Mazorchuk Lecture 3: Koszul duality

§1 Recall Quadratic duality theorem

A – positively graded algebra

a) $\exists$ pair (K', K) of adjoint functors

$$D^{\downarrow}(A\text{-lf Mod}) \underset{K'}{\overset{K}{\rightleftarrows}} D^{\uparrow}(A^{!}\text{-lf Mod})$$

b) $K^{(\prime)}(-[i]\langle j\rangle) = K^{(\prime)}(-)[i+j]\langle -j\rangle$

c) K : simples $\to$ injectives

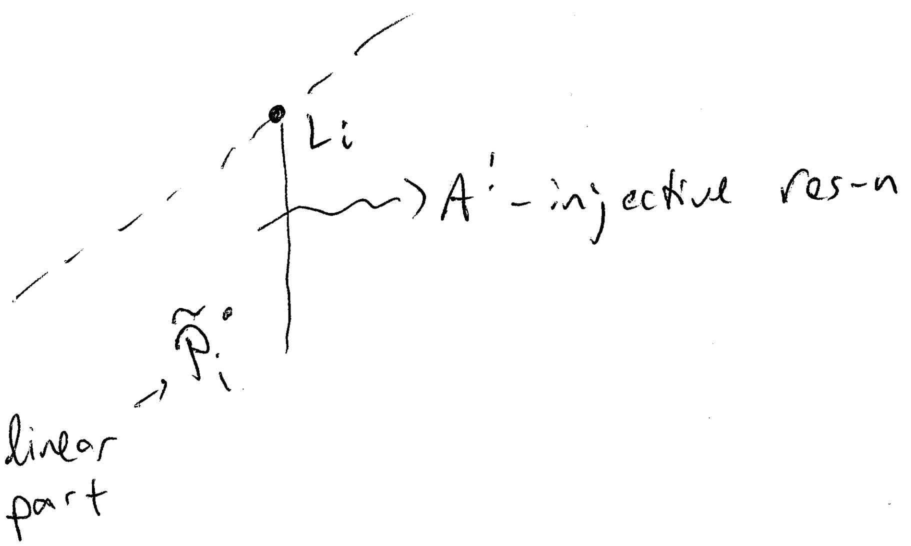
K' : simples $\to$ projectives

d) K': injectives $\to$ linear part of the projective res-n of a simple

K : proj $\to$ –"– injective core of a simple if A quadratic

$\mathbb{P}^{\bullet}$ – cat $N^{-1}\tilde{\mathcal{I}}_i^{\bullet}[j]\langle -j\rangle$ $\quad i = 1 \ldots n$, $j \in \mathbb{Z}$

$D^{\downarrow}(A\text{-lf Mod})$



<u>Example</u> $\Bbbk(1 \xrightarrow{\alpha} 2)$

projs: $P_1 \quad \begin{array}{c} 1 \\ \downarrow\alpha \\ 2 \end{array} \rightsquigarrow \begin{array}{c} 1 \\ 2 \end{array} \qquad P_2 \quad 2$

resolutions:

$\mathcal{P}^\circ_1 \qquad 0 \to 2 \to \begin{array}{c} 1 \\ 2 \end{array} \to 0$ (both already linear)

$\mathcal{P}^\circ_2 \qquad 0 \to 2 \to 0$

Homs between $\mathcal{P}^\circ_1$ and $\mathcal{P}^\circ_2$

$$\begin{tikzcd} 0 \arrow[r] & 2 \arrow[r] \arrow[d, dotted] & \begin{array}{c} 1 \\ 2 \end{array} \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & 2 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd} \quad = \quad \begin{array}{c} I^!_1 \\ \downarrow \\ I^!_2\langle 1 \rangle \end{array}$$

So $\mathcal{P}^\circ_1 = \tilde{\mathcal{P}}^\circ_1 = I^!_1$

$\mathcal{P}^\circ_2 = \tilde{\mathcal{P}}^\circ_2 = I^!_2$

$\rightsquigarrow \quad 1 \xleftarrow{\alpha^*} 2$

$$I^!_1 \begin{array}{c} 2 \\ 1 \end{array} \dashrightarrow 2 \; I^!_2$$

$\mathrm{Cone}\,(I^!_1 \to I^!_2\langle 1\rangle)$

$$\begin{tikzcd}[row sep=small] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[r] & 0 \\ 0 \arrow[r] & 2 \arrow[ur, "1"] \arrow[dr, "1"'] & \oplus & \\ & & 2 & \end{tikzcd} \quad \overset{\text{change basis}}{\rightsquigarrow} \quad 0 \to \begin{smallmatrix}1\\2\end{smallmatrix} \to 0$$

$$\begin{smallmatrix}1\\2\end{smallmatrix} = P_1$$

So $P^\bullet_1 = 0 \to P_1 \to 0$ simple obj in $\mathcal{LP}(P)$

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§2 Koszul algebras

<u>Def</u> A is Koszul if $\widetilde{P}^\bullet_i = P^\bullet_i \quad \forall i$.

Examples 1. $\Bbbk[x]$, $\Bbbk[x]/(x^2)$ $\deg x = 1$

2. $\Bbbk Q$ $\deg(\text{arrow}) = 1$

3. non-Koszul

$\Bbbk[x]$ $\deg x > 1$, $\Bbbk[x]/(x^n)$ $n > 2$.

