LINEAR COMPLEXES OF PROJECTIVES

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① $k = \bar{k}$

$A = \bigoplus_{i \in \mathbb{Z}} A_i$ — $\mathbb{Z}$-graded k-algebra

— positively graded
- A_0 - semisimple
- $A_i = 0 \quad i < 0$
- $\dim A_i < \infty$.

Examples: 1) $k[x]$, $\deg k = 0$, $\deg x = 1$

2) $k[x]$, $\deg k = 0$, $\deg x = 2$

3) $k[x]/(x^2)$ $\deg x = 1$

4) $k[x]/(x^m)$ $m > 2$, $\deg x = 1$.

5) Q - finite quiver
kQ path algebra $\deg \varepsilon_i = 0$
$\deg \alpha = 1$

where
ε_i : trivial path at $i \in Q_0$
α : an arrow

GOAL: Give an alternative approach to Koszul Duality theorem from the previous lecture following M-Ovsienko-Stroppper TAMS 2009

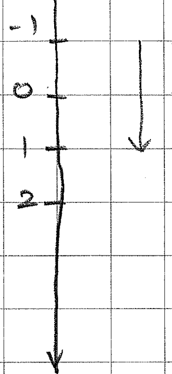
↑

via linear complexes of projectives.

② • $A\text{-gmod} \ni M = \bigoplus_{i \in \mathbb{Z}} M_i$

Morphisms are homogeneous of degree zero. (We want to have a preadditive structure on our category)

• $A\text{-LfMod}$ Locally finite dimensional modules – $\dim M_i < \infty$.



A picture of our grading convention.

"In rep. theory proj. modules usually have tops"

$M_i = 0$ for $i \ll 0$

$M_i = 0$ for $i \gg 0$

③ Indecomposable projective

$A_0 \ni e = e^2$ primitive.

$\Rightarrow \underbrace{Ae}_{\text{has grading}}$ - ind. proj $\in A\text{-gmod}$

Remark: "A-LfMod does not have enough projectives in general".

Ex: $Q: 1 \bullet \xrightarrow{\alpha} \bullet\, 2$ A - basis $\varepsilon_1, \varepsilon_2, \alpha$

prim. idem.

- $P_1 = A\varepsilon_1$ $\varepsilon_1 \longleftarrow 0$, $\alpha \longleftarrow 1$

- $P_2 = A\varepsilon_2$ $\varepsilon_2 \longleftarrow 0$

Shift:

$$\begin{array}{c|c} -1 & M\langle 1\rangle \\ 0 & M \end{array} \quad \Uparrow$$

A picture of $\langle 1\rangle$-grading shift.

X° $[1]$: homological shift (to the left)

