Perverse sheaves on the flag variety – Part 1

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12.8.15

Perverse Sheaves $\longleftrightarrow$ category $\mathcal{O}$

$$\mathrm{Perv}_G(G/B) \cong \mathcal{O}_0$$
$$\text{\rotatebox{90}{$\cong$}} \qquad \text{\rotatebox{90}{$\cong$}}$$
$$A\text{-mod} \qquad \text{where } A \text{ is self-Koszul}$$

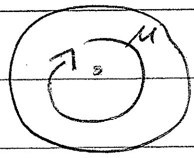
Perverse Sheaves (in general)

X complex algebraic variety

$\mathcal{F}$ sheaves of $\mathbb{C}$-vs (f.d.)

Def: A local system $\mathcal{L}$ is a locally constant sheaf
i.e. $\forall p \in X \ \exists U \ni p$ s.t. $\mathcal{L}|_U = \underline{\mathbb{C}}^k|_U$

Ex: $Loc(\mathbb{C}) =$ f.d. vs
$Loc(\mathbb{C}^*) = \{V \text{ vs} +$
$\mu : V \to V\}$



$Loc(X) = Rep(\pi_1(X), \mathbb{C})$

Def: $X = \coprod_{\lambda \in \Lambda} X_\lambda$ stratification if

1) $\forall \lambda \in \Lambda$: X_λ locally closed, smooth, connected
2) $\overline{X_\lambda} = \coprod_{\mu \in \Lambda_\lambda} X_\mu$ for some $\Lambda_\lambda \subseteq \Lambda$.

Ex: C curve (maybe singular)
$C = C_{smooth} \amalg \left(\coprod_{p \in Sing(X)} \{p\} \right)$

Def: $(X, \{X_\lambda\})$ stratified space.
$\mathcal{F}$ sheaf, then
$\mathcal{F}$ is said constructible if $\forall i_\lambda : X_\lambda \to X$
$i_\lambda^* \mathcal{F}$ is a local system

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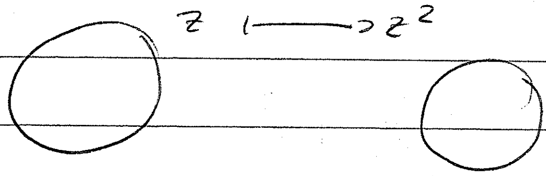
Ex: $\mathbb{C} \xrightarrow{f} \mathbb{C}$

$z \longmapsto z^2$



(with $\mathbb{C} = \mathbb{C}^* \sqcup \{0\}$)

$f_* \underline{\mathbb{C}}$ is constructible but not local system

$(f_* \underline{\mathbb{C}})_0 \cong \mathbb{C}$, $(f_* \underline{\mathbb{C}})_x \cong \mathbb{C}^2 \quad \forall x \neq 0.$

(X, Λ) stratified

$D^b_\Lambda(X)$ = derived category of sheaves whose cohomology sheaves are constructible wrt Λ

$\Downarrow$

$\ldots \to \mathcal{F}^1 \to \mathcal{F}^2 \to \ldots$ s.t. $H^i(\mathcal{F})$ constructible $\forall i$

Fact: Preserved under 6 functors (if you allow to change the stratification [..])

$$Sh(X) \subseteq D^b_\Lambda(X) \supseteq \underbrace{Perv_\Lambda(X)}_{\text{abelian subcat., "heart of a t-structure"}}$$

t-structure

$${}^pD^{\leq 0}_\Lambda = \{ \mathcal{F} \in D^b_\Lambda(X) \mid H^i(i_\lambda^* \mathcal{F}) = i_\lambda^* H^i(\mathcal{F}) = 0 \quad \forall i > -d_\lambda = \dim X_\lambda \}$$

$${}^pD^{\geq 0}_\Lambda = \{ \mathcal{F} \in D^b_\Lambda(X) \mid H^i(i_\lambda^! \mathcal{F}) = 0 \quad \forall i < d_\lambda \}$$

Thm: $({}^pD^{\leq 0}_\Lambda, {}^pD^{\geq 0}_\Lambda)$ is a t-structure

Cor: $Perv_\Lambda(X) := {}^pD^{\leq 0}_\Lambda \cap {}^pD^{\geq 0}_\Lambda$ is an abelian category.

