Generators for triangulated categories

⓪ Motivation

<u>Thm</u> : If $\mathcal{A}$ is an abelian category admitting all set indexed coproducts and P is a compact projective generator of $\mathcal{A}$, then the functor

$$\mathrm{Hom}(P,-) : \mathcal{A} \xrightarrow{\sim} \mathrm{Mod}\ \mathrm{End}\ P$$

Thm [Keller] : Suppose $\mathcal{T}$ is triangulated algebraic. If $\mathcal{T}$ admits all set-indexed coproducts and T is a compact generator for $\mathcal{T}$, there is an equivalence (of triangulated categories)

$$\mathcal{T} \simeq D(R\mathrm{Hom}(T,T))$$

① Generators and dimensions

<u>Def</u> : a) $\mathcal{T}$ Δ-cat. $\mathcal{I}_1, \mathcal{I}_2$ two strictly full subcat. (closed under isomorphism)

$\mathcal{I}_1 * \mathcal{I}_2$: objects M occur in a triangle

$$\begin{array}{ccccccc} M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & \Sigma M_1 \\ \cap & & & & \cap & & \\ \mathcal{I}_1 & & & & \mathcal{I}_2 & & \end{array}$$

b) Let $\mathcal{E}$ be a set of objects in $\mathcal{T}$. Denote by $\langle \mathcal{E} \rangle$ the smallest strictly full subcat. of $\mathcal{T}$ containing $\mathcal{E}$ and closed under finite $\oplus$, direct summands, shifts.

c) If moreover I_1, I_2 are closed under finite $\oplus$, define

$$I_1 \diamond I_2 := \langle I_1 * I_2 \rangle$$

Given $\mathcal{E}$ as in b):

$$\langle \mathcal{E} \rangle_0 := 0, \quad \langle \mathcal{E} \rangle_i = \langle \mathcal{E} \rangle_{i-1} \diamond \langle \mathcal{E} \rangle$$

$$\langle \mathcal{E} \rangle_\infty = \bigcup_{i \geq 0} \langle \mathcal{E} \rangle_i =: \text{thick}(\mathcal{E})$$

Def. 1) $\mathcal{E}$ generates $\mathcal{T}$ if given $C \in \mathcal{T}$ with
$\mathrm{Hom}_{\mathcal{T}}(D[i], C) = 0 \quad \forall D \in \mathcal{E}, i \in \mathbb{Z}$
$\Rightarrow C = 0$

2) $\mathcal{E}$ classically generates $\mathcal{T}$ if $\text{thick}(\mathcal{E}) = \mathcal{T}$

3) $\mathcal{E}$ strongly generates $\mathcal{T}$ if $\mathcal{T} = \langle \mathcal{E} \rangle_k$ for some $k \in \mathbb{N}$.

add "finitely" if $\mathcal{E}$ has one object.

Remark: If $\mathcal{T}$ is strongly generated, then all classical generators are strong.

Def. The dimension of $\mathcal{T}$ is the minimal integer $d \geq 0$ s.t. there is $M \in \mathcal{T}$ with

$$\mathcal{T} = \langle M \rangle_{d+1}$$

(∞ if no such d).

Examples. 1) If R is a ring, then R is a classical generator for $D_{perf}(R)$ (bounded complexes of f.g. proj)

$R \in D_{perf}(R) \subset D(R)$ is a strictly full triangulated subcat. closed under direct summands

$\Longrightarrow \langle R \rangle_\infty \subset D_{perf}(R)$.

Conversely, if $M^* \in D_{perf}(R)$ is represented by

$$0 \to M^a \to M^{a+1} \to \cdots \to M^b \to 0$$

then $M^* \in \langle R \rangle_{b-a+1}$, by induction on $b-a$

- $b-a=0$: $0 \to M^a \to 0 \in \langle R \rangle$ ✓

In general, we have triangle in $D(R)$

$$M^b[-b] \longrightarrow M^* \longrightarrow \sigma_{b-1} M^* \longrightarrow M^b[-b+1]$$

By induction hypothesis, $\sigma_{b-1} M^* \in \langle R \rangle_{b-a}$

and $M^b[-b] \in \langle R \rangle \Longrightarrow M^* \in \langle R \rangle_{b-a+1}$.