④ $\mathcal{LC}(P)$

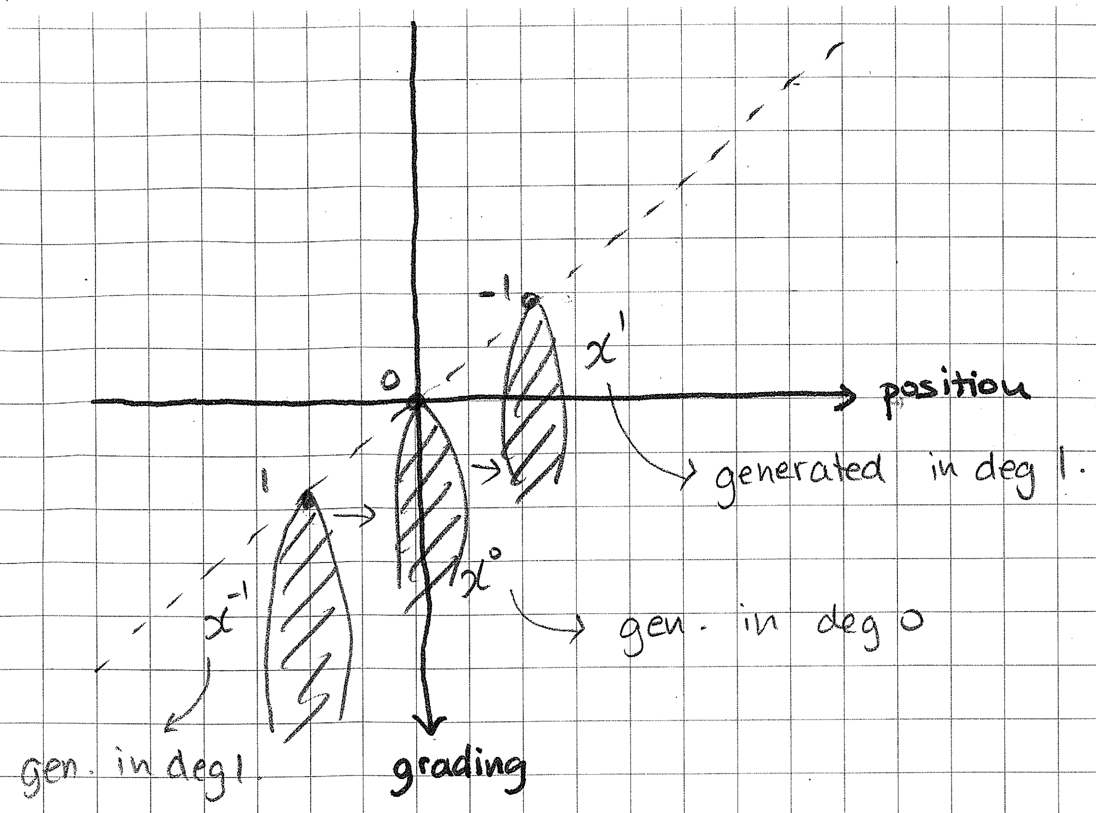
def: $\mathcal{LC}(P)$ category

objects $X^\circ \quad \cdots \longrightarrow X^{-1} \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$

s.t. $X^i \in \mathrm{add}({}_A A\langle i\rangle)$

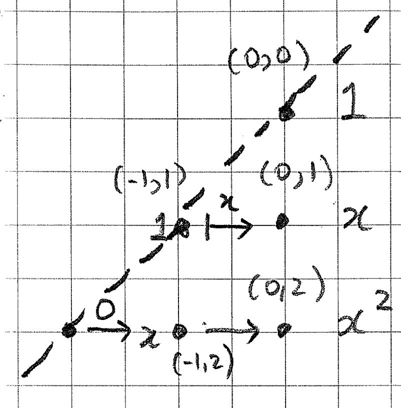
complex $d^2 = 0$.

morphisms: morphism of complexes.



$$\dashrightarrow x^{-1} \longrightarrow x^{0} \longrightarrow x^{1} \longrightarrow \cdots$$

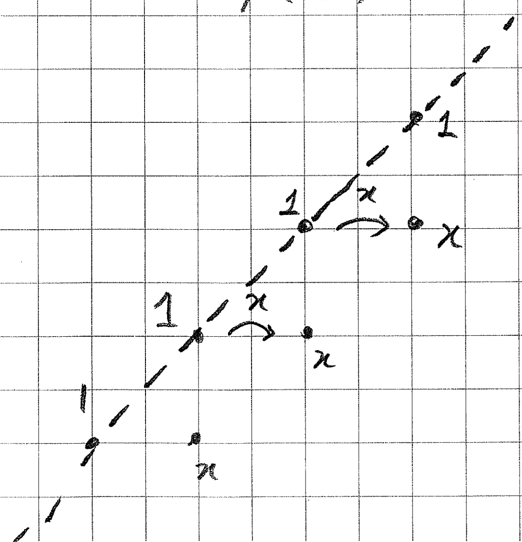
<u>Ex 1</u> : $k[x]$ $\quad \deg x = 1$.



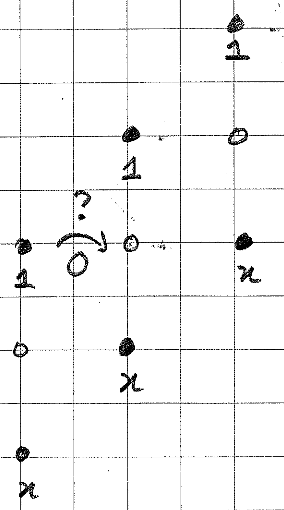
" This is a 'small' object "

<u>Ex 2</u> : $k[x]/(x^2)$ $\quad \deg x = 1$.

