$\boxed{\text{talk 10}}$ 1

<u>Koszul duality for category $\mathcal{O}$</u>

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<u>Plan</u>: 1) Introduction to category $\mathcal{O}$
2) Koszul duality for $\mathcal{O}_0$
3) Generalizations

<u>Setting</u>: $\mathfrak{g}$ s.s. Lie alg. $/\mathbb{C}$
$\cup$
$\mathfrak{h}$ Cartan subalgebra

$\leadsto \mathfrak{h}^* \supseteq \Phi$ root system $\qquad \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$
$\cup$
Δ ~~simple roots~~ $\qquad \mathfrak{n}^+ = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$

$R := \mathbb{Z}\Phi$
$\mathfrak{h} \supset \Phi^\vee$ dual root system
$\cup$
$\Delta^\vee$ dual basis

$\mathfrak{h}^* \supseteq P := \bigoplus_{\alpha \in \Delta} \mathbb{Z}\varpi_\alpha$ where $\langle \omega_\alpha, \beta^\vee \rangle = 0 \quad \forall \alpha, \beta \in \Delta$

$\leadsto$ Weyl group $W \curvearrowright \mathfrak{h}^*$ via $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \quad \forall \alpha \in \Delta$

<u>Def</u>: BGG category $\mathcal{O}$

$$\mathcal{O} = \left\{ M \in \mathfrak{g}\text{-mod} \;\middle|\; \begin{array}{l} M \text{ f.g. as } \mathfrak{g}\text{-module} \\ \mathfrak{h} \text{ acts s.s. on } M \\ \mathfrak{b} \text{ acts locally finite on } M \end{array} \right\}$$

$\boxed{\text{not nec. f.d.}}$

$\leadsto M = \bigoplus_{\alpha \in \mathfrak{h}^*} M_\alpha$

$\forall m \in M$: $\dim \mathcal{U}(\mathfrak{b}) m < \infty$

For $\lambda \in \mathfrak{h}^*$ define
Verma module

$$\Delta(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$$

$$= \mathcal{U}(\mathfrak{g}) \big/ \langle \mathfrak{n}^+, h - \lambda(h)\cdot 1 \mid h \in \mathfrak{h} \rangle \twoheadrightarrow L(\lambda)$$

irred. module

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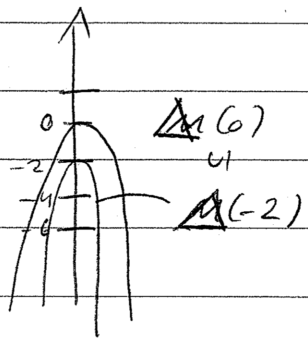
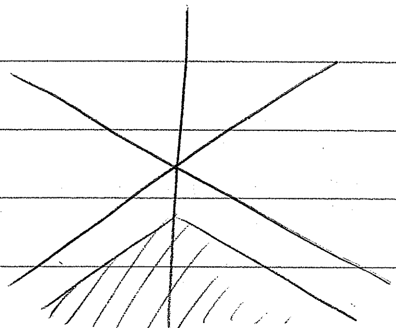
<u>Ex</u>: sl_2 : $P = \mathbb{Z} \supset 2\mathbb{Z} = R$



$\rightsquigarrow L(0) = \mathbb{C}$

sl_3 :



$\mathfrak{h}^* \supset W$

<u>Thm – Properties of cat. $\mathcal{O}$</u>

1) $\{\text{simple object in } \mathcal{O}\} / \cong \quad \longleftrightarrow \quad \{L(\lambda) \mid \lambda \in \mathfrak{h}^*\}$

2) $M \in \mathcal{O}$: M is Noeth. $U(\mathfrak{g})$-module of finite length

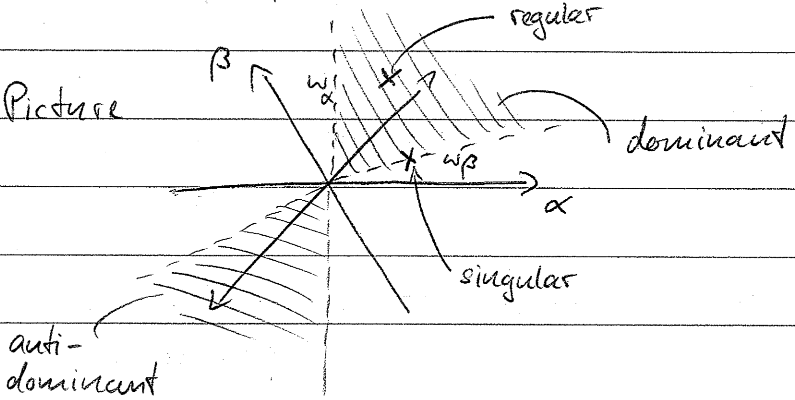
3) $\mathcal{O}$ is abelian.

