Koszulity & the Hilbert series of preproj. algebras

(after Etingof - Eu)

k field & $Q = (Q_0, Q_1, t, h)$ finite quiver

$\overline{Q}$ double of Q

$(\overline{Q}_0 = Q_0$ & $\overline{Q}_1 = Q_1 \cup \{a^* : j \to i \mid a : i \to j \in Q_1\}$

Ex: $Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3$

$$\overline{Q} = \begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "a^*"] \arrow[r, bend left, "b"] & 3 \arrow[l, bend left, "b^*"] \end{tikzcd}$$

$Q_0 \supseteq J$ "white" vertices

Def: $\Pi_{Q,J} = k\overline{Q} \Big/ \left(\sum_{t(a)\notin J} a^* a - \sum_{h(a)\notin J} a a^* \right)$ partial preproj. alg. of (Q, J)

$\Pi_{Q,\emptyset}$ preproj. algebra of Q
$\parallel$
Π_Q

Thm (Etingof - Eu) Q connected quiver

(a) $J \neq \emptyset \Rightarrow \Pi_{Q,J}$ is Koszul

(b) Q non-Dynkin $\Rightarrow \Pi_Q$ is Koszul

1. The Golod-Shafarevich inequality

$R := k^{Q_0}$ $\qquad Q_0 = \{1, 2, \ldots, n\}$

$V \in \operatorname{bimod} R \rightsquigarrow V = \bigoplus_{i,j \in Q_0} V_{ij}$, $\qquad V_{ij} \in \operatorname{mod} k$

$V \in \operatorname{gr}^+ \operatorname{bimod} R \rightsquigarrow V = \bigoplus_{d \geq 0} V[d]$, $\qquad V[d] \in \operatorname{bimod} R$

$V, W \in \mathrm{bimod}\, R \rightsquigarrow (V \otimes_R W)_{ij} = \bigoplus_k V_{ik} \otimes_K W_{kj}$

$V, W \in \mathrm{gr}^+ \mathrm{bimod}\, R \rightsquigarrow (V \otimes_R W)[d] = \bigoplus_{d'+d''=d} V[d'] \otimes_K W[d'']$

<u>Def</u>: $V \in \mathrm{gr}^+ \mathrm{bimod}\, R \rightsquigarrow h_V(t) = \sum_{d \geq 0} h_d t^d \in (K^{Q_0 \times Q_0})[[t]]$
$h_d = (\dim V[d]_{ij})$

<u>Lemma</u>: $U, V, W \in \mathrm{gr}^+ \mathrm{bimod}\, R$

(a) $h_R(t) = 1$

(b) $h_{V \otimes_R W}(t) = h_V(t)\, h_W(t)$

c) $0 \to U \to V \to W \to 0$ exact $\Rightarrow h_V(t) = h_U(t) + h_W(t)$

$V \in \mathrm{bimod}\, R$ & $E \subseteq V \otimes_R V \rightsquigarrow A := T_R V / (E)$

<u>Thm</u>: $C = (\dim V_{ij})$ & $D = (\dim E_{ij})$

(a) $\frac{1}{1 - Ct + Dt^2} \geq 0$ (termwise) $\Rightarrow h_A(t) \geq \frac{1}{1 - Ct + Dt^2}$

(b) $h_A(t) = \frac{1}{1 - Ct + Dt^2} \Rightarrow A$ is Koszul

<u>Proof</u>: (a) Consider the Koszul complex

$$0 \to L \to A \otimes_R E \to A \otimes_R V \to A \to R \to 0$$

$$\Rightarrow 0 \leq h_L(t) = h_{A \otimes_R E}(t) - h_{A \otimes_R V}(t) + h_A(t) - h_R(t)$$

$$= h_A(t)\,(h_E(t) - h_V(t) + 1) - 1$$

$$= h_A(t) \quad (1 - Ct + Dt^2) - 1$$

$$\Rightarrow h_A(t) \geq \frac{1}{1 - Ct + Dt^2}$$

(b) $h_L(t) = h_A(t)\,(1 - Ct + Dt^2) - 1$ & $h_A(t) = (1 - Ct + Dt^2)^{-1}$

$\Rightarrow h_L(t) = 0 \quad \Rightarrow L = 0$

$$0 \to A \otimes_R E \to A \otimes_R V \to A \to R \to 0 \quad \text{exact}$$

$$\rightsquigarrow \begin{array}{ccccccccc} 0 & \to & R \otimes_A A \otimes_R E & \to & R \otimes_A A \otimes_R V & \to & R \otimes_A A & \to & 0 \\ & & \cong & & \cong & & \cong & & \\ 0 & \to & E & \xrightarrow{0} & V & \xrightarrow{0} & R & \to & 0 \end{array}$$

$$\mathrm{Tor}_k^A(R,R) = \begin{cases} R & k=0 \\ V & k=1 \\ E & k=2 \\ 0 & k > 2 \end{cases} \quad \Rightarrow A \text{ is Koszul}$$

$\uparrow$ [Prop: A Koszul $\Leftrightarrow$ $\mathrm{Tor}_k^R(R,R)$ is conc. in deg. k] $\square$