" This is a 'big' object "



Ex 3 : $k[x]/(x^2)$ $\qquad \deg x = 2$



$\longrightarrow$ direct sum $\rightsquigarrow$ V. spaces.

Ex 4 : $k\left(1 \xrightarrow{\alpha} 2\right)$

$P_1 \quad \begin{matrix} \varepsilon_1 \\ \alpha \end{matrix} \qquad\qquad P_2 \quad \varepsilon_2$

$X_1 \quad \bullet \qquad\qquad \varepsilon_2$

$X_2 \quad \bullet \qquad\qquad \varepsilon_1 \longleftarrow 0$

$\in \mathcal{LC}(P)$

$\varepsilon_2 \xrightarrow{\cdot\alpha} \alpha \longleftarrow 1$

EXERCISE : Classify all indecomposable objects in $\mathcal{LC}(P)^b$ for $k\left(1 \xrightarrow{\alpha} 2\right)$

5. PROPERTIES

a) There are <u>no</u> homotopies between $X^\bullet, Y^\bullet \in \mathcal{LC}(P)$.

Why?

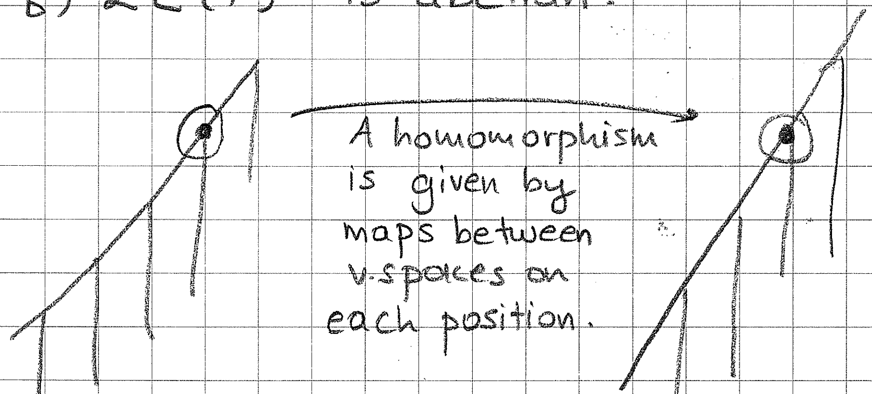
gen. in deg $-i$

$$\begin{tikzcd}
X^\bullet : \cdots & X^{i-1} \arrow[r] & X^{i} \arrow[r] \arrow[dl] & X^{i+1} \arrow[r] \arrow[dl] & \cdots \\
Y^\bullet : \cdots & Y^{i-1} \arrow[r] & Y^{i} \arrow[r] & Y^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

gen. in deg. $-(i-1)$

$$Y^{i-1}_{-i} = 0 .$$

b) $\mathcal{LC}(P)$ is abelian.



Use this info to construct kernels and cokernels in this category.

6. QUADRATIC DUALS

$$A = \bigoplus_{i \geq 0} A_i \Rightarrow A_1 \in A_0\text{-mod-}A_0 .$$

$A_0[A_1]$ – free tensor algebra

$$\parallel$$

$$\bigoplus_{i \geq 0} \underbrace{A_1 \otimes_{A_0} A_1 \otimes_{A_0} A_1 \cdots \otimes_{A_0} A_1}_{i \text{ times}}$$

graded assoc. algebra

$i = 0 \Rightarrow A_0 .$

$$I \hookrightarrow A_0[A_1] \xrightarrow{\text{can}} A$$

$$\begin{array}{l} I \\ \shortparallel \\ \ker(\text{can}) \\ \uparrow \text{homog.} \\ \cong \\ \bigoplus_{i \geq 0} I_i \end{array}$$

via can: $A_0 \xrightarrow{id} A_0$, $A_1 \xrightarrow{id} A_1$

$I_0 = I_1 = 0$

<u>DEF</u>: A is generated in degrees $0,1$ if can is surjective.

<u>Ex</u>: • $k[x]$, $k[x]/(x^m)$; $\deg x = 1$.

gen in degrees $0,1$.

