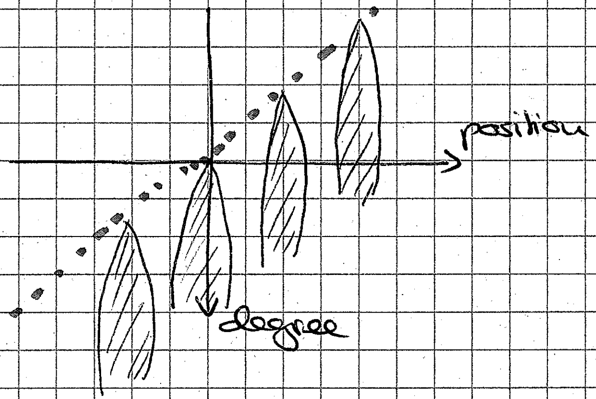
Quadratic Duality
(Volodymyr Mazorchuk Lecture II)

1 $\Bbbk = \overline{\Bbbk}$, $A = \bigoplus_{i \geq 0} A_i$, $\dim_{\Bbbk} A_i < \infty$, $A_0 = \bigoplus_{\text{finite}}$ copies of $\Bbbk$

$\mathcal{LC}(\mathcal{P})$ – linear compl. of A-proj.



Thm: [Martinez-Villa – Saorin]

$\mathcal{LC}(\mathcal{P}) \simeq A^!$-lf Mod

$A^!$: $I_2 \hookrightarrow A_1 \otimes_{A_0} A_1 \xrightarrow{\text{mult}} A_2$

$$A^! := A_0[A_1^*] / \mathrm{Im}(\text{mult}^*)$$

2 Objects (structural) in $\mathcal{LC}(\mathcal{P})$

2a simples: $A_0 \ni 1 = e_1 + \dots + e_n$

simple graded A-modules: $A_0 e_i =: L_i$, $i = 1, \dots, n$

ind. A-projectives: $A e_i = P_i$, $i = 1, \dots, n$

$P_i^\bullet$: $\dots 0 \to \underset{0}{\underline{P_i}} \to 0 \dots$

simple object in $\mathcal{LC}(\mathcal{P})$

$\mathbb{Z} \circlearrowright \mathcal{LC}(\mathcal{P})$ via $[1]\langle -1 \rangle$

shift of hom. position — grading shift

Simples in $\mathcal{LC}(\mathcal{P})$ are $P_i^\bullet[j]\langle -j \rangle$, $i = 1, \dots, n$, $j \in \mathbb{Z}$

2b) Injectives in $\mathcal{L}\mathcal{C}(P)$

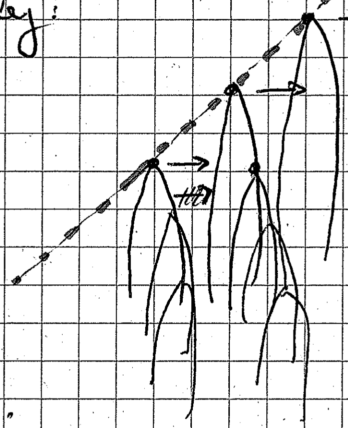
$i \in \{1, \dots, n\} \rightsquigarrow L_i \overset{f_x}{\rightsquigarrow} \mathcal{P}_i^\bullet$

min. proj. resol.

<u>Note:</u> $\mathcal{P}_i^\bullet \notin \mathcal{L}\mathcal{C}(P)$

Intuitively:

$\tilde{P}_i \to L_i \to$



<u>Lemma:</u> $\tilde{\mathcal{P}}_i^\bullet$ is a subcomplex of $\mathcal{P}_i^\bullet$.