2. Hilbert series of partial preproj. algebras

Def: $A = R \oplus A_+ = T_R V/(r_i)$ & $B = R \oplus B_+ = T_R W/(s_i)$

$$A *_R B = \bigoplus_{m=0}^{\infty} A \otimes_R (B_+ \otimes_R A_+)^{\otimes m} \otimes_R B$$

$\uparrow$ free product of A & B

Lemma: $h_A(t) = \frac{1}{1-\alpha}$ & $h_B(t) = \frac{1}{1-\beta}$

$$\Rightarrow h_{A *_R B}(t) = \frac{1}{1-\alpha-\beta}$$

Proof: $A = R \oplus A_+ \Rightarrow h_A(t) = h_R(t) + h_{A_+}(t)$

$$h_{A_+}(t) = \frac{1}{1-\alpha} - 1 = \frac{\alpha}{1-\alpha} \quad \& \quad h_{B_+}(t) = \frac{\beta}{1-\beta}$$

$$\Rightarrow h_{\bigoplus_{m=0}^{\infty} (B_+ \otimes_R A_+)^{\otimes m}}(t) = \sum_{m=0}^{\infty} \left(\frac{\beta}{1-\beta}\,\frac{\alpha}{1-\alpha}\right)^m = \left(1 - \frac{\beta}{1-\beta}\,\frac{\alpha}{1-\alpha}\right)^{-1}$$

$$h_{A *_R B}(t) = (1-\alpha)^{-1}\left(1 - \frac{\beta}{1-\beta}\,\frac{\alpha}{1-\alpha}\right)^{-1}(1-\beta)^{-1}$$
$$= \left((1-\alpha)(1-\beta) - \beta\alpha\right)^{-1}$$
$$= (1 - \alpha - \beta)^{-1}$$

Prop: $Q_0 = \{1, \ldots, n+1\} \supseteq J \ni n+1$

$$Q_1 = \left\{ n+1 \;\substack{\xrightarrow{a_1^{(i)}} \\ \vdots \\ \xrightarrow[a_{r_i}^{(i)}]{}}\; i \;\middle|\; i \in Q_0 \right\}$$

$$D = \mathrm{diag}(D_1, \ldots, D_{n+1}), \quad D_i = \begin{cases} 1 & i \notin J \\ 0 & i \in J \end{cases}$$

C = adjacency matrix of $\overline{Q}$ & $A = \Pi_{Q,J}$

$$\Rightarrow h_{\Pi_{Q,J}}(t) = \frac{1}{1 - Ct + Dt^2}$$

Proof: $\Pi_{Q,J} = \Pi_{Q^{(1)},J} \overset{*}{R} \cdots \overset{*}{R} \Pi_{Q^{(\,)},J}$

$Q^{(i)} = n+1 \;\substack{\longrightarrow \\ \vdots \\ \longrightarrow}\; i$ and vertex set Q_0

possibilities for $Q^{(i)}$

(a) $Q^{(n+1)} = n+1$

$$\Rightarrow h_{\Pi_{Q^{(n+1)},J}}(t) = 1 + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & r_{n+1} \end{array}\right] t + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & r_{n+1} \end{array}\right]^2 t^2 + \ldots$$

$$= \frac{1}{1 - \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & r_{n+1} \end{array}\right] t}$$

(b) $Q^{(i)} = n+1 \;\substack{\longrightarrow \\ \vdots \\ \longrightarrow}\; i$ $\quad (i \in J)$

$$\Rightarrow h_{\Pi_{Q^{(i)},J}}(t) = 1 + \left[\begin{array}{c|c} 0 & r_i \\ \hline r_i & 0 \end{array}\right] t + \left[\begin{array}{c|c} 0 & r_i \\ \hline r_i & 0 \end{array}\right]^2 t^2 + \ldots = \frac{1}{1 - \left[\begin{array}{c|c} 0 & r_i \\ \hline r_i & 0 \end{array}\right] t}$$

(c) $Q^{(i)} = n+1 \;\substack{\longrightarrow \\ \vdots \\ \longrightarrow}\; i$ $\quad (i \notin J)$

two cases: (i) $r = 1$

(ii) $r \geq 2$

Thm (Etingof - Eu) $Q_0 \supseteq J \neq \emptyset$ such that every connected component of Q has a white vertex $\Rightarrow$

(a) $h_{\Pi_{Q,J}}(t) = \frac{1}{1 - Ct + Dt^2}$ $\quad D = D_J$

b) $\Pi_{Q,J}$ is Koszul

3. Hilbert series of preproj. algebras

$k = \bar{k}$ & char $k = 0$

Prop: Q connected, extended Dynkin quiver (? is ok)
$\Rightarrow h_{\Pi_Q}(t) = \frac{1}{1 - Ct +}$

Proof: $Q \rightsquigarrow \Gamma < SL(2, k)$

Thm (Reiten - V. d. Bergh)
f = sum of the central idempotents corresp. to the irred. reps of $k\Gamma$
$A = k[x,y] \rtimes k\Gamma \Rightarrow \Pi_Q = fAf$

idea: take Koszul complex of $k[x,y]$
tensor with $- \otimes_k k\Gamma$
multiply both sides by f ☒

Thm: Q connected, non-Dynkin
$\Rightarrow$ a) $h_{\Pi_Q} = \frac{1}{1 - Ct + t^2}$

b) Π_Q is Koszul