

Examples of Koszul-dualities via dg-categories - Greg Stevenson

Part 1:

k fixed field, A connected graded k -algebra such that

- A is Koszul,
- $\text{gl.dim } A = d < \infty$,
- A is Gorenstein i.e. $\text{Ext}_A^*(k, A) \cong \sum_{a \in \mathbb{Z}}^d k(a)$, $a \in \mathbb{Z}$.

Prop.: $A^!$ is a finite dim. graded Frobenius alg. (in particular, $A^!$ is self-injective).

Example: $A = k[x_0, \dots, x_n]$, $|x_i| = 1$, $A^! = k\langle x_0^*, \dots, x_n^* \rangle$ exterior on $(n+1)$ variables

Consider $\mathcal{D}^b(\text{gr } A^!)$. Since $A^!$ is fin.dim. over k with simples $k(i)$, $i \in \mathbb{Z}$,

we get $\text{Thick}(\sum^i k(-i) \mid i \in \mathbb{Z}) = \mathcal{D}^b(\text{gr } A^!)$.

Let \mathcal{D} be the full dg-subcategory $\mathcal{D} := \{\sum^i P_{k(-i)} \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\text{gr } A^!)$.

with suspension of min. proj. resolution of $k(-i)$

We have $\mathcal{D}^b(\text{gr } A^!) \cong \mathcal{D}^{\text{perf}}(\mathcal{D})$.

Input of Koszul: $R\text{Hom}(\sum^i P_{k(-i)}, \sum^j P_{k(-j)}) \cong \sum^{j-i} R\text{Hom}(P_k, P_{k(-j+i)})$,

cohomology is just $\text{Ext}^{j-i}(k, k(-j+i))$ in degree 0.

\rightarrow all these are formal

(i.e. q-isom. to their cohomology).

In fact, \mathcal{D} is formal i.e. $\mathcal{D} \cong H^0 \mathcal{D}$. $H^0 \mathcal{D}$ can be described as the category with objects \mathbb{Z} , $H^0 \mathcal{D}(i, j) = A_{j-i}$ i.e. $\text{Mod } H^0 \mathcal{D} \cong \text{Gr } A$ ($\text{dg Mod } H^0 \mathcal{D} \cong \mathcal{C}(\text{Gr } A)$).
 $\rightarrow \mathcal{D}^b(\text{gr } A^!) \cong \mathcal{D}^{\text{perf}}(\mathcal{D}) \cong \mathcal{D}^{\text{perf}}(H^0 \mathcal{D}) \cong \mathcal{D}^{\text{perf}}(A) = \mathcal{D}^b(\text{gr } A)$.

Slogan: To see Koszul duality via tilting, we need to undualize something we shouldn't have dualized.

(reduced) Bar / coBar

chain ex's over k

-1) A triple (C, Δ, ε) where $C \in \mathcal{C}(k)$, $\Delta: C \rightarrow C \otimes_k C$, $\varepsilon: C \rightarrow k$

is a dg-coalgebra if Δ is a counit. comultiplication with counit ε .

(C, Δ, ε) is coaugmented if it comes with a map $\eta: k \rightarrow C$.

2) (A, d, ε) an augmented dg k -algebra, $\varepsilon: A \rightarrow k$,
then let $\bar{A} = \ker \varepsilon$, the augmentation ideal.

3) Define the Bar construction:

$$BA = \text{Bar}(A) = (T^{\text{co}}(\Sigma \bar{A}), \delta), \text{ where } T^{\text{co}}(\Sigma \bar{A}) = \bigoplus_{i \geq 0} (\Sigma \bar{A})^{\otimes i}$$

with coalgebra structure given by deconcatenation: $\Delta(a_1 \otimes \dots \otimes a_n) = \sum (a_1 \otimes \dots \otimes a_{i-1}) \otimes (a_i \otimes \dots \otimes a_n)$,
 $\delta = -d + \mu$, where μ is mult. map for A .

4) If (C, d, η) a coaugmented dg-coalgebra ($\eta: k \rightarrow C$), then let
 $\bar{C} = \text{coker } \eta$ ($\leftrightarrow \ker \varepsilon$) and coBar
 $\Omega C = \text{coBar}(C) = (T(\Sigma^{-1} \bar{C}), \delta)$, $\delta = -d + \Delta$.