$\forall j$ $\mathcal{P}_i^\bullet$ – gen. in deg. $-j$ or higher

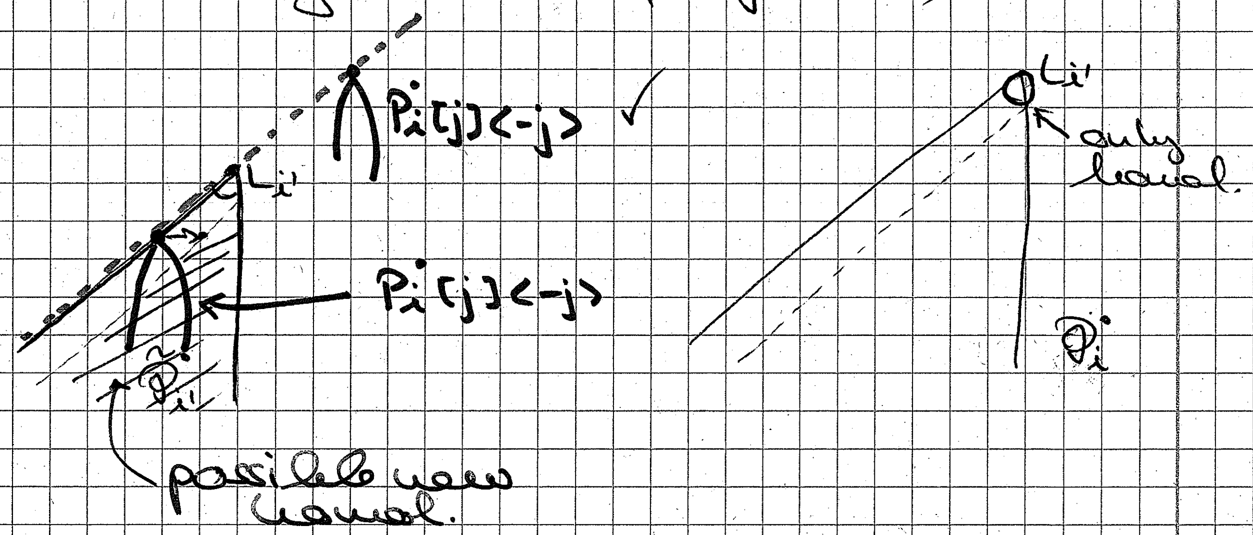
$\mathcal{P}_i^\bullet \supset \tilde{\mathcal{P}}_i^\bullet \leftarrow$ the diag. component of the resol.

<u>Prop:</u> Indec. injectives in $\mathcal{L}\mathcal{C}(P)$ are
$\tilde{\mathcal{P}}_i^\bullet[j]\langle -j\rangle$, $i = 1, \dots, n, j \in \mathbb{Z}$ $/\cong$

<u>Why?</u>

<u>Need:</u> $\mathrm{Ext}^1_{\mathcal{L}\mathcal{C}(P)}(P_i[j]\langle -j\rangle, \tilde{\mathcal{P}}_{i'}^\bullet[j']\langle -j'\rangle) = 0$.

(enough to show for $j' = 0$)



new basis elt: $v - v'$, $d(v - v') = 0$.

2c) Projectives in $\mathcal{LC}(P)$

A: $N := A^{\circledast} \otimes_A -$

$A^{\circledast}$ = restr. graded dual

$N: A\text{-proj} \xrightarrow{\sim} A\text{-inj}$

$\mathcal{LC}(P) \rightsquigarrow \mathcal{LC}(I)$

$\mathcal{LC}(I)$ = linear cplexes of A-injectives

$N: \mathcal{LC}(P) \to \mathcal{LC}(I)$ is an equiv.

For $\mathcal{LC}(I)$ indec. proj. are linear parts of inj. cores of A-simples.

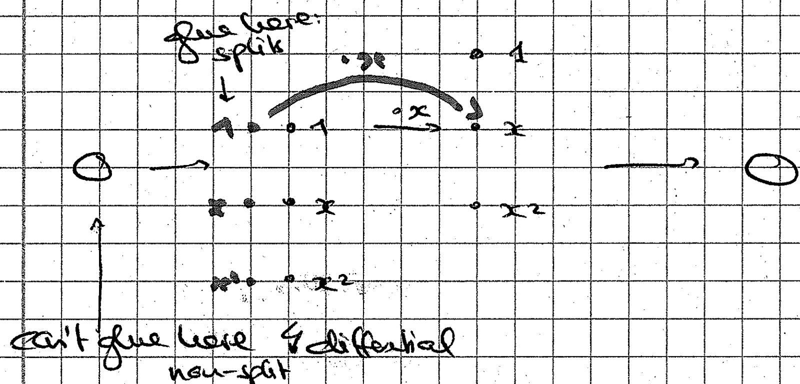
i.e. $L_i \rightsquigarrow I_i^{\bullet} \rightsquigarrow \tilde{I}_i^{\bullet}$ ← linear part (quotient)

