2nd talk

Necessary and sufficient conditions (Sondre Kvamme)

F field, $A = \bigoplus_{i=0}^{\infty} A_i$ graded F-algebra

s.t. (1) $\dim_F A_i < \infty$

(2) $A_0 = k = \bigoplus_{x \in W} F 1_x$ where: W finite set

$1_x \cdot 1_x = 1_x$

Note: A is left and right finite $\quad 1_x \cdot 1_y = 0 \quad x \neq y$

$(M[n])_m = M_{m-n}$

write A-Mod for the category of ungraded A-modules

have $\mathrm{Ext}^i_A(M,N)$, $\mathrm{Hom}_A(M,N)$

A-mod: category of graded A-modules

$\mathcal{E}xt^i_A(M,N)$, $\hom_A(M,N)$

$M \in k \otimes_F k$-Mod : $(a \otimes b) \cdot m = a \cdot m \cdot b \quad a,b \in k, \; m \in M$

$\dim_F M < \infty$

set $[M]_{x,y} = \dim_F 1_x \cdot M \cdot 1_y \quad \forall x,y \in W$

$M = \bigoplus_{i=0}^{\infty} M_i$ graded $k \otimes_F k$-module, with $\dim_F M_i < \infty$

Hilbert series:

define $W \times W$-matrix $P(M,t)$ with $P(M,t)_{x,y} = \sum_{i=0}^{\infty} [M_i]_{x,y} t^i$

Lemma: Assume A is Koszul. Then

$$P(A,t) \cdot P(A^!, -t) = I$$

proof: have Koszul complex

$$\cdots \longrightarrow A \otimes_k {}^*(A^!_2) \longrightarrow A \otimes_k {}^*(A^!_1) \longrightarrow A \otimes_k {}^*(A^!_0) \longrightarrow 0 \qquad (*)$$

is exact, since A is Koszul.

$$(*) \Rightarrow P(A_0,t)_{x,y} = \sum_{j=0}^{\infty} P(A \otimes_k {}^*(A^!_j), t)_{x,y} (-1)^j$$

$$[A_0]_{xy} = \sum_{\substack{j=0 \\ i=0}}^{\infty} (-1)^j \, t^{i+j} \, [A_i \otimes_k {}^*(A^!_j)]_{xy}$$

$$(A \otimes_k {}^*(A^!_i))_n = \begin{cases} A_{n-i} \otimes {}^*(A^!_i) & n \geq i \\ 0 & n < i \end{cases}$$

Note: $[A_0]_{xy} = \delta_{xy}$ by definition

$M \in k \underset{F}{\otimes} k\text{-Mod}$ $\quad 1_x \cdot M \cdot 1_y \cong {}^*(1_y M 1_x)$

$\Rightarrow [{}^*M]_{xy} = [M]_{yx}$

$M, N \in k \underset{F}{\otimes} k\text{-Mod} \Rightarrow M \underset{k}{\otimes} N \cong \bigoplus_{z \in W} M 1_z \underset{F}{\otimes} 1_z N$

$\Rightarrow [M \underset{k}{\otimes} N]_{xy} = \sum_{z \in W} [M]_{xz} \cdot [N]_{zy}$ since $k \cong F \times \ldots \times F$

$$\delta_{xy} = \sum_{\substack{j=0 \\ i=0}}^{\infty} (-1)^j t^{i+j} \sum_{z \in W} [A_i]_{xz} \cdot [A_j^!]_{zy} = \sum_{z \in W} \left(\sum_{i=0}^{\infty} [A_i]_{xz} t^i \right) \cdot \left(\sum_{j=0}^{\infty} (-1)^j [A_j^!]_{yz} \right)$$

$$= \sum_{z \in W} P(A,t)_{xz} \cdot P(A_j^!, -t)^T_{zy} = \left(P(A,t) \cdot P(A^!, -t)^T \right)_{xy}$$