2) (Beilinson) Let $X = \mathbb{P}^n$. The object

$$G = \mathcal{O}(-n+1) \oplus \dots \oplus \mathcal{O}$$

is a strong generator for $D^b(\mathrm{coh}\, X)$.
In fact, $\dim D^b(\mathrm{coh}\, X) = n$.
Let $\Delta \subset X \times X$ denote the diagonal.
$p_1, p_2 : X \times X \longrightarrow X$ the projections
For sheaves $\mathcal{F}, \mathcal{G}$, define

$$\mathcal{F} \boxtimes \mathcal{G} := p_1^* \mathcal{F} \otimes_{X \times X} p_2^* \mathcal{G}$$

There is a Koszul-type resolutions

$$0 \to \mathcal{O}(-n) \boxtimes \Omega^n(n) \to \dots \to \mathcal{O}(-1) \boxtimes \Omega^1(1) \to \mathcal{O}_{X \times X} \to \mathcal{O}_\Delta \to 0$$

$\Rightarrow \mathcal{O}_\Delta$ can be generated by

$$\{\mathcal{O}(-n) \boxtimes \Omega^n(n), \mathcal{O}(-n+1) \boxtimes \Omega^{n-1}(n-1), \dots, \mathcal{O} \boxtimes \mathcal{O}\}$$

Now for every $A \in D^b(\mathrm{coh}\, X)$, define

$$A \cong \Phi^{\mathcal{O}_\Delta}(A) := Rp_{2*}(Lp_1^* A \overset{L}{\otimes} \mathcal{O}_\Delta)$$

projection formula: $RF_*(\mathcal{E} \overset{L}{\otimes} LF^* \mathcal{F})$
$\cong RF_* \mathcal{E} \overset{L}{\otimes} \mathcal{F}$

is generated by

$$\{\Phi^{\mathcal{O}(-n)\boxtimes\Omega^n(n)}(A), \dots, \Phi^{\mathcal{O}\boxtimes\mathcal{O}}(A)\}$$

$\Rightarrow$ It suffices to show that $\forall i$: $\Phi^{\mathcal{O}(-i)\boxtimes\Omega^i(i)}(A)$ is in $\langle \mathcal{O}(-i)\rangle$

This also shows that $D^b(\mathrm{coh}\, X) = \langle \mathcal{O}(-n+1)\oplus\dots\oplus\mathcal{O}\rangle_{n+1}$

$\Rightarrow \dim D^b(\mathrm{coh}\, X) \leq n$.

<u>Thm</u> (Rouquier) If X is reduced, separated of finite type, we have

$$\dim (D^b(\mathrm{coh}\, X)) \geq \dim X$$

$\Rightarrow \dim D^b(\mathrm{coh}\, \mathbb{P}^n) = n$.

<u>Some results</u> [Rouquier]

1) If $\mathcal{T}'$ dense, full subcat of $\mathcal{T} \Rightarrow \dim \mathcal{T}' = \dim \mathcal{T}$

2) $F : \mathcal{T} \longrightarrow \mathcal{T}'$ triangulated with dense image.
If $\mathcal{T} = \langle \mathcal{E}\rangle_d$ then $\mathcal{T}' = \langle F(\mathcal{E})\rangle_d$.

then $\dim \mathcal{T}' \leq \dim \mathcal{T}$.

3) $\mathcal{T} = \mathcal{T}_1 \diamond \mathcal{T}_2 \Rightarrow \dim \mathcal{T} \leq 1 + \dim \mathcal{T}_1 + \dim \mathcal{T}_2$.

② Keller's theorem

Def. A Frobenius category is an exact category (has a class of adm. s.e.s) which has enough projectives and injectives and the class of proj. coincides with the class of inj.

The stable category $\underline{F}$ is obtained from Frobenius category F by quotienting by the morphisms factoring through a proj-inj.

Remark : $\underline{F}$ has a structure of Δ-cat.
The suspension functor Σ of $\underline{F}$ is obtained by choosing a s-e-s

$$0 \to L \to \underset{\substack{\cap \\ \text{inj-proj}}}{I} \to \Sigma(L) \to 0 \quad \forall L$$

Examples : $\mathcal{A}$ is an abelian cat. with enough injectives, then $D^b(\mathcal{A})$ is algebraic
- The derived category of a dg-cat is algebraic.