$I_i^{\bullet}$ = inj. cores

Proj. in $\mathcal{LC}(P)$ are $N^{-1}\tilde{I}_i^{\bullet}[j]\langle -j \rangle$, $j \in \mathbb{Z}$, $i = 1, \dots, n$

Ex 1: $\mathbb{k}[x]$, $\deg x = 1$

$0 \to A[1]\langle -1 \rangle \xrightarrow{\cdot x} A \to 0$



Ex 2: $1 \rightleftarrows 2 \rightleftarrows 3$

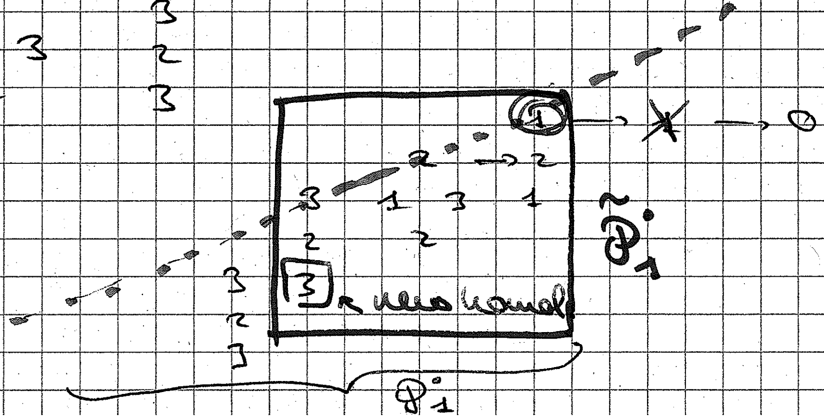
$\to \to = 0$

$\leftarrow \leftarrow = 0$

$\circlearrowright = \circlearrowleft$

proj.

$$\begin{matrix} 1 \\ 2 \\ 1 \end{matrix} \qquad \begin{matrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{matrix} \qquad \begin{matrix} 3 \\ 2 \\ 3 \end{matrix}$$



③ **Idea:**

$$\mathbb{P}^\circ := \bigoplus_{\substack{i=1,\dots,n \\ j\in\mathbb{Z}}} N^{-1}\tilde{I}^\circ_i[j]\langle -j\rangle$$

"progen" for $\mathcal{LC}(P)$

$\mathrm{Hom}(\mathbb{P}^\circ, -) : D^{\mathbb{Z}}(A) \longrightarrow D^{\mathbb{Z}}(A')$

Catch: $\mathbb{P}^\circ \notin \mathcal{LC}(P)$

④ Grading vs $\mathbb{Z}$-action

$\mathcal{C}$ - $\Bbbk$-linear small cat.

$\mathbb{Z} \curvearrowright \mathcal{C}$ strict action
free ($\mathrm{Stab}_{\mathbb{Z}}\lambda = \{0\}\ \forall \lambda \in \mathcal{C}$)

Quotient: $\mathcal{C}/\mathbb{Z}$ **objects:** $\mathbb{Z}\lambda,\ \lambda\in\mathcal{C}$

morphisms: $\mathcal{C}/_{\mathbb{Z}}(\mathbb{Z}\lambda,\mathbb{Z}\mu) := \bigoplus_{\substack{\lambda'\in\mathbb{Z}\lambda\\ \mu'\in\mathbb{Z}\mu}} \mathcal{C}(\lambda',\mu') \Big/ \underset{\forall i\in\mathbb{Z}}{(f - i\cdot f)}$

$$\cong \bigoplus_{\mu'\in\mathbb{Z}\mu} \mathcal{C}(\lambda,\mu')$$

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Chose a rep. $\hat{\lambda}\in\mathbb{Z}\lambda\ \forall\mathbb{Z}\lambda$.
This equips $\mathcal{C}/\mathbb{Z}$ with the str. of a $\mathbb{Z}$-graded category.

$$\mathcal{C}/_{\mathbb{Z}\,i}(\mathbb{Z}\lambda,\mathbb{Z}\mu) = \mathcal{C}(\hat{\lambda}, i\hat{\mu})$$