A left Noetherian + (1) and (2)

$E = E(A) = \mathrm{Ext}^\bullet_A(k,k)$ is a graded F-algebra $\quad E_i = \mathrm{Ext}^i_A(k,k)$

left noetherian $\Rightarrow \dim_F(\mathrm{Ext}^i_F(k,k)) < \infty$

$\Rightarrow P(E,-t)$ makes sense

$E_0 = k^{op} = k$

<u>THM</u>: A, $E(A)$ as above. Then TFAE:

(i) A is Koszul

(ii) $P(A,t)\,P(E,-t) = I$

proof: "(i) $\Rightarrow$ (ii)": A Koszul $\Rightarrow (A^!)^{op} \cong E$

$P(A^!, -t)^T = P(E,-t)$ result follows from lemma.

"(ii) $\Rightarrow$ (i)": construct graded exact sequence

$$\ldots P^2 \to P^1 \to P^0 \to k \to 0$$

s.t. P^i graded projective $\quad P^i = \bigoplus_{j=0}^{\infty} P^i_j$

and $A \cdot P^i_i = P^i$.

We do this inductively:

$i=0$: $P^0 = A$

assume P^i is constructed: $K^i = \ker(P^i \to P^{i-1})$

choose graded $V^{i+1} \in k \underset{F}{\otimes} k$-mod s.t.

$K^i = A_{>0} \cdot K^i \oplus V^{i+1}$ as graded $k \underset{F}{\otimes} k$-modules

$V^{i+1} \overset{j}{\hookrightarrow} K^i$ inclusion

$$d_{i+1} : A \underset{k}{\otimes} V^{i+1} \xrightarrow{1 \otimes j} A \underset{k}{\otimes} K^i \longrightarrow K^i$$
$$a \otimes k \mapsto ak$$

obviously d_i morphisms $/ A \underset{F}{\otimes} k$

check: d_i surjective.

Have exact sequence $V^0 = k$

$$\cdots \to \underset{\underset{P^2}{\shortparallel}}{A \underset{k}{\otimes} V^2} \longrightarrow \underset{\underset{P^1}{\shortparallel}}{A \underset{k}{\otimes} V^1} \to \underset{\underset{P^0}{\shortparallel}}{A \underset{k}{\otimes} V^0} \longrightarrow k \to 0 \quad = P^\bullet$$

Note: $A \underset{k}{\otimes} V^i \cong \bigoplus_{z \in W} A 1_z \underset{F}{\otimes} 1_z V^i$ projective $/ A \underset{F}{\otimes} k$

Only need to show: $V_j^i = 0$ for $j \neq i$

check: $K^i \subset A_{>0} \cdot P^i \rightsquigarrow P^{i+1}$ gets mapped to $A_{>0} \cdot P^i$

$$d_{i+1}(x) = \sum_j a_j x_j \qquad \begin{matrix} x \in P^{i+1} \\ a_j \in A_{>0} \end{matrix}$$

Consider the complex $\operatorname{Hom}_A(P^\bullet, k)$

$$\bar{d}_i : \operatorname{Hom}_A(P_i, k) \longrightarrow \operatorname{Hom}_A(P_{i+1}, k)$$

$$\bar{d}_i(f)(x) = f(d_{i+1} x) = \sum_j a_j f(x_j) = 0$$

$$\operatorname{Ext}_A^i(k,k) = \operatorname{Hom}_A(A \underset{k}{\otimes} V^i, k) \cong \operatorname{Hom}(V^i, k) = (V^i)^*$$

similarly $\mathcal{E}xt_A^i(k, k[j]) = \hom_A(A \underset{k}{\otimes} V^i, k[j]) \cong \hom_A(V^i, k[j])$
$= (V_j)^*$

check: $[E_i]_{xy} = [V^i]_{xy}$

$$\Rightarrow P(E, -t) = P(V, -t) \quad \Rightarrow \quad \underset{(\star)}{P(A,t) \cdot P(V, -t) = I}.$$