Thm [Keller] let $\mathcal{F}$ be a k-linear Frobenius cat. with arbitrary ~~direct~~ sums. Suppose that $\underline{\mathcal{F}}$ admits a set of compact classical generators $\mathcal{X}$. Then there is a DG-cat $\mathcal{A}$ and an equivalence $G : \underline{\mathcal{F}} \to D\mathcal{A}$ giving rise to an equivalence between the full subcat. formed by $\mathcal{X}$ and the one formed by the free modules $\mathcal{A}(-, A)$ for each $A \in \mathcal{A}$.

set of generators for $\mathcal{D}\mathcal{A}$

Constructions of $\mathcal{G}$ and $\mathcal{A}$

1) Let $\widetilde{\mathcal{F}}$ be the cat of acyclic complexes

$$P = (\cdots \to P^n \xrightarrow{d} P^{n-1} \longrightarrow \cdots) \quad n \in \mathbb{Z}$$

with proj. components $P^k \in \mathcal{F}$.

$\Rightarrow$ $\widetilde{\mathcal{F}}$ has a structure of Frobenius cat.
The proj./inj. are given by the null homotopic complexes in $\widetilde{\mathcal{F}}$.
The functor $P \mapsto Z^0 P$ induces an equivalence

$$G_1 : \underline{\widetilde{\mathcal{F}}} \xrightarrow{\sim} \underline{\mathcal{F}}$$

2) For each $X \in \mathcal{X}$, choose $\widetilde{X} \in \widetilde{\mathcal{F}}$ with $Z^0 \widetilde{X} = X$.
Let $\mathcal{A}$ be the dg-cat whose objects are the $\widetilde{X}$ and whose morphisms

$$\mathcal{A}(\widetilde{X}, \widetilde{Y}) := \mathcal{H}om(\widetilde{X}, \widetilde{Y})$$

with components

$$\prod_{p \in \mathbb{Z}} \widetilde{\mathcal{F}}(\widetilde{X}^p, \widetilde{Y}^{n+p}), \quad n \in \mathbb{Z}$$

and the differentials are given by

$$d(f^p) = d \circ f^p - (-1)^n f^{p+1} \circ d.$$

Note: $\tilde{\mathcal{F}}(\tilde{X}, \tilde{Y}[n]) \xrightarrow{\sim} H^n \mathcal{A}(\tilde{X}, \tilde{Y})$

The composition of exact functors

$$\tilde{\mathcal{F}} \longrightarrow C\mathcal{A} \xrightarrow{loc} D\mathcal{A}$$

$$P \longmapsto (\tilde{X} \mapsto \mathrm{Hom}(\tilde{X}, P)) \in C\mathcal{A}$$

vanishes on projectives

Get $G_2 : \underline{\tilde{\mathcal{F}}} \longrightarrow D\mathcal{A}$. It is an equivalence.

③ Special case / Applications.

a) If $\mathcal{X}$ has only one object T, then $\mathcal{A}$ is the dg-endo algebra of T.

b) If A is a finite dimensional algebra over k. Let $A^* = \mathrm{Hom}_k(A, k)$. It is an A-bimodule.

Define B the graded algebra with $B^0 \simeq A$, $B^1 = A^*$, $B^p = 0$ for $p \neq 0, 1$

Consider $\mathcal{F} = \mathrm{Grmod}\, B$. $\mathcal{F}$ is Frobenius

Consider in $\mathcal{F}$ the B-module T given by A considered as a graded B-module concentrated in degree 0.

$\Longrightarrow$ T is a compact generators. Moreover,

$$\mathrm{Hom}_{\underline{\mathcal{F}}}(T, T[n]) = 0 \quad \forall n \neq 0$$

$$\mathrm{Hom}_{\underline{\mathcal{F}}}(T, T) \simeq A.$$

$\Rightarrow \mathrm{RHom}(T, T) \cong A$.

Thm [Happel] If A is f.d. then there is Δ-equivalence

$$\underline{\mathrm{Grmod}\ B} \simeq D(A)$$

3) If T is a tilting complex in $\underline{\mathcal{F}} = D(B)$, B k-alg.

$$D(B) \simeq D(\mathrm{End}(T, T)).$$