14.08.15

Ivan Yudin: Koszul duality of strict polynomial functors

H. Krause "Koszul, Ringel and Serre duality for strict polynomial functors"

There is a cat. $\mathcal{SP}_d$ of s. pol. fun. of degree d.

$$R\text{-mod} \ni V \longmapsto S^d V \in R\text{-mod}$$
$$V \longmapsto \Lambda^d V$$

(R commu. ring)

more generally: $V \longmapsto S^{\lambda_1} V \otimes \ldots \otimes S^{\lambda_n} V = S^\lambda V$

$V \longmapsto \Lambda^{\lambda_1} V \otimes \ldots \otimes \Lambda^{\lambda_n} V$, $\lambda \in \Lambda(n,d)$

more exotic $V \longmapsto \Gamma^d V$

$V \longmapsto \Gamma^\lambda V = \Gamma^{\lambda_1} V \otimes \ldots \otimes \Gamma^{\lambda_n} V$, $\lambda \in \Lambda(n,d)$

This cat. $\mathcal{SP}_d$ is abelian, Γ^λ are projectives $(\lambda \in \Lambda(d,d))$
S^λ are injectives
Λ^λ are tilting in the sense of theory of height w. cat.

<u>Theorem</u> 1) There is a tensor product on $\mathcal{SP}_d$
2) $\Lambda^d \otimes^L - : \mathcal{D}(\mathcal{SP}_d) \longrightarrow \mathcal{D}(\mathcal{SP}_d)$ is an autoeq.
3) $(\Lambda^d \otimes^L -)^2 = S^d \otimes^L -$
4) $\Lambda^d \otimes^L \Gamma^\lambda \cong \Lambda^\lambda$
$\Lambda^d \otimes^L \Lambda^\lambda \cong S^\lambda$

$\mathcal{SP}_d \cong \underbrace{S(n,d)}_{\text{schur algebra}}\text{-mod}, \quad n \geq d$

$S(n,d)$ is quasi-hered. algebra

$\{L_\lambda \mid \lambda \in \Lambda^+(n,d)\}$ simple modules

$\{\Delta_\lambda \mid \lambda \in \Lambda^+(n,d)\}$ standard modules

$\{P_\lambda \mid \lambda \in \Lambda^+(n,d)\}$ projective modules

$\{\nabla_\lambda \mid \lambda \in \Lambda^+(n,d)\}$ costandard modules

$\underset{\substack{\text{filtered by} \\ \Delta_\lambda}}{\mathcal{F}(\Delta)} \cap \underset{\substack{\text{filtered by} \\ \nabla_\lambda}}{\mathcal{F}(\nabla)} \ni$ direct sums of Tilting

$\bigoplus_{\lambda \in \Lambda^+} \underbrace{T(\lambda)}_{\text{ind. tilting}} = T$

$$\mathcal{D}(S(n,r)\text{-mod}) \xrightarrow{\text{Ringel duality}} \mathcal{D}(\mathrm{End}_{S(n,r)}(T)) \qquad n \geq r$$

Morita eq. to $S(n,r)$

Ringel dual funct. is inverse to $\Lambda^d \otimes -$

Divided powers

$\mathbb{Q}[x_1, \dots, x_n]$
$\overset{\cup}{p}, k \in \mathbb{N}, \quad p^{(k)} = \frac{1}{k!} p^k$

Let $L = \mathbb{Z}\langle x^{(\lambda)} = x_1^{(\lambda_1)} \dots x_n^{(\lambda_n)} \mid \lambda \in \coprod_{d \geq 0} \Lambda(n,d) \rangle$

$$p^{(k)} p^{(l)} = \underbrace{\binom{k+l}{k}}_{\in \mathbb{N}} p^{(k+l)}$$

$$(p+q)^{(k)} = \sum_{j=0}^{k} p^{(j)} q^{(k-j)}$$

$\Rightarrow$ L is a $\mathbb{Z}$ subring of $\mathbb{Q}[x_1, \dots, x_n]$ and L does not depend on choice of the basis $x_1 \dots x_n \in \mathbb{Q}\langle x_1, \dots, x_n \rangle$

L is graded, non Noeth.