Recall: A graded with A_0 semisimple

$\Rightarrow \underset{\underset{(V_j^i)^*}{\cong}}{\operatorname{ext}_A^i(k, k[j])} = 0$ for $i > j$

$\Rightarrow V_j^i = 0$ for $i > j$

we show $V_j^n = 0$ for $n < j$ by induction on n:

$n=0$ $V^0 = k$ ✓

$n>0$ $(A \underset{k}{\otimes} V^i)_n = \bigoplus_{j=0}^{n} A_{n-j} \otimes V_j^i$

for $i < n$ $\quad (A \otimes_k V^i)_n = A_{n-i} \otimes_k V^i_j = A_{n-i} \otimes_k V^i$

pf: $(A \otimes_k V^n)_n = A_0 \otimes_k V^n_n$ since $V^n_j = 0$ for $j < n$.

degree n part of $P^\cdot$ $\quad n \geq 1$

$$0 \to A_0 \otimes_k V^n_n \to A_1 \otimes_k V^{n-1} \to \dots \to A_{n-1} \otimes_k V^1 \to A_n \otimes_k V^0 \to 0$$

Hence: (i) $\displaystyle\sum_{j=0}^{n-1} (-1)^j A_{n-j} \otimes_k V^j + (-1)^n A_0 \otimes_k V^n_n = 0$ in Grothendieck group $K_0(k \otimes_F k)$

$$(\star) \iff \delta_{xy} = \sum_{z \in W} P(A,t)_{xz} \cdot P(V,-t)_{zy} = \sum_{z \in W} \left(\sum_{i=0}^{\infty} [A_i]_{xz} t^i \right) \cdot \left(\sum_{j=0}^{\infty} [V_j]_{zy} (-1)^j t^j \right) =$$

$$= \sum (-1)^j t^{i+j} \sum_{z \in W} [A_i]_{xy} [V_j]_{zy} = \sum_{\substack{i=0 \\ j=0}}^{\infty} (-1)^j t^{i+j} \sum_{z \in W} [A_i]_{xz} [V_j]_{zy}$$

$$= \sum_{\substack{i=0 \\ j=0}}^{\infty} (-1)^j t^{i+j} [A_i \otimes_k V_j]_{xy}$$

taking coefficient in front of t^n, $n = i+j \geq 1$ we get

$$\sum_{j=0}^{n} (-1)^j [A_{n-j} \otimes_k V_j]_{xy} = 0 \quad \forall x,y \in W \implies \sum_{j=0}^{n} (-1)^j A_{n-j} \otimes_k V_j = 0 \text{ in } K_0(k \otimes_F k) \quad \text{(ii)}$$

$$\text{(i)} + \text{(ii)} \implies (-1)^n A_0 \otimes_k V_n = (-1) A_0 \otimes_k V^n_n \text{ in } K_0(k \otimes_F k) \implies V_n = V^n_n$$

as modules.

$A = \bigoplus_{i=0}^{\infty} A_i$ Noetherian graded algebra, $A_0 = k$, k a field

$\operatorname{gldim}(A)$ = global dimension of A not considering grading (in Mod A)

THM (3): $\operatorname{gldim}(A)$ = projective dimension of k
$= \sup\{ i \mid \operatorname{Ext}^i_A(k,k) \neq 0 \}$

Corollary: A as above is Koszul, then A has finite global dimension iff $A^!$ is a finite dimensional algebra

proof: A has finite global dimension $\overset{\text{THM(3)}}{\iff}$ $\exists n : \operatorname{Ext}^i_A(k,k) = 0$ for $i > n$.

$\iff \dim_k \left(\bigoplus_{i=0}^{\infty} \operatorname{Ext}^i_A(k,k) \right) = \dim_k E < \infty$

$\iff \dim_k A^! < \infty$ $\quad\square$

$E \cong (A^!)^{op}$