4) $\mathrm{Hom}_{\mathcal{O}}(M, N)$ is f.d. for all M, N in $\mathcal{O}$.

Define $\rho = \frac{1}{2} \sum_{\alpha \in \phi^+} \alpha \quad \rightsquigarrow$ dot action $w \cdot \lambda := w(\lambda + \rho) - \rho$

$\lambda \in \mathfrak{h}^*$, $W \cdot \lambda$ is called the "linkage" class.

Picture



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$Z(\mathfrak{h}) \subseteq U(\mathfrak{g})$ centre

for $\lambda \in \mathfrak{h}^* \rightsquigarrow \chi_\lambda$

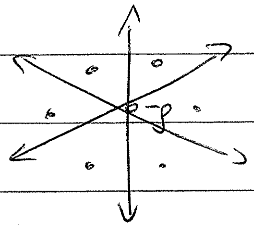
Thm (HC-isom) $Z(\mathfrak{h}) \cong S(\mathfrak{h})^W$

$Z(\mathfrak{h}) \circlearrowright \Delta(\lambda)$

$$\rightsquigarrow \mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathcal{O}_{\chi_\lambda} \quad \text{block decomposition}$$

$\rightsquigarrow \mathcal{O}_0$ = principal block

Ex: In $\mathcal{O}_0$ for sl_3



are L, Δ
$\uparrow$
P proj. cover

(2) Koszul duality for $\mathcal{O}_0$

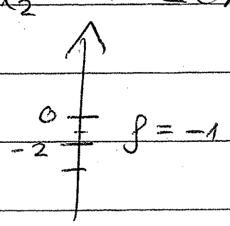
Set $L := \bigoplus_{w \in W} L(w \cdot 0)$, $\bar{P} := \bigoplus_{w \in W} P(w \cdot 0)$

$\Rightarrow \bar{P}$ is a proj. generator for $\mathcal{O}_0$

$A := \mathrm{End}_{\mathcal{O}_0}(\bar{P})$ f.d. algebra $/\mathbb{C}$

Lemma: $\mathcal{O}_0 \longrightarrow \mathrm{Mod}^{fg}\text{-}A$ is an equivalence of categories.
$M \longmapsto \mathrm{Hom}_{\mathcal{O}}(\bar{P}, M)$

Ex: sl_2 (h, e, f): $L(0) = \mathbb{C}$, $L(-2) = \Delta(-2)$ $\quad M \in \mathcal{O}_0$



$$\begin{matrix} M_0 \\ f \downarrow \uparrow e \\ M_{-2} \\ \downarrow \uparrow \\ \vdots \end{matrix}$$

$A := \mathbb{C}(\overset{a}{\bullet \rightleftarrows \bullet}_{b}) / \circlearrowleft$

$\mathcal{O}_0 \xrightarrow{\cong} \mathrm{mod}^{fg} A$

$M \longmapsto \left(\begin{matrix} M_0 \\ \uparrow \downarrow \\ M_2 \end{matrix} \right)$

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Thm (Sörgel)

$$A = \mathrm{End}_{\mathcal{O}_0}(\bar{P}) \cong \underbrace{\mathrm{Ext}^{\bullet}_{\mathcal{O}_0}(L, L)}_{\text{natural graded ring}}$$ is isom. of $\mathbb{C}$-algebras

$$e_{P(w\cdot 0)} \longmapsto e_{L(w^{-1}w_0\cdot 0)}$$

$\Rightarrow A$ is graded.