Given a free R-module V, we define

$$\Gamma^d V = R \otimes_{\mathbb{Z}} (\underbrace{L_{x_1, \dots, x_n}}_{\text{any basis of } V})_d$$

$$\Gamma V = R \otimes_{\mathbb{Z}} L_{x_1 \dots, x_n} \quad \text{is a ring of div. powers}$$

$$\Gamma^d V \xrightarrow{\psi} T^d V$$
$$x^{(\lambda)} \longmapsto \sum_{\substack{i \in I(n,d) : wt(i) = \lambda \\ \{(i_1 \dots i_n) \mid 1 \leq i_j \leq d\}}} x^{\otimes i}$$

$$wt(i)_s = \#\{j \mid i_j = s\}$$

$\operatorname{Im} \psi \subseteq (T^d V)^{\Sigma_d} = (V^{\otimes d})^{\Sigma_d}$

If $R = k$ is an inf. field, then $\operatorname{Im} \varphi = (V^{\otimes d})^{\Sigma_d}$

Remark If V is a free R-module of fin. rank, then

$$\Gamma^d(V) \xrightarrow[\text{nat.}]{\cong} S^d(V^*)^*$$

weak monoidal structure on Γ^d

Given V, W free R-modules, we will construct

$$\Gamma^d(V) \otimes \Gamma^d(W) \underset{\text{nat. transf.}}{\longrightarrow} \Gamma^d(V \otimes W)$$

$$S^d(V \otimes W) \longrightarrow S^d(V) \otimes S^d(W)$$
$z_{ij} = x_i \otimes y_j \qquad \overset{\cup}{x_1 \dots x_n} \qquad y_1, \dots, y_n$

$$z_{ij} \longmapsto x_i \otimes y_j$$

$$z^\omega = \prod_{ij} z_{ij}^{\omega_{ij}} \qquad \omega_{ij} \in \Lambda(n,m|d) = \{\omega \in M_{nm}(\mathbb{N}) \mid \Sigma \omega_{ij} = d\}$$

$$z^\omega \longmapsto \Big(\prod_i x_i^{\overset{\omega^{(2)}}{\sum_j \omega_{ij}}}\Big) \otimes \Big(\prod_j y_j^{\overset{\omega^{(1)}}{\sum_i \omega_{ij}}}\Big)$$

$$\rho : x^{(\lambda)} \otimes y^{(\mu)} \longmapsto \sum_{\substack{\omega \in \Lambda(n,m|d) \\ \omega^{(1)} = \mu \\ \omega^{(2)} = \lambda}} z^{(\omega)}$$

<u>Roby</u> [1980] C.R. Sej. Par. A
Ferrad "Un foncteur norme" Bull. SMF, 126 (1998)

Suppose A is k-algebra, then

$$\Gamma^d(A) \otimes \Gamma^d(A) \xrightarrow{\rho} \Gamma^d(A \otimes A) \xrightarrow{\Gamma^d(\mathrm{mult.})} \Gamma^d(A)$$

<u>γ_d</u> objects $0, R, R^{\otimes 2}, \dots$

$\gamma_d(R^m, R^n) = \Gamma^d(R\text{-mod}(R^m, R^n))$ we get composition since Γ^d is weakly monoidal.

$$\underset{(e_{ij})_{i=1\ j=1}^{m_1\ n_1}}{R\text{-mod}(R^{m_1}, R^{n_1})} \otimes \underset{(f_{ij})_{i=1\ j=1}^{m_2\ n_2}}{R\text{-mod}(R^{m_2}, R^{n_2})} \xrightarrow{\otimes} R\text{-mod}(R^{m_1 m_2}, R^{n_1 n_2})$$

$$e^{(\omega)}, \omega \in \Lambda(m,n|d) \otimes f^{(\tau)}, \tau \in \Lambda(m_2, n_2|d)$$

$$\Gamma^d R\text{-mod}(\dots) \otimes \Gamma^d R\text{-mod}(\dots) \xrightarrow{\rho} \Gamma^d(R\text{-mod}(\cdots) \otimes R\text{-mod}(\dots))$$

$$\xrightarrow{\text{Kronecker}\otimes} \Gamma^d(R\text{-mod}(\dots))$$

$$= \sum (e \otimes f)^{t(\nu)}$$

$$\nu \in \Lambda(m_1, n_1, m_2, n_2|d) \xrightarrow[t]{\sim} \Lambda(m_1, m_2, n_1, n_2|d)$$

$$\nu^{(1,2)} = \tau$$

$$\nu^{(3,4)} = \omega$$

$$SP_d \cong \gamma_d\text{-Mod} \cong \mathrm{Lin\,Func}(\gamma_d R\text{-Mod})$$

$F \in SP_d$

$$\begin{array}{c} \text{fr.}R\text{-mod} \\ \Psi \\ V \longmapsto F(V) \end{array}$$

$$\begin{array}{ccccc} \text{free}R\text{-mod} & \xrightarrow{\otimes^d} & \gamma_d & \xrightarrow{F^{SP_d}} & R\text{-mod} \\ R^n & \longmapsto & (R^n)^{\otimes d} & & \end{array}$$

$$\mathrm{Hom}(R^m, R^n) \longrightarrow R\text{-mod}(G((R^m))\, G(R^n))$$

If R is a field,
these are affine spaces

Kan extension

$$\begin{tikzcd} A \arrow[r, "J"] \arrow[d, "F"'] & \tilde{A} \arrow[dl, dashed, "\exists ?\ \mathrm{Lan}_J F"] \\ \mathcal{C} & \end{tikzcd}$$

