Summer school on Koszul duality, Bad Driburg 13/8/15

Külshammer: Differential graded categories and derived categories

1 Triangulated categories

Def.: An additive category $\mathcal{T}$ is called triangulated if $\exists\ \Sigma : \mathcal{T} \to \mathcal{T}$ automorphism of $\mathcal{T}$, $\Delta = \{X \to Y \to Z \to \Sigma X\}$ closed under $\cong$ s.t.

(TR1): $X \xrightarrow{id_X} X \to 0 \to \Sigma X \in \Delta \quad \forall X \in \mathcal{T}$

(TR3): $X \to Y \to Z \to \Sigma X \in \Delta \iff Y \to Z \to \Sigma X \xrightarrow{-} \Sigma Y \in \Delta$

(TR5): Octahedral axiom

(TR2): $\underbrace{X \xrightarrow{f} Y}_{\text{given}} \xrightarrow{\exists} Z \xrightarrow{\exists} \Sigma X \in \Delta$

(TR4):
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dashed, "\exists"'] & \Sigma X \arrow[d, "\Sigma f"] & \in \Delta \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' & \in \Delta \end{tikzcd}$$

Ex. · A k-algebra (k field) $\rightsquigarrow$ $D(A)$ derived cat. is triangulated with Σ = shifting to the left

· A self-injective k-algebra $\rightsquigarrow$ $\underline{\mathrm{mod}}\, A$ stable category is triangulated with $\Sigma X = \mathrm{Coker}\,(X \hookrightarrow I(X))$

Another motivation: A k-algebra, $e \in A$ idempotent.

$$\mathrm{mod}\,(A/AeA) \rightleftarrows \mathrm{mod}\,A \rightleftarrows \mathrm{mod}\,(eAe)$$

"standard recollement"

In general $\nexists$ $D(A/AeA) \rightleftarrows D(A) \rightleftarrows D(eAe)$

$\exists$ recollement $D(B) \rightleftarrows D(A) \rightleftarrows D(eAe)$

B dg algebra with $H^0(B) = A/AeA$.

2. dg categories

Notation: • complex of VS

$$\cdots \to C_i \xrightarrow{d} C_{i+1} \to \cdots \qquad d^2 = 0$$

• C, D complexes

$$C \otimes D = \bigoplus_{k+l=i} C_k \otimes D_l \xrightarrow{d_C \otimes 1 + (-1)^k 1 \otimes d_D} \bigoplus_{k+l=i+1} C_k \otimes D_l$$

$$\cdot \; \mathcal{H}om(C,D) = \begin{cases} \mathcal{H}om(C,D)_i = \mathrm{Hom}^i_{gr}(C,D) \ni f: C_k \to D_{k+i} \\ d(f) = d_D \circ f - (-1)^{|f|} f \circ d_C \end{cases}$$

Def: A (dg) category $\mathcal{A}$ consists of

objects: class of objects of $\mathcal{A}$

morphisms: $\forall x, y \in \mathcal{A}$ a set $\mathcal{A}(x,y)$ (a complex $\mathcal{A}(x,y)$)

composition: $\mathcal{A}(y,z) \times \mathcal{A}(x,y) \xrightarrow{\circ} \mathcal{A}(x,z)$ associative ($\mathcal{A}(y,z) \otimes_k \mathcal{A}(x,y) \to \mathcal{A}(x,z)$ morph. of complexes)

units: $1_x \in \mathcal{A}(x,x)$.

Remark i.e.: $d(fg) = m(d_\otimes(f \otimes g)) = m(df \otimes g + (-1)^{|f|} f \otimes dg)$

$$= (df) \circ g + (-1)^{|f|} f \circ dg$$

EX.: (i) A a k-algebra, then A is a dg category

objects: $\{*\}$

morphisms: $\mathrm{End}(*)_i = \begin{cases} A & i = 0 \\ 0 & \text{otherwise} \end{cases}$

differential $= 0$.

(ii) $\mathrm{Dif}(k)$ the dg cat. of $\underbrace{\text{complexes of vector spaces}}_{\text{objects}}$

morphisms: $\mathrm{Hom}_{\mathrm{Dif}(k)}(x,y) = \mathcal{H}om(x,y)$.

(iii) $\begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, dashed, "\varphi"'] & 2 \end{tikzcd}$ $\quad |a| = 0, \quad |\varphi| = 1, \quad d(a) = \varphi, \quad d(\varphi) = 0$

3 Associated categories

$\mathcal{A}$ dg category:

underlying category $Z^0(\mathcal{A})$; same objects but $\mathrm{Hom}_{Z^0(\mathcal{A})}(x,y) = Z^0(\mathcal{A}(x,y))$

homotopy category $H^0(\mathcal{A})$; $\mathrm{Hom}_{H^0(\mathcal{A})}(x,y) = H^0(\mathcal{A}(x,y))$

Ex.: (i) A k-algebra $\rightsquigarrow Z^0 A = A, \quad H^0 A = A.$

(ii) $Z^0(\mathrm{Dif}(k)) = \mathcal{C}(k)$ cat. of chain complexes

$H^0(\mathrm{Dif}(k)) = \mathcal{H}(k)$ homotopy category

(iii) $Z^0\left(\begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, dashed, "\varphi"'] & 2 \end{tikzcd}\right) = 1 \quad 2$