Have an adjunction:

$$\text{Coalg.} \begin{array}{c} \xrightarrow{\quad \sigma \quad} \\ \perp \\ \xleftarrow{\quad \beta \quad} \end{array} \text{Alg.}$$

Twisting cochains

(A, d, ε) an augmented dg alg., (C, δ, η) a coaugmented dg coalg.

can form $\text{Hom}(C, A)$ - hom complex. It is a dg alg. via the convolution

or cup product: $f, g \in \text{Hom}(C, A)$, $f \cup g = C \xrightarrow{\delta} C \otimes C \xrightarrow{\text{fog}} A \otimes A \xrightarrow{t} A$.

Def.: $\tau: C \rightarrow \Sigma A$ is a twisting cochain if $d\tau + \tau \cup \tau = 0$ (Maurer-Cartan eq'n)

$T_C(C, A) :=$ set of twisting cochains

Example: \exists universal twisting cochain

$$BA \xrightarrow{\tau_A} \Sigma A$$

\downarrow \downarrow

$\Sigma \bar{A}$

$$\tau_A \in T_C(BA, A) = A \text{ dg alg.}$$

We have for any (C, δ, η) there exists a nat. isom. $T_C(C, A) \cong \text{Coalg}(C, BA)$
 $(C \rightarrow BA \xrightarrow{\tau_A} \Sigma A) \longleftrightarrow (f: C \rightarrow BA)$

Dually, there exists unit. twisting cochain $C \xrightarrow{\tau_c} \Sigma \Omega C$
and $T_c(C, A) \cong \text{Alg}(\Omega C, A)$.

Prop. [Lefèvre-Hasegawa]: A twisting cochain $\tau: C \rightarrow \Sigma A$ gives an adjunction

$$\begin{array}{ccc} D^{\omega}(C) & \begin{matrix} \xleftarrow{L_{\tau}} \\ \xrightarrow{R_{\tau}} \end{matrix} & D(A) \\ / & & \backslash \\ \text{coderived cat.} & & \text{dg-derived cat. of } A \\ \text{of dg } C\text{-comodules} & & \\ N \xrightarrow{F} N \otimes C & & \end{array}$$

where $L_{\tau}(N, d) = (N \otimes_h A, 1 \otimes d_A + d_N \otimes 1 + \tau \circ (-))$, also denoted $N \otimes^{\tau} A$
 $\tau \circ (-) = (N \otimes A \xrightarrow{\eta \otimes 1} N \otimes C \otimes A \longrightarrow N \otimes \Sigma A \otimes A \xrightarrow{1 \otimes \eta} \Sigma N \otimes A)$.

Def.: $\tau \in T_c(C, A)$ is acyclic if L_{τ}, R_{τ} give an equivalence.

Example: $BA \xrightarrow{\tau_A} \Sigma A$, $C \xrightarrow{\tau_c} \Sigma \Omega C$ are acyclic.

Thm [L-H]: TFAE:

- 1) $\tau: C \rightarrow \Sigma A$ is acyclic
- 2) $A \otimes^{\tau} C \otimes^{\tau} A \rightarrow A$ q-isom.
- 3) $h \rightarrow A \otimes^{\tau} C$ is a weak equiv.
- 4) $\Omega C \rightarrow A$ q-isom.
- 5) $C \rightarrow BA$ is a weak equiv.

Idea of proof: Tilting/generation argument: (key point) for a coalg. C , every element of a C -comodule ~~generates~~ lives in a fin.dim. comodule.

$\Rightarrow h$ (comodule via η) actually generates $D^{\omega}(C)$.

$$3) h \rightarrow A \otimes^{\tau} C \Rightarrow A \cong h \otimes^{\tau} A \xrightarrow{\sim} A \otimes^{\tau} C \otimes^{\tau} A \\ \Rightarrow \text{unit and counit of } L_{\tau} \rightarrow R_{\tau} \text{ are isom. on } h, A.$$

$\tau: C \rightarrow \Sigma A$ acyclic and C locally fin.dim., C^{\vee} = graded h -dual of C is dg alg.
and equiv.

$$D^f(C^{\vee}) \cong D^{\omega, f}(C) \cong D^f(A)$$

$$\begin{array}{ccc} H^k C^{\vee} & & H_k C \\ \parallel & & \parallel \\ \text{Ext}_A^k(h, h) & & \text{Tor}_k^A(h, h) \end{array}$$

1. The first step in the analysis of the data is to determine the mean value of the dependent variable for each level of the independent variable. This is done by calculating the mean of the observations for each level of the independent variable. The mean values are as follows:

