

14.08.15

Ivan Yudin: Koszul duality of strict polynomial functors

H. Krause "Koszul, Ringel and Serre duality for strict polynomial functors"

There is a cat.  $SP_d$  of s-pol fun. of degree  $d$ .

$$R\text{-mod} \ni V \mapsto S^d V \in R\text{-mod} \\ V \mapsto \Lambda^d V \quad \leftarrow \text{comm. ring}$$

more generally:  $V \mapsto S^{\lambda} V \otimes \dots \otimes S^{\mu} V = S^{\lambda \cup \mu} V$   
 $V \mapsto \Lambda^{\lambda} V \otimes \dots \otimes \Lambda^{\mu} V$ ,  $\lambda \in \Lambda(n, d)$

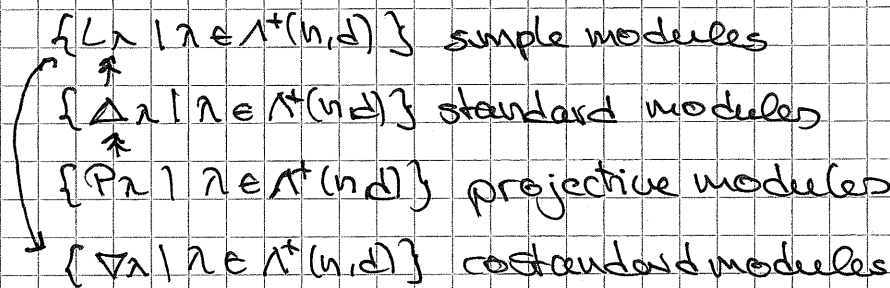
more exotic  $V \mapsto P^{\lambda} V$   
 $V \mapsto P^{\lambda \cup \mu} V = P^{\lambda} V \otimes \dots \otimes P^{\mu} V$ ,  $\lambda \in \Lambda(n, d)$

This cat.  $SP_d$  is abelian,  $P^{\lambda}$  are projectives ( $\lambda \in \Lambda(n, d)$ )  
 $S^{\lambda}$  are injectives  
 $\Lambda^{\lambda}$  are tilting in the sense of theory of height w. cat.

- Theorem**
- 1) There is a tensor product on  $SP_d$
  - 2)  $\Lambda^d \otimes - : \mathcal{D}(SP_d) \rightarrow \mathcal{D}(SP_d)$  is an autoeq.
  - 3)  $(\Lambda^d \otimes -)^2 = S^d \otimes -$
  - 4)  $\Lambda^d \otimes P^{\lambda} \cong \Lambda^{\lambda}$   
 $\Lambda^d \otimes \Lambda^{\lambda} \cong S^{\lambda}$

$SP_d \cong \underline{S}(n, d)\text{-mod}$ ,  $n \geq d$   
 solw algebra

$S(n, d)$  is quasi-hered. algebra



$\mathcal{F}(\Delta)$  filtered by  $\Delta_{\lambda}$   $\cap$   $\mathcal{F}(\nabla)$  filtered by  $\nabla_{\lambda}$  direct sums of T-tilting

$\bigoplus_{\lambda \in \Lambda^+} T(\lambda) = T$   
 ind. tilting

Ringel duality

$\mathcal{D}(S(n, r)\text{-mod}) \xrightarrow{\sim} \mathcal{D}(\text{Eucl. mod } (T))$   $n \geq r$

Monit. eq. to  $S(n, r)$

Ringel dual funct. is inverse to  $\Lambda^d \otimes -$

# Divided powers

$$\mathbb{Q}[x_1, \dots, x_n]$$

$$p \mid k \in \mathbb{N}, p^{(k)} = \frac{1}{k!} p^k$$

$$\text{let } L = \mathbb{Z} \langle x^{(n)} = x_1^{(n)}, \dots, x_n^{(n)} \mid n \in \mathbb{N}, n \geq 0 \rangle$$

$$p^{(k)} p^{(r)} = \binom{k+r}{k} p^{(k+r)}$$

$$(p+q)^{(k)} = \sum_{j=0}^k p^{(j)} q^{(k-j)}$$

$\Rightarrow L$  is a  $\mathbb{Z}$  subring of  $\mathbb{Q}[x_1, \dots, x_n]$  and  $L$  does not depend on choice of the basis  $x_1, \dots, x_n \in \mathbb{Q}[x_1, \dots, x_n]$

$L$  is graded, non Noeth.