• $k[x]$; $\deg x = 2$.

not!!

$$I_2 \xhookrightarrow{\text{incl.}} A_1 \otimes_{A_0} A_1 \xrightarrow[\text{mult.}]{\text{can.}} A_2$$

left exact

exact if A is gen. in deg $0,1$.

Apply $\mathrm{Hom}_k(-,k)$

$$A_2^* \xrightarrow{\text{mult}^*} (A_1 \otimes_{A_0} A_1)^* \twoheadrightarrow I_2^*$$

$$\cong$$

$$A_1^* \otimes_{A_0} A_1^*$$

where $A_1^* \in A_0\text{-mod-}A_0$.

$$A_0[A_1^*]_2 \supset \mathrm{Image}(\text{mult}^*).$$

DEF: The quadratic dual of A is

$$A^! := A_0[A_1^*] \Big/ \big(\underbrace{\mathrm{Image}(\mathrm{mult}^*)}_{\deg 2}\big)$$

quadratic algebra.

EXERCISE: If A is gen. in deg $0, 1$
and
A is quadratic,

(Show that) then $(A^!)^! \cong A$.

Ex 1: $k[x]$; $\deg x = 1$

$A_0 = k\{1\}$, $A_1 = k\{x\}$.

$$I_2 = 0 \hookrightarrow k \otimes_k k \xrightarrow[\sim]{\text{mult}} k$$

Take dual

$$k^* \overset{\sim}{\hookrightarrow} k^* \otimes_k k^* \twoheadrightarrow 0^*$$

So,

$$A^! = k[x^*] \Big/ \left((x^*)^2 \right)$$

Ex 2 : $k[x]/(x^2)$

$$I_2 \xrightarrow[\;]{\sim} k \otimes_k k \xrightarrow{\text{mult}} 0$$

dualize

$$0 \xrightarrow[\mathrm{Im}=0]{} k \otimes_k k \cong k$$

$$A^! = k[x^*]$$

Ex 3 : $k[x]/(x^2)$ $\quad \deg x = 2$

$\Rightarrow A_1 = 0$.

$$0 \longrightarrow 0 \otimes_k 0 \longrightarrow 0$$

$$A^! = k[0^*]/(0) = k$$

Ex 4 : $k(1 \xrightarrow{\alpha} 2)$

$A_0 = k\{\varepsilon_1\} \oplus k\{\varepsilon_2\}$

$A_1 = k\{\alpha\}$

$$\underbrace{A_1 \otimes_{A_0} A_1}_{= \, 0} = k\Big(\underbrace{\alpha \otimes_{A_0} \alpha}\Big)$$

$$\alpha \otimes_{A_0} \alpha = \alpha\varepsilon_1 \otimes \alpha = \alpha \otimes \varepsilon_1\alpha = \alpha \otimes 0 = 0$$

$$I_2 \hookrightarrow \underbrace{A_1 \otimes_{A_0} A_1}_{= \, 0} \to A_2 = 0$$

So, $I_2 = 0$.

$$A^! = A_0[A_1^*]$$
$$\cong k(1 \xleftarrow{\alpha^*} 2)$$

EXERCISE: Compute $(\mathrm{Sym}(V))^!$.

⑦ MAIN THEOREM

thm (R Martinez-Villa – M. Saorin, 2004, PJM)

Let A be as above.
Then,
$$\mathcal{LC}(P) \cong A^! - \mathrm{lf\,Mod}.$$

Why?

a) simples in $\mathcal{LC}(P)$:

$$\cdots\ 0 \longrightarrow Ae \longrightarrow 0 \longrightarrow \cdots$$

e-primitive
up to shift $[1]\langle -1\rangle$

b) Ext^1's between simples.

$$\cdots \longrightarrow Ae \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow Ae'\langle -1\rangle \cdots$$

$$\mathrm{Ext}^1_{\mathcal{LC}(P)}\left(Ae'[1]\langle -1\rangle^{\bullet}, Ae^{\bullet}\right) \cong \mathrm{Hom}_{A\text{-grmod}}\left(Ae'\langle -1\rangle, Ae\right)$$

$$\rightsquigarrow A_0[A_1^*] \longrightarrow \mathcal{B}$$

quadratic

alg. of $\mathcal{LC}(P)$