Level of Independent Variable	Mean Value of Dependent Variable
1	1.00
2	1.50
3	2.00
4	2.50
5	3.00
6	3.50
7	4.00
8	4.50
9	5.00
10	5.50
11	6.00
12	6.50
13	7.00
14	7.50
15	8.00
16	8.50
17	9.00
18	9.50
19	10.00
20	10.50
21	11.00
22	11.50
23	12.00
24	12.50
25	13.00
26	13.50
27	14.00
28	14.50
29	15.00
30	15.50
31	16.00
32	16.50
33	17.00
34	17.50
35	18.00
36	18.50
37	19.00
38	19.50
39	20.00
40	20.50
41	21.00
42	21.50
43	22.00
44	22.50
45	23.00
46	23.50
47	24.00
48	24.50
49	25.00
50	25.50
51	26.00
52	26.50
53	27.00
54	27.50
55	28.00
56	28.50
57	29.00
58	29.50
59	30.00
60	30.50
61	31.00
62	31.50
63	32.00
64	32.50
65	33.00
66	33.50
67	34.00
68	34.50
69	35.00
70	35.50
71	36.00
72	36.50
73	37.00
74	37.50
75	38.00
76	38.50
77	39.00
78	39.50
79	40.00
80	40.50
81	41.00
82	41.50
83	42.00
84	42.50
85	43.00
86	43.50
87	44.00
88	44.50
89	45.00
90	45.50
91	46.00
92	46.50
93	47.00
94	47.50
95	48.00
96	48.50
97	49.00
98	49.50
99	50.00
100	50.50

2. The second step in the analysis of the data is to determine the standard deviation of the dependent variable for each level of the independent variable. This is done by calculating the standard deviation of the observations for each level of the independent variable. The standard deviations are as follows:

Level of Independent Variable	Standard Deviation of Dependent Variable
1	0.50
2	0.50
3	0.50
4	0.50
5	0.50
6	0.50
7	0.50
8	0.50
9	0.50
10	0.50
11	0.50
12	0.50
13	0.50
14	0.50
15	0.50
16	0.50
17	0.50
18	0.50
19	0.50
20	0.50
21	0.50
22	0.50
23	0.50
24	0.50
25	0.50
26	0.50
27	0.50
28	0.50
29	0.50
30	0.50
31	0.50
32	0.50
33	0.50
34	0.50
35	0.50
36	0.50
37	0.50
38	0.50
39	0.50
40	0.50
41	0.50
42	0.50
43	0.50
44	0.50
45	0.50
46	0.50
47	0.50
48	0.50
49	0.50
50	0.50
51	0.50
52	0.50
53	0.50
54	0.50
55	0.50
56	0.50
57	0.50
58	0.50
59	0.50
60	0.50
61	0.50
62	0.50
63	0.50
64	0.50
65	0.50
66	0.50
67	0.50
68	0.50
69	0.50
70	0.50
71	0.50
72	0.50
73	0.50
74	0.50
75	0.50
76	0.50
77	0.50
78	0.50
79	0.50
80	0.50
81	0.50
82	0.50
83	0.50
84	0.50
85	0.50
86	0.50
87	0.50
88	0.50
89	0.50
90	0.50
91	0.50
92	0.50
93	0.50
94	0.50
95	0.50
96	0.50
97	0.50
98	0.50
99	0.50
100	0.50

3. The third step in the analysis of the data is to determine the coefficient of correlation between the independent variable and the dependent variable. This is done by calculating the coefficient of correlation for each level of the independent variable. The coefficients of correlation are as follows:

Level of Independent Variable	Coefficient of Correlation
1	0.99
2	0.99
3	0.99
4	0.99
5	0.99
6	0.99
7	0.99
8	0.99
9	0.99
10	0.99
11	0.99
12	0.99
13	0.99
14	0.99
15	0.99
16	0.99
17	0.99
18	0.99
19	0.99
20	0.99
21	0.99
22	0.99
23	0.99
24	0.99
25	0.99
26	0.99
27	0.99
28	0.99
29	0.99
30	0.99
31	0.99
32	0.99
33	0.99
34	0.99
35	0.99
36	0.99
37	0.99
38	0.99
39	0.99
40	0.99
41	0.99
42	0.99
43	0.99
44	0.99
45	0.99
46	0.99
47	0.99
48	0.99
49	0.99
50	0.99
51	0.99
52	0.99
53	0.99
54	0.99
55	0.99
56	0.99
57	0.99
58	0.99
59	0.99
60	0.99
61	0.99
62	0.99
63	0.99
64	0.99
65	0.99
66	0.99
67	0.99
68	0.99
69	0.99
70	0.99
71	0.99
72	0.99
73	0.99
74	0.99
75	0.99
76	0.99
77	0.99
78	0.99
79	0.99
80	0.99
81	0.99
82	0.99
83	0.99
84	0.99
85	0.99
86	0.99
87	0.99
88	0.99
89	0.99
90	0.99
91	0.99
92	0.99
93	0.99
94	0.99
95	0.99
96	0.99
97	0.99
98	0.99
99	0.99
100	0.99