Given a free  $R$ -module  $V$ , we define

$$P^d V = R \otimes_{\mathbb{Z}} (L x_1, \dots, x_n)^d$$

any basis of  $V$

$$P^* V = R \otimes_{\mathbb{Z}} L x_1, \dots, x_n \text{ is a ring of div. powers}$$

$$P^d V \xrightarrow{\psi} P^d V$$

$$x^{(n)} \mapsto \sum x_i \otimes i$$

$$i \in I(d): \text{wt}(i) = d$$

$$\{(i_1, \dots, i_n) \mid 1 \leq i_j \leq d\}$$

$$\text{wt}(i)_s = \#\{j \mid i_j = s\}$$

$$\text{Im } \psi \subseteq (P^d V)^{\mathbb{Z}} = (V^{\otimes d})^{\mathbb{Z}}$$

$$\text{If } R = k \text{ is an inf. field, then } \text{Im } \psi = (V^{\otimes d})^{\mathbb{Z}}$$

Remark If  $V$  is a free  $R$ -module of fin. rank, then

$$P^d(V) \xrightarrow[\text{nat.}]{\cong} S^d(V^*)$$

## weak monoidal structure on $P^d$

Given  $V, W$  free  $R$ -modules, we will construct

$$P^d(V) \otimes P^d(W) \xrightarrow{\quad} P^d(V \otimes W)$$

$\uparrow$   
nat. transf.

$$S^d(V \otimes W) \longrightarrow S^d(V) \otimes S^d(W)$$

$$z_{ij} = x_i \otimes y_j$$

$$x_1, \dots, x_n$$

$$y_1, \dots, y_n$$

$$z_{ij} \longmapsto x_i \otimes y_j$$

$$z^\omega = \prod z_{ij}^{\omega_{ij}} \quad \omega_{ij} \in \Lambda(n, m|d) = \{\omega \in M_{nm}(k) \mid \sum \omega_{ij} = d\}$$

$$z^\omega \mapsto \left( \prod_i x_i^{\sum_j \omega_{ij}} \right) \otimes \left( \prod_j y_j^{\sum_i \omega_{ij}} \right)$$

$$S: X^{(n)} \otimes Y^{(m)} \mapsto \sum_{\substack{\omega \in \Lambda(n, m|d) \\ \omega^{(1)} = \pi \\ \omega^{(2)} = \lambda}} z^\omega$$

Roby [1980] C.R. Séj. Par. A  
 Fermat "Un foncteur norme" Bull. SMF, 126 (1998)

Suppose  $A$  is  $k$ -algebra, then

$$\Gamma^d(A) \otimes \Gamma^d(A) \xrightarrow{S} \Gamma^d(A \otimes A) \xrightarrow{\Gamma^d(\text{mult.})} \Gamma^d(A)$$

$\gamma_d$  objects  $\mathbb{R}, \mathbb{R}^{\otimes 2}, \dots$

$\gamma_d(\mathbb{R}^m, \mathbb{R}^n) = \Gamma^d(\mathbb{R}\text{-mod}(\mathbb{R}^m, \mathbb{R}^n))$  we get composition since  $\Gamma^d$  is weakly monoidal.