Under the equivalence $\mathcal{O}_0 \xrightarrow{\cong} \mathrm{mod}^{fg} A$

$$L \longleftrightarrow A_0$$

$$\Rightarrow E(A) = \mathrm{Ext}^{\bullet}_A(A_0, A_0) = \mathrm{Ext}^{\bullet}_{\mathcal{O}_0}(L, L)$$

Excursion: Grading on $\mathcal{O}_0$ $\quad \mathcal{O}_0^{\mathbb{Z}} := \mathrm{mod}^{f.g.}_{\mathbb{Z}} - A$

$$1_w A \underset{\text{graded lift}}{\rightsquigarrow} P(w\cdot 0)$$

$$1_w A_0 \rightsquigarrow L(w\cdot 0)$$

$$1_w \left(A / I_{\cdot w}\right) \rightsquigarrow \Delta(w\cdot 0)$$

$I_{\cdot w}$: 2-sided ideal gen. by all 1_y, $y < w$

Lemma (Bezrukavnikov)

A $\mathbb{Z}_{\geq 0}$-gr $\mathbb{C}$-alg. with A_0 ss, $\dim A_i < \infty$ for $i \in \{0,1\}$

If $A \cong E(A)$ as gr $\mathbb{C}$-alg., then A is Koszul.

Proof: Note $V \in \mathrm{Mod}^{fg} - A_0 \rightsquigarrow \dim_{\mathbb{C}} V^* = \dim_{\mathbb{C}} V$

Set $I := A_{>0}$ consider ses $I \hookrightarrow A \twoheadrightarrow A_0$

$$\begin{array}{ccc} \mathrm{Ext}^1(A_0, A_0) & = & \left(I / I^2\right)^* \\ \uparrow & & \uparrow \\ \mathrm{Ext}^1_{A,\mathbb{Z}}(A_0, A_0\langle 1\rangle) & = & A_1^* \end{array}$$

$A \cong E(A)$

$\Rightarrow \dim A_1^* = \dim \mathrm{Ext}^1_A(A_0, A_0)$

"so both inclusions are equalities"

$$\Rightarrow \mathrm{Ext}^1_A(A_0, A_0) = \mathrm{Ext}^1_{A,\mathbb{Z}}(A_0, A_0\langle 1\rangle)$$

and $A_1 = I / I^2$

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$\Rightarrow A$ generated by A_1 as A_0-algebra.

$E(A) \cong A \Rightarrow E(A)$ –"– $Ext_A^1(A_0, A_0)$ as A_0-algebra

$$\Rightarrow Ext_A^i(A_0, A_0) = Ext^i_{A,\mathbb{Z}}(A_0, A_0\langle i\rangle) \quad \forall i \in \mathbb{Z}$$

"first talk" $\Rightarrow$ A is Koszul $\square$

Sketch of proof

Thm (Sörgel) refinement of BB-loc + Riemann-Hilbert corresp.

$$\begin{tikzcd} \mathcal{O}_0 \arrow[d, "\cong"', "L(x^{-1}w_0\cdot 0)\,\mapsto\, IC_x"] \arrow[dr, "\mathbb{V}"] & \\ Perv_\Lambda(G/B) \arrow[r, "\mathbb{H}"'] & C\text{-mod} \end{tikzcd}$$

(a) Define $\mathbb{V} : \mathcal{O}_0 \longrightarrow \text{mod } \underbrace{End_{\mathcal{O}_0}(P(w_0\cdot 0))}_{\cong\, C}$

$\mathbb{V} = Hom_{\mathcal{O}_0}(P(w_0\cdot 0), -)$

$\cong C$ ← Sörgel's Endomorphismensatz

Thm: $\mathbb{V}$ is fully faithful on projectives

$$\Rightarrow End_{\mathcal{O}_0}(\bar{P}) \cong End_{C\text{-mod}}\Big(\bigoplus_{w\in W} \mathbb{V}(P(w\cdot 0))\Big)$$

(b) Use $\mathcal{O}_0 \xrightarrow{\sim} Perv_\Lambda(G/B)$ to reduce to

$$Ext^\cdot_{Perv_\Lambda(G/B)}\Big(\bigoplus_x IC_x, \bigoplus_x IC_x\Big) = End_{D^b_\Lambda(G/B)}\Big(\bigoplus_x IC_x\Big)$$

$$= End_{C\text{-mod}}\Big(\bigoplus_{x\in W} \mathbb{H}(IC_x)\Big)$$

$\mathbb{H}(IC_e) = \mathbb{C}$

$\mathbb{V}(P(0)) = \mathbb{C}$

In $\mathcal{H}$ (Hecke algebra) for $ws > w$

$$\underline{H}_w \cdot \underline{H}_s = \underline{H}_{ws} + \sum_{\substack{y<w \\ ys<w}} \mu(-,-)\, \underline{H}_y$$

KL-basis

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$\rightsquigarrow$ inductively

$$\mathbb{H}^{\cdot}(\mathbb{IC}_x) = \mathbb{V}(P(x\cdot\theta))$$