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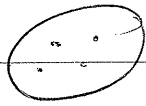
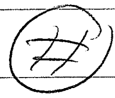
$\mathcal{F} \in \mathrm{Perv}_\Lambda(X)$
$d = \dim X$

$H^1(\mathcal{F}) = 0$

$H^0(\mathcal{F})$ supported on  X_λ with $\dim X_\lambda = 0$

$H^{-1}(\mathcal{F})$ —"—  —"— 1

$\vdots$

$H^{-d+1}(\mathcal{F})$  on divisors

$H^{-d}(\mathcal{F})$ on all of X

$H^{-d-1}(\mathcal{F}) = 0$ (this follows from ${}^p D_\Lambda^{\geq 0}$)

Simple objects in $\mathrm{Perv}_\Lambda(X)$

$\mathcal{L}$ irred local system on X_λ

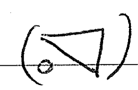
$IC(\overline{X}_\lambda, \mathcal{L})$ "minimal" extension of $\mathcal{L}$ to a perverse sheaf on $\overline{X_\lambda}$

$IC(\overline{X}_\lambda, \mathcal{L})|_{X_\lambda} \cong \mathcal{L}[d_\lambda]$, $d_\lambda = \dim X_\lambda$

$\mathcal{L}$ irred. $\Rightarrow$ $i_{\lambda *} IC(\overline{X}_\lambda, \mathcal{L})$ is simple in $\mathrm{Perv}_\Lambda(X)$

Thm: All simple perverse sheaves are of this form.
All the objects in $\mathrm{Perv}_\Lambda(X)$ have finite length.

Flag variety

G reductive group$/\mathbb{C}$ $\qquad G = Gl_n(\mathbb{C})$
$\cup$ $\qquad \cup$
B Borel subgroup $\qquad B =$ 
$\cup$ $\qquad \cup$
T max. torus $\qquad T =$ 

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Bruhat Decomposition, $W = N_G(T)/T$

$$G/B = \coprod_{w \in W} \underbrace{BwB/B}_{=: X_w} \cong \mathbb{C}^{\ell(w)} \quad \text{Schubert cell}$$

$\overline{X_w}$ Schubert variety

$\rightsquigarrow \Lambda = \{X_w\}_{w \in W}$ is a stratification of $X = G/B$

Since $X_w = \mathbb{C}^{\ell(w)}$ all the local systems on X_w are trivial.
All the simple objects in $\mathrm{Perv}_\Lambda(G/B)$ are

$$i_{w*} IC(\overline{X_w}, \underline{\mathbb{C}}) =: IC_w, \quad w \in W$$

$i_w : X_w \to X \overset{\text{have}}{\rightsquigarrow} i_{w!}\mathcal{F}, \ i_{w*}\mathcal{F}$ for a sheaf $\mathcal{F}$

Since i_w is an affine map it follows that
for $\mathcal{F} = \underline{\mathbb{C}}_w[d_w]$ $\quad d_w = \dim X_w = \ell(w)$

"shift"
$\mathcal{F}[1]$

$$\begin{array}{c} \to \mathcal{F}^{-1} \to \mathcal{F}^{0} \to \mathcal{F}^{1} \to \dots \\ \downarrow [1] \\ \mathcal{F}^{0} \to \mathcal{F}^{1} \to \mathcal{F}^{2} \end{array}$$

Prop: $i_{w!}\mathbb{C}_w[d_w]$, $i_{w*}\mathbb{C}_w[d_w]$ are perverse sheaves

$$i_{w!}\mathbb{C}_w[d_w] \twoheadrightarrow IC_w \hookrightarrow i_{w*}\mathbb{C}_w[d_w]$$

Verma module $\quad$ irred module $\quad$ dual Verma module

Thm (Borel picture) $\quad H^*(G/B) = S^\cdot(\mathfrak{h}^*) / S(\mathfrak{h}^*)^W_+ S(\mathfrak{h}^*) =: C$ (coinvariant algebra) where

$\mathfrak{h}^* = \mathrm{Lie}(T)^* \subset S^\cdot(\mathfrak{h}^*)$ has $\deg = 2$
$S^\cdot(\mathfrak{h}^*)^W_+$ invariants of degree > 0

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$\mathcal{F} \in D^b_\Lambda(X) =: D$