$$\mathbb{R}\text{-mod}(\mathbb{R}^{m_1}, \mathbb{R}^{n_1}) \otimes \mathbb{R}\text{-mod}(\mathbb{R}^{m_2}, \mathbb{R}^{n_2}) \xrightarrow{\otimes} \mathbb{R}\text{-mod}(\mathbb{R}^{m_1+m_2}, \mathbb{R}^{n_1+n_2})$$

$$\Gamma^d \mathbb{R}\text{-mod}(\dots) \otimes \Gamma^d \mathbb{R}\text{-mod}(\dots) \xrightarrow{S} \Gamma^d(\mathbb{R}\text{-mod}(\dots) \otimes \mathbb{R}\text{-mod}(\dots))$$

$$\xrightarrow{\text{Kronecker } \otimes} \Gamma^d(\mathbb{R}\text{-mod}(\dots))$$

$$= \sum_i (k \otimes 1 \oplus 1 \otimes v_i)$$

$$v \in \Lambda(m_1, m_2, n_1, n_2|d) \xrightarrow{\cong} \Lambda(m_1, m_2, n_1, n_2|d)$$

$$v^{(1,1)} = 0 \\ v^{(3,4)} = \omega$$

$$SP_d \cong \gamma_d\text{-Mod} \cong \text{LinFunc}(\gamma_d \mathbb{R}\text{-Mod})$$

$$F \in SP_d$$

$$F: \mathbb{R}\text{-mod}$$

$$V \mapsto F(V)$$

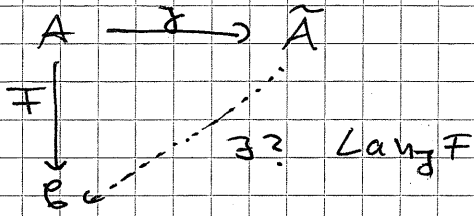
$$\text{free } \mathbb{R}\text{-mod} \xrightarrow{\otimes^d} \gamma_d \xrightarrow{F \in SP_d} \mathbb{R}\text{-mod}$$

$$\mathbb{R}^n \mapsto (\mathbb{R}^n)\text{-mod}$$

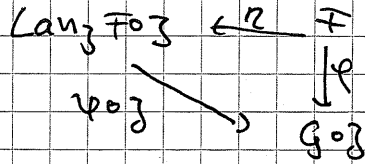
$$\text{Hom}(\mathbb{R}^m, \mathbb{R}^n) \longrightarrow \mathbb{R}\text{-mod}(G(\mathbb{R}^m), G(\mathbb{R}^n))$$

If  $\mathbb{R}$  is a field, these are affine spaces

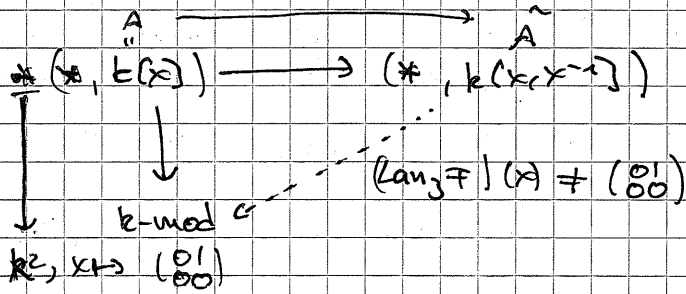
# Kan extension



$\text{Lan}_J F: \tilde{A} \rightarrow C$  is called left Kan extension, if  $\exists \eta: F \rightarrow \text{Lan}_J F \circ J$  and if  $G: \tilde{A} \rightarrow C$ ,  $\psi: F \rightarrow G \circ J$ , then  $\exists! \varphi: \text{Lan}_J F \rightarrow G$