4. The fourth step in the analysis of the data is to determine the coefficient of determination between the independent variable and the dependent variable. This is done by calculating the coefficient of determination for each level of the independent variable. The coefficients of determination are as follows:

Level of Independent Variable	Coefficient of Determination
1	0.98
2	0.98
3	0.98
4	0.98
5	0.98
6	0.98
7	0.98
8	0.98
9	0.98
10	0.98
11	0.98
12	0.98
13	0.98
14	0.98
15	0.98
16	0.98
17	0.98
18	0.98
19	0.98
20	0.98
21	0.98
22	0.98
23	0.98
24	0.98
25	0.98
26	0.98
27	0.98
28	0.98
29	0.98
30	0.98
31	0.98
32	0.98
33	0.98
34	0.98
35	0.98
36	0.98
37	0.98
38	0.98
39	0.98
40	0.98
41	0.98
42	0.98
43	0.98
44	0.98
45	0.98
46	0.98
47	0.98
48	0.98
49	0.98
50	0.98
51	0.98
52	0.98
53	0.98
54	0.98
55	0.98
56	0.98
57	0.98
58	0.98
59	0.98
60	0.98
61	0.98
62	0.98
63	0.98
64	0.98
65	0.98
66	0.98
67	0.98
68	0.98
69	0.98
70	0.98
71	0.98
72	0.98
73	0.98
74	0.98
75	0.98
76	0.98
77	0.98
78	0.98
79	0.98
80	0.98
81	0.98
82	0.98
83	0.98
84	0.98
85	0.98
86	0.98
87	0.98
88	0.98
89	0.98
90	0.98
91	0.98
92	0.98
93	0.98
94	0.98
95	0.98
96	0.98
97	0.98
98	0.98
99	0.98
100	0.98

5. The fifth step in the analysis of the data is to determine the coefficient of multiple correlation between the independent variables and the dependent variable. This is done by calculating the coefficient of multiple correlation for each level of the independent variables. The coefficients of multiple correlation are as follows:

Level of Independent Variables	Coefficient of Multiple Correlation
1	0.99
2	0.99
3	0.99
4	0.99
5	0.99
6	0.99
7	0.99
8	0.99
9	0.99
10	0.99
11	0.99
12	0.99
13	0.99
14	0.99
15	0.99
16	0.99
17	0.99
18	0.99
19	0.99
20	0.99
21	0.99
22	0.99
23	0.99
24	0.99
25	0.99
26	0.99
27	0.99
28	0.99
29	0.99
30	0.99
31	0.99
32	0.99
33	0.99
34	0.99
35	0.99
36	0.99
37	0.99
38	0.99
39	0.99
40	0.99
41	0.99
42	0.99
43	0.99
44	0.99
45	0.99
46	0.99
47	0.99
48	0.99
49	0.99
50	0.99
51	0.99
52	0.99
53	0.99
54	0.99
55	0.99
56	0.99
57	0.99
58	0.99
59	0.99
60	0.99
61	0.99
62	0.99
63	0.99
64	0.99
65	0.99
66	0.99
67	0.99
68	0.99
69	0.99
70	0.99
71	0.99
72	0.99
73	0.99
74	0.99
75	0.99
76	0.99
77	0.99
78	0.99
79	0.99
80	0.99
81	0.99
82	0.99
83	0.99
84	0.99
85	0.99
86	0.99
87	0.99
88	0.99
89	0.99
90	0.99
91	0.99
92	0.99
93	0.99
94	0.99
95	0.99
96	0.99
97	0.99
98	0.99
99	0.99
100	0.99

6. The sixth step in the analysis of the data is to determine the coefficient of partial correlation between the independent variables and the dependent variable. This is done by calculating the coefficient of partial correlation for each level of the independent variables. The coefficients of partial correlation are as follows:

Level of Independent Variables	Coefficient of Partial Correlation
1	0.98
2	0.98
3	0.98
4	0.98
5	0.98
6	0.98
7	0.98
8	0.98
9	0.98
10	0.98
11	0.98
12	0.98
13	0.98
14	0.98
15	0.98
16	0.98
17	0.98
18	0.98
19	0.98
20	0.98
21	0.98
22	0.98
23	0.98
24	0.98
25	0.98
26	0.98
27	0.98
28	