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Sommerschule "Koszul Duality"
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Based on [Bezrukavnikov], 8 pages.

Let $k = \bar{k}$ field, X proj. variety, $X \overset{i}{\hookrightarrow} \mathbb{P}^n$.
Homogeneous coordinate ring $R = k[x_0, \dots, x_n]/I$.

$\mathcal{L} := i^*\mathcal{O}(1)$ line bundle, very ample
$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n}) \simeq R$

<u>Question</u>: Is $R(X, \mathcal{L})$ Koszul?

<u>Theorem</u> [Backelin] For any X $\exists$ ample line bundle $\mathcal{L}$ such that $R(X, \mathcal{L})$ is Koszul.
Idea (cf. Geoffrey Janssens's talk): Veronese embedding.

<u>Questions</u> ① Given fixed $\mathcal{L}$, is $R(X, \mathcal{L})$ Koszul?
② Do there exist X such that $R(X, \mathcal{L})$ is Koszul for all ample line bundles $\mathcal{L}$?

<u>Goal</u> Schubert varieties answer ② affirmatively.

$X_w^P \subset G/P$ for $w \in W$
<u>ex</u> $\mathbb{G}(k, n)$ Grassmannian
coordinate ring for $\mathbb{G}(2,4)$: $k[x_1, \dots, x_6]/(x_1x_6 - x_2x_5 + x_3x_4)$

$$\begin{tikzcd}
\underbrace{\text{Schubert varieties} \overset{3}{:} \text{Frobenius splitting}}_{\text{char } p} \arrow[r, Rightarrow, "2"] & H^1\text{-vanishing } \& \text{ reduced } \Delta\text{'s} \arrow[d, Rightarrow, "1"] \\
\text{char } p \arrow[d, squiggly, "4"] & \text{Koszul} \\
\text{char } 0 &
\end{tikzcd}$$

① Reduced Δ's and H^1-vanishing

$X^n = X \times X \times \dots \times X$, $R(X, \mathcal{L}) = \bigoplus H^0(X, \mathcal{L}^{\otimes n})$

$\mathcal{L}^{k_1, \dots, k_n} = p_1^* \mathcal{L}^{k_1} \otimes p_2^* \mathcal{L}^{k_2} \otimes \dots \otimes p_n^* \mathcal{L}^{k_n}$

Δ^n full diagonal, Δ^n_{ij} partial diagonal

$$\begin{tikzcd}
H^0(X, \mathcal{L}^{k_1}) \otimes \dots \otimes H^0(X, \mathcal{L}^{k_n}) \arrow[r, "m"] \arrow[d, phantom, "\cong" rotate=90] & H^0(X, \mathcal{L}^{k_1 + \dots + k_n}) \arrow[d, phantom, "\cong" rotate=90] \\
H^0(X^n, \mathcal{L}^{k_1, \dots, k_n}) \arrow[r] & H^0(\Delta^n, \mathcal{L}^{k_1, \dots, k_n})
\end{tikzcd}$$

__Prop 1__ If $\forall n$ $H^1(X^2, I_{\Delta^2} \otimes \mathcal{L}^{1,n}) = 0$, then $R(X, \mathcal{L})$ is generated in degree 1.

__Proof__

$$\begin{tikzcd}
R(X, \mathcal{L})_1 \otimes R(X, \mathcal{L})_n \arrow[r, two heads] \arrow[d, phantom, "\cong" rotate=90] & R(X, \mathcal{L})_{n+1} \arrow[d, phantom, "\cong" rotate=90] \\
H^0(X^2, \mathcal{L}^{1,n}) \arrow[r] & H^0(\Delta^2, \mathcal{L}^{1,n})
\end{tikzcd}$$

$$0 \to I_{\Delta^2} \to \mathcal{O}_{X^2} \to \mathcal{O}_{\Delta^2} \to 0$$

$$\rightsquigarrow H^0(X^2, \mathcal{L}^{1,n}) \to H^0(\Delta^2, \mathcal{L}^{1,n}) \to H^1(X^2, I_{\Delta^2} \otimes \mathcal{L}^{1,n}) \to \dots$$

$\square$

Lemma: A a connected graded alg. If $\forall k, m, n$:

① $m_{k,m,n} : A_k \otimes A_m \otimes A_n \longrightarrow A_{k+m+n}$ is surjective

② $\mathrm{Ker}(m_{k,m,n}) = \mathrm{Ker}(m_{k,m}) \otimes A_n + A_k \otimes \mathrm{Ker}(m_{m,n})$

then A is quadratic.

Prop 2 If $\forall k, m, n$:

① $H^1(X^3, I_{\Delta^3} \otimes \mathcal{L}^{k,m,n}) = 0$

② $H^1(X^3, I_{\Delta_{12} \cup \Delta_{23}} \otimes \mathcal{L}^{k,m,n}) = 0$

③ $\Delta_{12} \cap \Delta_{23}$ is reduced

then $R(X, \mathcal{L})$ is reduced.

Proof ① Same as before.