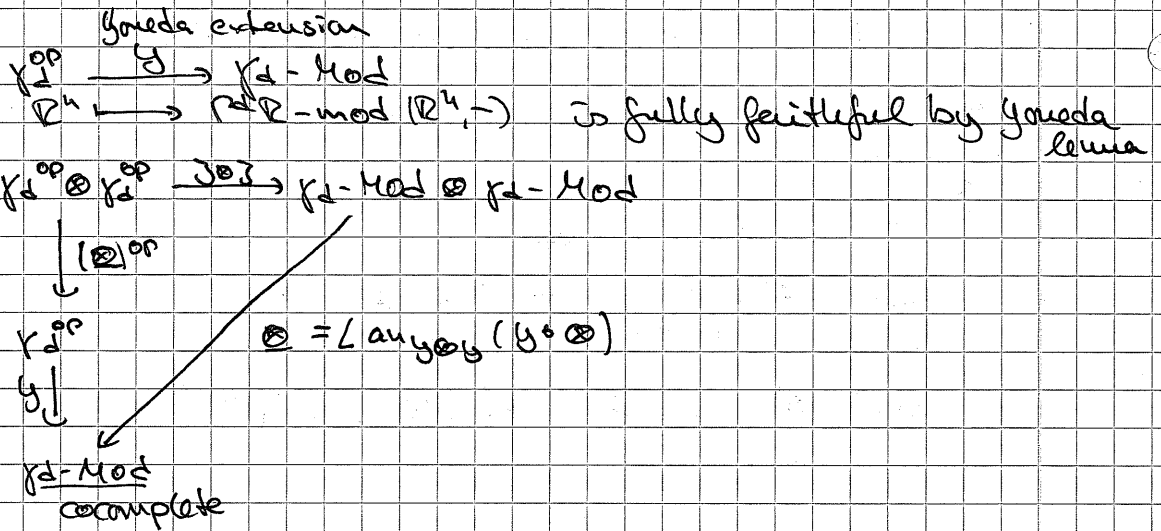


**Theorem** If  $C$  is cocomplete and  $A$  is (skeletal) small, then  $\text{Lan}_J F$  exists



**Theorem** If  $J$  is fully faithful, then

$\eta: F \rightarrow \text{Lan}_J F \circ J$  is a nat. isomorphism



**Exercise** To detect as many Kan extensions as you can in the next article you read.

There is a functor

$$\tau: \begin{array}{ccc} \mathcal{A} \text{-Mod} & \longrightarrow & \mathcal{A} \otimes \mathcal{B} \text{-Mod} \\ \mathbb{R}^m & \longmapsto & (\mathbb{R}^m \otimes \mathbb{R}^m) \end{array}$$

$$\begin{array}{ccc} \mathcal{A} \text{-Mod} \otimes \mathcal{B} \text{-Mod} & \xrightarrow{\hat{\otimes}} & \mathcal{A} \otimes \mathcal{B} \text{-Mod} \\ F \otimes G & \longmapsto & (F \otimes G) \otimes \tau \end{array}$$

$$(F_1 \otimes G_1) \hat{\otimes} (F_2 \otimes G_2)$$

$\downarrow$  nat. transf.  $\pi$

$$(F_1 \hat{\otimes} F_2) \otimes (G_1 \hat{\otimes} G_2)$$

Theorem (Krause)

(if  $F_1 = \mathbb{1}^d, F_2 = \mathbb{1}^e$ , then

$$(\mathbb{1}^d \otimes G_1) \hat{\otimes} (\mathbb{1}^e \otimes G_2) \xrightarrow{\pi} (\mathbb{1}^d \hat{\otimes} \mathbb{1}^e) \otimes (G_1 \hat{\otimes} G_2)$$

$$\xrightarrow{\mathbb{1} \otimes \text{id}} \mathbb{1}^{d+e} \otimes (G_1 \hat{\otimes} G_2) \quad \text{is an isom.}$$

"Koszul duality":  $\Lambda^d \hat{\otimes} \Lambda^d \cong S^d$

$$S^d \hat{\otimes} - : \begin{array}{ccc} \mathcal{P}^{\mathbb{R}^d} & \xrightarrow{\quad} & \mathcal{S}^{\mathbb{R}^d} \\ \text{proj.} & & \text{inj.} \end{array}$$

Koszul complex:  $0 \rightarrow \Lambda^d U \rightarrow \dots \rightarrow \bigoplus_{\substack{\mathbb{R}^{\lambda} U \\ \text{reiner}}} \rightarrow V \otimes \dots \rightarrow S^d V \rightarrow 0$

$$\Lambda^d \otimes \mathbb{R}^{\lambda} \cong \Lambda^{\lambda}$$

$$\dots \rightarrow \bigoplus \mathbb{R}^{\lambda} U \rightarrow V \otimes \dots \rightarrow \Lambda^d U \rightarrow 0$$

