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Examples of Koszul-dualities via dg-categories - Greg Stevenson

Part 1:

k fixed field, A connected graded k-algebra such that

- A is Koszul,
- gl.dim $A = d < \infty$,
- A is Gorenstein i.e. $\underline{Ext}^*_A(k, A) \cong \Sigma^d k(a)$, $a \in \mathbb{Z}$.

cohom. shift — grading shift

<u>Prop.:</u> $A^!$ is a finite dim. graded Frobenius alg. (in particular, $A^!$ is self-injective).

<u>Example:</u> $A = k[x_0, \dots, x_n]$, $|x_i| = 1$, $A^! = k\langle x_0^*, \dots, x_n^*\rangle$ exterior on $(n+1)$ variables

Consider $D^b(\mathrm{gr}\, A^!)$. Since $A^!$ is fin. dim. over k with simples $k(i)$, $i \in \mathbb{Z}$, we get

$$\mathrm{Thick}(\Sigma^i k(-i) \mid i \in \mathbb{Z}) = D^b(\mathrm{gr}\, A^!).$$

Let $\mathcal{D}$ be the full dg-subcategory $\mathcal{D} := \{\Sigma^i P_{k(-i)} \mid i \in \mathbb{Z}\} \subseteq D^b(\mathrm{gr}\, A^!)$.

ith suspension of min. proj. resolution of $k(-i)$

We have $D^b(\mathrm{gr}\, A^!) \cong D^{perf}(\mathcal{D})$.

Input of Koszul: $\quad \mathrm{RHom}(\Sigma^i P_{k(-i)}, \Sigma^j P_{k(-j)}) \cong \Sigma^{j-i}\, \mathrm{RHom}(P_k, P_{k(-j+i)})$,

cohomology is just $\mathrm{Ext}^{j-i}(k, k(-j+i))$ in degree 0.

$\Rightarrow$ all these are <u>formal</u>
(i.e. q-isom. to their cohomology).

In fact, $\mathcal{D}$ is formal i.e. $\mathcal{D} \cong H^0\mathcal{D}$. $H^0\mathcal{D}$ can be described as the category with objects $\mathbb{Z}$, $H^0\mathcal{D}(i,j) = A_{j-i}$ i.e. $\mathrm{Mod}\, H^0\mathcal{D} \cong \mathrm{Gr}\, A$ ($\mathrm{dg\,Mod}\, H^0\mathcal{D} \simeq \mathcal{C}(\mathrm{Gr}\, A)$).

$\Rightarrow D^b(\mathrm{gr}\, A^!) \cong D^{perf}(\mathcal{D}) \cong D^{perf}(H^0\mathcal{D}) \cong D^{perf}(A) = D^b(\mathrm{gr}\, A)$.

<u>Slogan:</u> To see Koszul duality via tilting, we need to undualize something we shouldn't have dualized.

(reduced) Bar / co Bar

-1) A triple (C, Δ, ε) where $C \in \mathcal{C}(k)$ (chain cx's over k), $\Delta : C \to C \otimes_k C$, $\varepsilon : C \to k$ is a dg-coalgebra if Δ is a coassoc. comultiplication with counit ε.
(C, Δ, ε) is coaugmented if it comes with a map $\eta : k \to C$.

2) (A, d, ε) an augmented dg k-algebra, $\varepsilon : A \to k$, then let $\bar{A} = \ker \varepsilon$, the augmentation ideal.

3) Define the Bar construction:

$$BA = \mathrm{Bar}(A) = (T^{co}(\Sigma \bar{A}), \delta), \quad \text{where } T^{co}(\Sigma \bar{A}) = \bigoplus_{i \geq 0} (\Sigma \bar{A})^{\otimes i}$$

with coalgebra structure given by deconcatenation: $\Delta(a_1 \otimes \dots \otimes a_n) = \sum (a_1 \otimes \dots \otimes a_{i-1}) \otimes (a_i \otimes \dots \otimes a_n)$,
$\delta = -d + \mu_{\sqcup}$, where μ is mult. map for A.