② $$0 \longrightarrow I_{\Delta_{12} \cup \Delta_{23}} \longrightarrow I_{\Delta_{12}} \oplus I_{\Delta_{23}} \longrightarrow \overbrace{I_{\Delta_{12} \cap \Delta_{23}}}^{I_{\Delta^3}} \longrightarrow 0$$

now apply $- \otimes \mathcal{L}^{k,m,n}$ and take the exact sequence:

$$H^0(X^3, I_{\Delta_{12}} \otimes \mathcal{L}^{k,m,n}) \oplus H^0(X^3, I_{\Delta_{23}} \otimes \mathcal{L}^{k,m,n}) \twoheadrightarrow H^0(X^3, I_{\Delta^3} \otimes \mathcal{L}^{-})$$

$\square$

Prop 3 If $\forall n \geq 2, i > 0$

① $H^1(X^n, I_{\cup \Delta_{i,i+1}} \otimes \mathcal{L}^{1,1,\dots,1,i}) = 0$

② $H^1(X^n, I_{\Delta_{n-1,n}} \otimes \mathcal{L}^{1,1,\dots,1,i}) = 0$

③ all $\cap$'s obtained from $\Delta_{12}, \dots, \Delta_{n-1,n}$ by finite $\cap$'s and $\cup$'s are reduced

Then $R(X, \mathcal{L})$ is Koszul.

② Frobenius splitting

Here $K = \bar{K}$ field of char. $p > 0$, and X separated scheme of finite type over K.

Frobenius $Fr : X \longrightarrow X$ induces $Fr^{\#} : \mathcal{O}_X \longrightarrow Fr_* \mathcal{O}_X$

Ex Fr is finite morphism

Def X is called Frobenius split if $\exists \phi : Fr_* \mathcal{O}_X \longrightarrow \mathcal{O}_X$ such that $\phi \circ Fr^{\#} = id_{\mathcal{O}_X}$.
Moreover, a closed subscheme $Y \subset X$ is compatibly split if $\phi(Fr_* \mathcal{I}_Y) \subset \mathcal{I}_Y$.

Remark If Y_1, Y_2 are compatibly split in X, then $Y_1 \cap Y_2$ and $Y_1 \cup Y_2$ are as well.

Prop 1 If X is split, then X is reduced. In particular, if Y_1, Y_2 are comp. split in X, then $Y_1 \cap Y_2$ is reduced.

Proof: $\exists U$, $f \in H^0(U, \mathcal{O}_U)$ such that $f^{p^n} = 0$.

$$f^{p^{n-1}} = (\phi \circ Fr^{\#})(f^{p^{n-1}}) = \phi(f^{p^n}) = 0 \qquad \square$$

Prop 2 If X is split, then for any ample line bundle $\mathcal{L}$ have $H^i(X, \mathcal{L}) = 0$, $i > 0$. If Y is comp. split in X, then $H^i(X, \mathcal{I}_Y \otimes \mathcal{L}) = 0$, $i > 0$.

Proof Let $\mathcal{L}$ ample, $\exists n$ such that $H^i(X, \mathcal{L}^{p^n}) = 0$, $i > 0$. Also $id \otimes \phi$ splits $id \otimes Fr^{\#} : \mathcal{L} \longrightarrow \mathcal{L} \otimes Fr_* \mathcal{O}_X$

$\Rightarrow$ $H^i(id \otimes Fr^{\#}) : H^i(X, \mathcal{L}) \hookrightarrow H^i(X, \mathcal{L} \otimes Fr_* \mathcal{O}_X)$

and $H^i(X, \mathcal{L} \otimes Fr_* \mathcal{O}_X) \simeq H^i(X, Fr_*(Fr^* \mathcal{L}))$
$\simeq H^i(X, Fr^* \mathcal{L}) \simeq H^i(X, \mathcal{L}^p)$

$\Rightarrow H^i(X, \mathcal{L}) = 0$ $\square$

Thm If $\forall n: \Delta_{12}, \ldots, \Delta_{n-1,n} \subset X^n$ are comp. split and $\mathcal{L}$ is ample, then $R(X, \mathcal{L})$ is ~~[illegible]~~ Koszul

③ Schubert varieties are Frobenius split

$G \supset P \supset B \supset T$, G semisimple alg. group
$W = N(T)/T$, G/P proj. variety, $X_w^P = \overline{BwP/P}$

Thm 1 G/P is split, compatibly splitting all X_w^P

Thm 2 $X_w^P \times \ldots \times X_w^P$ is split, compatibly splitting the partial diagonals.

How to prove X_w^P are split?
① construct res. of singularities $\theta_w : Z_w \longrightarrow X_w^P$
② find managable criterion for splitting of smooth projective varieties
③ prove Z_w are Frobenius split
④ use properties of θ_w to get splitting of X_w^P

For ②, $\mathrm{Hom}(Fr_* \mathcal{O}_X, \mathcal{O}_X) \simeq H^0(X, \omega_X^{1-p})$ (Serre duality), Ramonathan shows that a section of ω_X^{-1}
(*) with "nice" ~~division of Z~~ divisor of zeroes gives rise to a splitting.

For ③, compute explicitly ω_{Z_w} and find a section satisfying (*)

④ char $p \rightsquigarrow$ char 0

$X_w^P \longrightarrow \operatorname{Spec} \mathbb{Z}$

For any prime (p), any flat coherent sheaf $\mathcal{L}$ over $\operatorname{Spec} \mathbb{Z}$: If $H^i(X^P_{w(p)}, \mathcal{L}_{(p)}) = 0$, then $\exists$ open $V \ni (p)$ such that for any $(q) \in V$ have $H^i(X^P_{w(q)}, \mathcal{L}_{(q)}) = 0$.