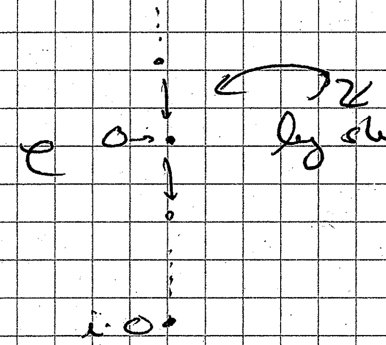
Conversely: $\mathcal{C}$ $\mathbb{Z}$-graded

$$\mathcal{C}^{\mathbb{Z}} := \mathcal{C}\times\mathbb{Z} \circlearrowleft \mathbb{Z}$$

$$\mathcal{C}^{\mathbb{Z}}((\lambda,i),(\mu,j)) := \mathcal{C}_{j-i}(\lambda,\mu).$$

$$\mathcal{C}^{\mathbb{Z}}/_{\mathbb{Z}} \cong \mathcal{C}.$$

Ex: $k[x]$, $\deg x = 1$



by shift by 1 $\qquad \mathcal{C}/\mathbb{Z}$: 1 object

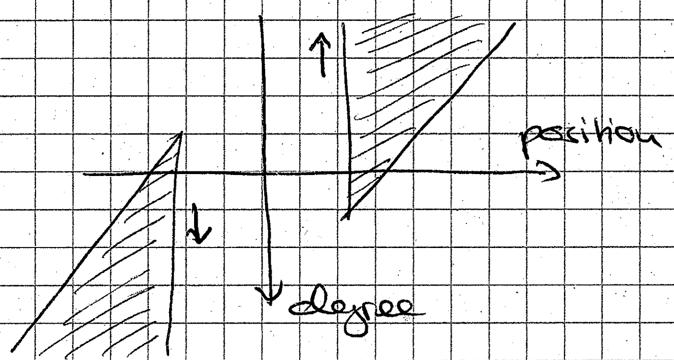
$\mathbb{P}^\bullet$ — the full subcategory in $D(A\text{-grMod})$ with objects $N^{-1}\tilde{I}_i^\bullet[j]\langle -j\rangle$ $\quad i=1,\dots,n$, $j\in\mathbb{Z}$.

$\mathbb{P}^\bullet \rightsquigarrow A \otimes A^!$ (shifts)

$\circlearrowleft$
$\mathbb{Z}$ via $[1]\langle -1\rangle$

Now we can define $\mathrm{Hom}_A(\mathbb{P}^\bullet, -)$ or $\mathbb{P}^\bullet \otimes_{A^!}$

5 Main theorem



Thm: (Quadratic duality)
The above gives rise to a pair
$$D^\downarrow(A\text{-fgMod}) \underset{K^!}{\overset{K}{\rightleftarrows}} D^\uparrow(A^!\text{-fgMod})$$
of functors s.t. a) $(K, K^!)$ an adj. pair.
b) $K(x^\bullet[i]\langle j\rangle) \cong (Kx^\bullet)[i+j]\langle -j\rangle$
$K^!(y^\bullet[i]\langle j\rangle) \cong (K^! y^\bullet)[i+j]\langle -j\rangle$

c) K sends simples to injectives

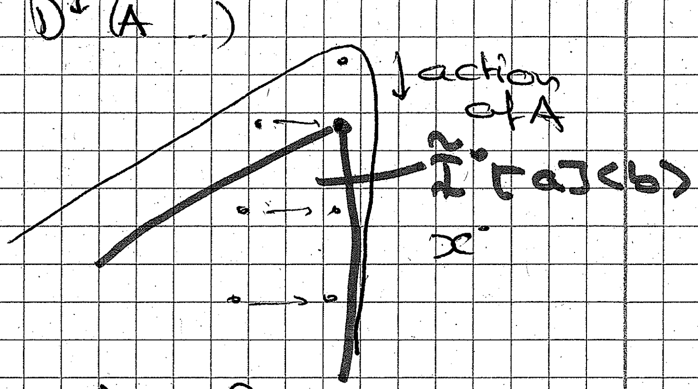
$K^!$ sends simples to projectives

d) K: proj $\longrightarrow$ lin. part of inj. res of the corresp. simple (provided that A is quadratic)

$K^!$: inj $\longrightarrow$ lin. part of proj. res. of simple

Intuitive feeling of what is going on.

1) $D(A)$ is "gener" by $L_i[j]\langle k\rangle$



choose basis for each basis elt take its inj. res & linear part of that