$\mathrm{Lan}_J F : \tilde{A} \to \mathcal{C}$ is called left Kan extension, if $\exists\ \eta : F \to \mathrm{Lan}_J F \circ J$
and if $G : \tilde{A} \to \mathcal{C}$, $\psi : F \to G \circ J$, then $\exists!\ \varphi : \mathrm{Lan}_J F \to G$

$$\begin{tikzcd} \mathrm{Lan}_J F \circ J \arrow[dr, "\varphi \circ J"'] & F \arrow[l, "\eta"'] \arrow[d, "\psi"] \\ & G \circ J \end{tikzcd}$$

<u>Theorem</u> If $\mathcal{C}$ is cocomplete and A is (skelet.) small, then $\mathrm{Lan}_J F$ exists

$$\begin{tikzcd} A \arrow[rr] & & \tilde{A} \\ (*, k[x]) \arrow[rr] \arrow[d] & & (*, k[x, x^{-1}]) \arrow[dll, dashed] \\ k\text{-mod} & & \end{tikzcd}$$

$*$ $\longrightarrow$ $k^2,\ x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$(\mathrm{Lan}_J F)(x) \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

<u>Theorem</u> If J is fully faithful, then

$\eta : F \to \mathrm{Lan}_J F \circ J$ is a nat. isomorphism

Yoneda extension

$$\gamma_d^{op} \xrightarrow{y} \gamma_d\text{-Mod}$$

$\mathbb{R}^n \longmapsto \gamma_d\mathbb{R}\text{-mod}(\mathbb{R}^n, -)$ is fully faithful by Yoneda lemma

$$\begin{tikzcd} \gamma_d^{op} \otimes \gamma_d^{op} \arrow[r, "y \otimes y"] \arrow[d, "(\otimes)^{op}"'] & \gamma_d\text{-Mod} \otimes \gamma_d\text{-Mod} \arrow[dd, "\underline{\otimes} = \mathrm{Lan}_{y \otimes y}(y \circ \otimes)"] \\ \gamma_d^{op} \arrow[d, "y"'] & \\ \gamma_d\text{-Mod} & \end{tikzcd}$$

$\underline{\gamma_d\text{-Mod}}$
cocomplete

<u>Exercise</u> To detect as many Kan extensions as you can in the next article you read.

There is a functor

$$\tau : \Gamma^{d+e} \longrightarrow \Gamma^{d} \otimes \Gamma^{e}$$
$$R^m \longmapsto (R^m \text{"}\otimes\text{"} R^m)$$

$$\Gamma^d\text{-Mod} \otimes \Gamma^e\text{-Mod} \xrightarrow{\hat{\otimes}} \Gamma^{d+e}\text{-Mod}$$
$$F \otimes G \longrightarrow (F \boxtimes G) \circ \tau$$

$$\begin{array}{c} (F_1 \boxtimes G_1)\ \hat{\otimes}\ (F_2 \boxtimes G_2) \\ \Big\downarrow \exists \text{ nat. transf. } \pi \\ (F_1 \hat{\otimes} F_2) \boxtimes (G_1 \hat{\otimes} G_2) \end{array}$$

<u>Theorem</u> (Krause)
If $F_1 = \Lambda^d$, $F_2 = \Lambda^e$, then

$$(\Lambda^d \boxtimes G_1)\ \hat{\otimes}\ (\Lambda^e \boxtimes G_2) \xrightarrow{\pi} (\Lambda^d \hat{\otimes} \Lambda^e) \boxtimes (G_1 \hat{\otimes} G_2)$$
$$\xrightarrow{\Lambda \otimes \mathrm{id}} \Lambda^{d+e} \boxtimes (G_1 \hat{\otimes} G_2)$$ is an isom.

"Koszul duality": $\Lambda^d \otimes^L \Lambda^d \cong S^d$

$$S^d \otimes - : \underset{\text{proj.}}{\Gamma^\lambda} \longmapsto \underset{\text{inj}}{S^\lambda}$$

Koszul complex: $0 \to \Lambda^d V \to \ \cdots \ \bigoplus_{\lambda \in \Lambda(2,d)} \Lambda^\lambda V \to V^{\otimes d} \to S^d V \to 0$

$$\uparrow \Lambda^d \otimes -$$

$\Lambda^d \otimes \Gamma^\lambda \cong \Lambda^\lambda$ $\qquad \cdots \to \bigoplus \Gamma^\lambda V \longrightarrow V^{\otimes d} \longrightarrow \Lambda^d V \longrightarrow 0$