$\mathbb{IH}^\cdot(\mathcal{F}) = R\underbrace{\Gamma^\cdot}_{\text{global section}}(\mathcal{F}) = \mathrm{Hom}_D(\mathbb{C}_X, \mathcal{F})$

$\mathrm{Hom}_D(\mathbb{C}_X, \mathbb{C}_X)$

$\mathbb{IH}(\mathbb{C}_X) = H^\cdot(G/B) = C$

$\mathbb{IH} : D^b_\Lambda(X) \longrightarrow C\text{-Mod}$

<u>Thm</u> (Erweiterungssatz, proved by Sörgel)

$$\mathrm{Hom}_{D^b_\Lambda(X)}(IC_x, IC_y) \xrightarrow[\sim]{\mathbb{IH}} \mathrm{Hom}_{C\text{-Mod}}(\mathbb{IH}^\cdot(IC_x), \mathbb{IH}^\cdot(IC_y))$$

i.e.

$\mathbb{IH}$ is a fully faithful functor on the simple perverse sheaves and their shifts.

<u>Proof (only of injectivity)</u>

$\exists$ spectral sequence E_1^{pq} s.t.

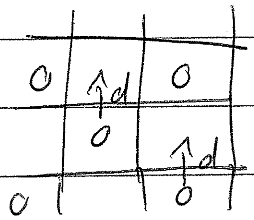
$$E_1^{pq} = \mathbb{IH}^{p+q}(i_p^! IC_x) \Longrightarrow E_\infty^{pq} = \mathbb{IH}^{p+q}(IC_x)$$

where $i_p : X_p = \coprod_{\ell(w) = \dim X - p} X_w \hookrightarrow X$

<u>Parity vanishing</u>: $\mathcal{H}^i(IC_x) = 0$ if $i + \ell(x)$ is odd

$\Longrightarrow E_1^{pq}$ vanishes as a "chess-board"

$\Longrightarrow$ the spectral sequence degenerates at 1



$\Longrightarrow$ all differentials are 0

$$\mathbb{IH}^n(IC_x) \cong \bigoplus_{p+q=n} E_1^{pq}$$

Take $f : IC_x \to IC_y[i]$ s.t. $\mathbb{IH}(f) = 0 : \mathbb{IH}^\cdot(IC_x) \longrightarrow \mathbb{IH}^\cdot(IC_y[i])$

$\Longrightarrow i_p^! f = 0 \quad \forall p \quad , \ i_p^! f : i_p^! IC_x \longrightarrow i_p^! IC_y[i]$

take $a_p : \overline{X_p} \hookrightarrow X$

$$X_p \xrightarrow[\text{open}]{u_p} \overline{X_p} \xleftarrow[\text{closed}]{j_p} \overline{X_{p+1}} \qquad X_p = \overline{X_p} \setminus \overline{X_{p+1}}$$

$$X_p \xrightarrow[\text{open}]{u_p} \overline{X_p} \xleftarrow[\text{closed}]{j_p} \overline{X_{p+1}}$$

$u_p^* = u_p^!$

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$$\begin{tikzcd}
u_{p*} u_p^* (a_p^! \mathbb{C}_x) = u_{p*} i_p^! \mathbb{C}_x \arrow[r] \arrow[d, "u_{p*} i_p^!(f)"] & a_p^! \mathbb{C}_x \arrow[r] \arrow[d, "a_p^! f"] & j_{p!} j_p^! (a_p^! \mathbb{C}_x) \arrow[r, "+1"] \arrow[d, "a_{p+1}^! f"] & {} \\
u_{p*} u_p^* i_p^! \mathbb{C}_y[i] \arrow[r] & a_p^! \mathbb{C}_y[i] \arrow[r] & j_{p!} j_p^! a_p^! \mathbb{C}_y[i] &
\end{tikzcd}$$

distinguished triangle

by induction $i_p^! f = 0 + a_{p+1}^! f = 0$

$\Rightarrow a_p^! f = 0 \Rightarrow a_0^! f = 0 \Rightarrow f = 0.$ $\square$

$\mathbb{H} \,\hat{=}\, K \longrightarrow C\text{-Mod}$ is fully faithful

$K = \langle \mathbb{C}_x \rangle_{\oplus, [1]}$

$\rightsquigarrow$ $\mathbb{H}$ gives an equivalence with essential image ("Sörgel modules").