, x_n] \xrightarrow{\psi} k[t_1, \ldots, t_d]$$

$$x_i \longmapsto t^{\alpha_i}$$

& by this $k[\Lambda] \cong \dfrac{k[x_1, \ldots, x_n]}{\ker(\psi)}$

We call $I_\Lambda := \ker(\psi)$ the 'Toric ideal', & we want to know when we can grade $k[\Lambda]$ by the natural grading on $k[x_1, \ldots, x_n]$; & furthermore, when is it Koszul?

__Proposition__: The following are equivalent

i) $k[\Lambda]$ is a graded algebra (where x_i is in degree 1)

ii) $I_\Lambda \trianglelefteq_{gr} k[x_1, \ldots, x_n]$ —— " ——

iii) $\forall \lambda \in \Lambda$, $\Lambda(\lambda)$ is a pure simplicial complex.

We recall...

Def^n: For $\mu, \lambda \in \Lambda$

i) $\mu \leq \lambda$ iff $\lambda - \mu \in \Lambda$

ii) $]\mu, \lambda[:= \{ f \in \Lambda : \mu < f < \lambda \}$

iii) $\Delta(\lambda) := \Delta(]0, \lambda[)$, order complex of $]0, \lambda[$

(i.e, the groundset is $\{ \mu \in]0, \lambda[\}$ &

faces $= \{ \{\mu_1, \_, \mu_k\} \subseteq]0, \lambda[\}$

with $\mu_1 < \_ < \mu_k$)

Theorem: (Peeva, Reiner, Sturmfels)

$K[\Lambda]$ is Koszul iff, $\forall \lambda \in \Lambda$, $\Delta(\lambda)$ is a

Cohen-Macaulay poset... (definition below)

Def^n: For Δ a simplicial complex, Δ is Cohen-Macaulay if, $\hat{H}_i(lk_F, k) = 0$ $\forall F \in \Delta$ &

$i < \dim(lk_F)$, where we let

$lk_F = \{ G \in \Delta \mid G \cup F \in \Delta, G \cap F = \emptyset \}$

<u>Theorem</u>: Δ is Cohen-Macaulay if and only if $\dfrac{K[x_1, \ldots, x_n]}{I_\Delta}$ is a Cohen-Macaulay ring, where we let I_Δ denote the ideal generated by all square free monomials $x_{i_1} \cdot \ldots \cdot x_{i_r}$ with $\{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta$

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§2 Algebras with monomial relations

<u>Proposition</u>: If I is generated by quadratic monomials,

$\dfrac{K\langle x_1, \ldots, x_n\rangle}{I}$ is Koszul

<u>Reference</u>: 'Confluence property'

Def^n: Let Δ be a simplicial complex (say for example on $\{1, \ldots, n\}$), then the algebras of the form $K[\Delta] := \dfrac{K[x_1, \ldots, x_n]}{(x^F : F \notin \Delta)}$ are called

Stanley-Reisner rings

Proposition: As in the above, let $I_\Delta := (x^F : F \notin \Delta)$

Then $K[\Delta]$ is Koszul iff I_Δ is quadratic

iff Δ is a flag complex

F is a face of Δ iff each pair in F occurs in some face

Def^n A commutative algebra containing a poset P, say A, satisfies the ASL-property if

ASL-1] $p_1, \ldots, p_n \in P$, $p_i < p_{i+1}$ $\forall i \in \{1, \ldots, n-1\}$. Then 1 together with above monomials in the p_i's are a k-basis for A.

ASL-2] For $\ell, n \in P$ incomparable, if μ is a standard monomial with $b_\mu \neq 0$ in $\ell n = \sum b_\mu \mu$, there exists a σ with $\sigma \leq \ell$ & $\sigma \leq n$

EX] coordinate rings of Grassmanians and their Shubert Varieties

prop: quadratic ASL-algebras are Koszul