4) If (C, d, η) a coaugmented dg-coalgebra $(\eta : k \to C)$, then let $\bar{C} = \mathrm{coker}(\eta)$ $(\leftrightarrow \ker \varepsilon)$ and coBar

$$\Omega C = \mathrm{coBar}(C) = (T(\Sigma^{-1} \bar{C}), \delta), \quad \delta = -d + \Delta.$$

Have an adjunction:

$$\begin{tikzcd} \mathrm{Coalg.} \arrow[r, shift left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathrm{Alg.} \arrow[l, shift left, "B"] \end{tikzcd}$$

Twisting cochains

(A, d, ε) an augmented dg alg., (C, δ, η) a coaugmented dg coalg.

Can form $\mathrm{Hom}(C, A)$ - hom complex. It is a dg alg. via the convolution or cup product: $f, g \in \mathrm{Hom}(C, A)$, $f \cup g = C \xrightarrow{\delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A$.

Def.: $\tau : C \to \Sigma A$ is a twisting cochain if $d\tau + \tau \cup \tau = 0$ (Maurer-Cartan eq'n)
$Tc(C, A) :=$ set of twisting cochains

Example: $\exists$ universal twisting cochain

$$\begin{tikzcd} BA \arrow[rr, "\tau_A"] \arrow[dr] & & \Sigma A, \\ & \Sigma \bar{A} \arrow[ur] & \end{tikzcd}$$

$\tau_A \in Tc(BA, A) = A$ dg alg.

We have for any (C, δ, η) there exists a nat. isom. $Tc(C, A) \cong \mathrm{Coalg}(C, BA)$
$(C \to BA \xrightarrow{\tau_A} \Sigma A) \leftarrow\!\shortmid (f : C \to BA)$

Dually, there exists univ. twisting cochain $C \xrightarrow{\tau_C} \Sigma\Omega C$

and
$$Tc(C,A) \cong Alg(\Omega C, A).$$

Prop. [Lefèvre-Hasegawa]: A twisting cochain $\tau : C \to \Sigma A$ gives an adjunction

$$D^{co}(C) \underset{R_\tau}{\overset{L_\tau}{\rightleftarrows}} D(A)$$

coderived cat. of dg C-comodules $N \xrightarrow{\rho} N \otimes C$

dg-derived cat. of A

where $L_\tau((N,d)) = (N \otimes_k A,\ 1 \otimes d_A + d_N \otimes 1 + \tau \cap (-))$, also denoted $N \otimes^\tau A$

$$\tau \cap (-) = \left(N \otimes A \xrightarrow{\rho \otimes 1} N \otimes C \otimes A \longrightarrow N \otimes \Sigma A \otimes A \xrightarrow{1 \otimes \mu} \Sigma N \otimes A\right).$$

Def.: $\tau \in Tc(C,A)$ is acyclic if L_τ, R_τ give an equivalence.

Example: $BA \xrightarrow{\tau_A} \Sigma A$, $C \xrightarrow{\tau_C} \Sigma\Omega C$ are acyclic.

Thm [L-H]: TFAE:

1) $\tau : C \to \Sigma A$ is acyclic
2) $A \otimes^\tau C \otimes^\tau A \to A$ q-isom.
3) $k \to A \otimes^\tau C$ is a weak equiv.
4) $\Omega C \to A$ q-isom.
5) $C \to BA$ is a weak equiv.

Idea of proof: Tilting / generation argument: (key point) for a coalg. C, every element of a C-comodule lives in a fin. dim. comodule.

$\Rightarrow$ k (comodule via η) actually generates $D^{co}(C)$.

3) $k \to A \otimes^\tau C \Rightarrow A \cong k \otimes^\tau A \xrightarrow{\sim} A \otimes^\tau C \otimes^\tau A$

$\Longrightarrow$ unit and counit of $L_\tau \dashv R_\tau$ are isom. on k, A.

$\tau : C \to \Sigma A$ acyclic and C locally fin. dim., $C^\vee$ = graded k-dual of C is dg alg.

and equiv.
$$D^f(C^\vee) \cong D^{co,f}(C) \cong D^f(A)$$

$$\begin{array}{cc} H^* C^\vee & H_* C \\ \| & \| \\ Ext^*_A(k,k) & Tor^A_*(k,k) \end{array}$$