$H^0\left(\begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, dashed, "\varphi"'] & 2 \end{tikzcd}\right) = 1 \quad 2$

Def.: • $\mathcal{A}$ small dg category, $\mathcal{B}$ dg category. A dg functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is given by:

$x \in \mathcal{A} \rightsquigarrow Fx \in \mathcal{B}$

morphism $\mathrm{Hom}_{\mathcal{A}}(x,y) \longrightarrow \mathrm{Hom}(Fx, Fy)$ of complexes

$F(fg) = F(f)F(g), \quad F(1) = 1.$

• $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ dg category

objects: dg functors

morphisms: $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{A},\mathcal{B})}(F,G) = \{ \alpha_x \in \mathcal{B}(Fx, Gx) \}$

$d\left((\alpha_x)_{x\in\mathcal{A}}\right) = \left(d_{\mathcal{B}}\alpha_x\right)_{x\in\mathcal{A}}.$

Modules are functors:

module $\quad A \otimes_k M \longrightarrow M \quad$ k-lin., associative, unital

$\updownarrow$

representation $\quad A \longrightarrow \mathrm{End}(M) \quad$ k-alg. hom.

$\updownarrow$

functor:
$$\begin{tikzcd} \ast \arrow[loop left, "a \in A"] \arrow[r] & \mathrm{Mod}(k) \end{tikzcd}$$
$$\ast \mapsto M$$
$$a \mapsto f_a$$

A module over a small category $\mathcal{C}$ is a functor $\mathcal{C} \to \mathrm{Mod}\, k$

<u>Def.:</u>
- The dg category $\mathrm{Dif}\, \mathcal{A} := \mathrm{Fun}(\mathcal{A}, \mathrm{Dif}(k))$ of (left) dg modules
- category of dg modules $Z^0(\mathrm{Dif}\, \mathcal{A}) =: \mathcal{C}\mathcal{A}$
- homotopy category $H^0(\mathrm{Dif}\, \mathcal{A}) =: \mathcal{H}(\mathcal{A})$

$\longrightarrow \mathcal{A}^{op}$ for right modules

<u>Ex.:</u> A k-alg $\rightsquigarrow$ $\mathcal{C}A$ complexes of A-mod.
$\mathcal{H}A$ homotopy cat.

<u>Thm</u> (i) $\mathcal{C}\mathcal{A}$ is a Frobenius category.
(ii) $\mathcal{H}\mathcal{A} = \underline{\mathcal{C}\mathcal{A}}$ is a triangulated category.

<u>Def.</u> (i) A morphism $f: M \to N$ of dg modules is a qis if $H^i M x \xrightarrow{H^i f x} H^i N x$ is isomorphism.

(ii) A dg module M is acyclic if $0 \to M$ is a qis

(iii) A dg module M is $\mathcal{H}$-projective if $\mathrm{Hom}_{\mathrm{Dif}\,\mathcal{A}}(M, -)$ preserves qis

<u>TFAE:</u>
(1) M $\mathcal{H}$-projective

(2) $\mathrm{Hom}_{\mathrm{Dif}\,\mathcal{A}}(M, -)$ preserves acyclics

(3) $\mathrm{Hom}_{\mathcal{H}\mathcal{A}}(M, \mathcal{H}_{ac}) = 0$.

(4) in $\mathcal{H}(\mathcal{A})$:
$$\begin{tikzcd} & M \arrow[d] \arrow[dl, dashed, "\exists!"'] \\ L \arrow[r, "qis"'] \arrow[ur, phantom, "\circlearrowleft"] & N \end{tikzcd}$$

(5) M is homotopy equ. to N

$0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N$, $\quad N_{i+1}/N_i \cong \bigoplus_{i,x} \mathcal{A}(x,-)[i]$

with $\bigcup N_i = N$, $N_{i-1} \hookrightarrow N_i$ splits as graded modules.

Derived category:

"Def.:" $\mathcal{D}\mathcal{A} = \mathcal{H}\mathcal{A}[qis^{-1}]$

Problem: Hom's might not be sets.

Solution: Model categories

Cor.: $\mathcal{D}\mathcal{A} \simeq \mathcal{H}_p\mathcal{A}$ (Compare $D^- A \simeq K^-(\mathrm{Proj}\, A)$)

Compact objects: $M \in \mathcal{C}$ is compact if

$\mathrm{Hom}_{\mathcal{C}}(M, -)$ commutes with filtered colimits

Prop. $M \in \mathcal{D}\mathcal{A}$ is compact iff it is perfect, i.e. isomorphic to a bounded complex of repr. functors.

Pretriangulated dg categories

Idea: Lift structure of triangulated cat. to dg level.

Roughly: $\mathcal{A}$ is pretriangulated if $H^0\mathcal{A}$ is triangulated. One calls $\mathcal{A}$ a dg enhancement of $H^0\mathcal{A}$

Thm.: Algebraic triang. cat.s have enhancements

Thm.: Derived cat. have a unique enhancement.

Recall: Def.: $\mathcal{T}$ algebraic if

(1) $\mathcal{T} \simeq \underline{\mathcal{E}}$ for $\mathcal{E}$ Frobenius cat.

equivalently

(2) $\mathcal{T} \hookrightarrow \mathcal{K}\mathcal{A}$ for $\mathcal{A}$ an additive cat.