§3 Koszul duality

<u>Theorem</u> TFAE

1. A is Koszul
2. (K', K) are inverse equivalences
3. K: projectives $\to$ simples + A quadratic
4. K': injectives $\to$ simples

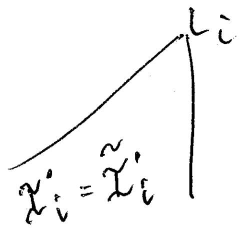
<u>Why is this true?</u>

vector space duality

1. A quadratic $\Rightarrow$ $K' = \mathbb{D} \circ K_{(A^!)} \circ \mathbb{D}$

$\Rightarrow$ $K'K \simeq \mathrm{Id}$

Each obj in $\mathcal{K}^{\downarrow}(A\text{-lfMod})$ is
"iterated" cone of simples + "linear pt."
Reduces to checking for simples.
On simples: $K'K(L_i)$



$\cap$

$D^{\uparrow}(A^!\text{-lfMod})$

$L_i \rightsquigarrow \tilde{\mathcal{L}}_i^{\circ}$
$\simeq \mathrm{Id}$

<u>Why "part 2"?</u>

$\rightsquigarrow$ parallels with Rickard's Thm.
$\mathbb{P}^{\circ} \rightsquigarrow$ "tilting complex"

$\mathrm{Hom}_D(\mathbb{P}^{\circ}, \mathbb{P}^{\circ}[j]) = 0$, $j \neq 0$, if A Koszul.

P° "generates" $D^\downarrow(A-\text{lf Mod})$

up to taking some limits

<u>Problem</u>: P° — not bounded in any way

<u>Good</u>: $D^\downarrow(A-\text{lf Mod})$ is "small"

Rickard's thm + some care $\Rightarrow$ $(1 \Rightarrow 2)$.

Examples.

1) $\circlearrowleft\bullet$ $\qquad \bullet \longrightarrow \bullet$

Path algebras are Koszul, loc fin

[BGS] $\Rightarrow$ Koszul duality thm

2) $\cdots \bullet \longrightarrow \bullet \longrightarrow \bullet \cdots$

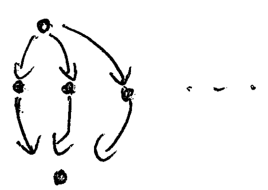
not loc finite

need to use [MOS]

3) 

path alg is Koszul

Koszul duality thm unknown here

4)  all squares commute

no Koszul duality

<u>Moral</u> Koszul alg $\not\Rightarrow$ Koszula duality

in general.

Remark 1) What happens if A_0 is not semisimple?

Look: [Dag Madsen. Adv. Math.]

2) $\mathcal{O}_0 \simeq A\text{-mod}$ A Koszul, $A \simeq A^!$

parabolic - singular Koszul duality.

$$D^b(\mathcal{O}_\lambda^{\mathbb{Z}}) \simeq D^b((\mathcal{O}_0^{\mathfrak{p}})^{\mathbb{Z}})$$

§4 Serre subcategories & centr subalg

A - fin dim alg / $\mathbb{k}$ $e \in A$ idempotent

$B(A,e) := eAe$ "centralizer subalg corresponding to e"

$C(A,e) := A/_{AeA}$

$B(A,e)\text{-mod} \hookrightarrow A\text{-mod} \hookleftarrow C(A,e)\text{-mod}$

Point: $eAe = \mathrm{End}_A(Ae)^{op}$ $(Ae \in A\text{-mod-}eAe)$

$$\mathrm{Hom}_A(Ae,-) : A\text{-mod} \rightleftarrows eAe\text{-mod}$$

$Ae \otimes_{eAe} -$

inverse equivalence btw eAe-mod and full subcategory in A-mod consisting of N:

$X_1 \to X_0 \to N \to 0$, $X_1, X_0 \in \mathrm{add}(Ae)$

$$A \xrightarrow{\pi} A/_{AeA}$$

$$\pi^* : A/_{AeA}\text{-mod} \to A\text{-mod}$$

π^* is an equiv between $A/_{AeA}$-mod and the Serre subcategory of A-mod gen by simples L s.t. $eL = 0$.

What does this have to do with Koszul duality?

§5 Koszul duality and §4

A - positively graded

$A_0 = k \oplus \cdots \oplus k \ni e$ idemp.

$e \rightsquigarrow eAe = B(A,e)$

$A \rightsquigarrow \mathcal{LC}(P)$

$$\mathcal{X}^\bullet : \quad \cdots \mathcal{X}^{i-1} \to \mathcal{X}^i \to \mathcal{X}^{i+1}$$

$\mathcal{X}^i \in \mathrm{add}({}_A A\langle i\rangle)$

Let's consider now $\mathcal{LC}(P_e) \subset \mathcal{LC}(P)$ defined by the condition $\mathcal{X}^i \in \mathrm{add}({}_A Ae\langle i\rangle)$

$$\begin{array}{c} \mathcal{LC}(P) \simeq A^!\text{-lfMod} \\ \cup \\ \mathcal{LC}(P_e) \longrightarrow \ ? \end{array}$$

ind. proj. A-mods $\rightsquigarrow$ simple objects in $\mathcal{LC}(P)$.

$\mathcal{LC}(P_e)$ – like Serre subcat gen by simple objects in Ae up to shifts
(like because not finite length)

<u>Corollary</u> A positively graded, $e \in A_0$ idemp.
Then $B(A,e)^! \simeq C(A^!, 1-e)$

Back to category $\mathcal{O}$: $\mathcal{O}_0 \simeq A$-mod

$$B(A,e) \subset A \simeq A^! \twoheadrightarrow C(A^!,e)$$

$B(A,e)$: singular block
$C(A^!,e)$: parabolic block.

So parabolic-singular duality can be derived from self-duality!

What is the advantage over [BGS] ?

Rmk. BGS relies on Koszul resolution

$$A \otimes_{\mathbb{k}} (A^!)^*$$

but A not always known explicitly.