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Derived invariants for surface algebras

(jt Yvonne Grimeland)

$k = \bar{k}$, Λ f.d. k-alg. Gen. Qu. When are two such algs derived equiv?

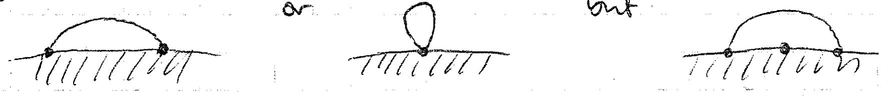
(Happel – know this for hered algs.)

Surface algs and main result

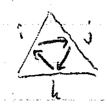
S oriented surface with boundary ∂S, M a set of marked pts on ∂S, and at least one marked pt on each conn. component of ∂S.

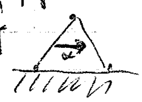
Δ = ideal triangulation = max collection of arcs that do not intersect.

do not allow or but is allowed.



Assoc to Δ a quiver Q^Δ, $Q_0^\Delta = \{\text{vertices}\} = \{\text{arcs}\}$
$Q_1^\Delta = \{\text{arrows}\} = \{\text{oriented angles}\}$
$Q_2^\Delta = \{\text{internal triangles}\}$



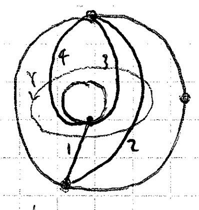
Defn. $d : Q_1^\Delta \to \mathbb{Z}$ is an admissible cut if
- $\mathrm{Im}(d) \subseteq \{0,1\}$.
- $\forall \alpha\beta\gamma \in Q_2^\Delta$, $d(\alpha) + d(\beta) + d(\gamma) = 1$.
- if  then $d(\alpha) = 0$.

Defn. [David, Roesler, Schiffler] $\Lambda = (\Delta, d)$ surface alg. $\Lambda = kQ^\Lambda / I$

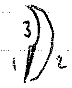
where $Q_0^\Lambda = Q_0^\Delta$, $Q_1^\Lambda = \{\alpha \in Q_1^\Delta : d(\alpha) = 0\}$

and $I = \{\alpha\beta : \exists\gamma, \alpha\beta\gamma \in Q_2^\Delta\}$.

Example.



γ simple closed curve.

one internal triangle 

$$Q^\Delta \quad \begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[dr] & & 3 \arrow[ll] \arrow[dl] \\ & 4 & \end{tikzcd} \qquad \Lambda = \begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[dr] & & 3 \arrow[ll] \arrow[dl] \\ & 4 & \end{tikzcd}$$

γ oriented simple closed curve on S. $\rightsquigarrow \bar{\gamma}^\Delta \in \mathbb{Z}Q_1^\Delta$ In example $\bar{\gamma}^\Delta = \alpha_{14} - \alpha_{34} + \alpha_{31}$

Thm [AG] $\Lambda = (\Delta, d)$, $\Lambda' = (\Delta', d')$ surface algs.

Tfae
1) $\mathcal{D}^b\Lambda \simeq \mathcal{D}^b\Lambda'$ ($\Rightarrow$ same surface S)
2) $\exists\, \Phi : S \to S$ orientation-preserving homeo, $\Phi(M) = M$ st. $\forall\gamma$ simple closed curve, $d(\bar{\gamma}^\Delta) = d'(\overline{\Phi(\gamma)}^{\Delta'})$.

In example, see that $\Lambda \underset{der}{\simeq} \Lambda'$, where $\Lambda' = \begin{tikzcd} & 2 & \\ 1 \arrow[ur] \arrow[dr] & & 3 \arrow[ll] \arrow[dl] \\ & 4 & \end{tikzcd}$

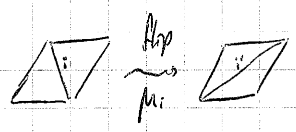
but not der. equiv. to $\Lambda'' = \begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[dr] & & 3 \arrow[dl] \\ & 4 & \end{tikzcd}$

About the pf.

Complex $C_\bullet$ $\quad \mathbb{Z}Q_2^\Delta \to \mathbb{Z}Q_1^\Delta \to \mathbb{Z}Q_0^\Delta$

$\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z}) \rightsquigarrow C^\bullet \quad \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_2}$.

graded flip / mutation.



can extend this to flip on pairs (Λ, d).

Easy Lemma. If $(\Delta'', d'') = \mu_i(\Delta, d)$, then $\forall \gamma$, $d(\bar{\gamma}^\Delta) = d''(\bar{\gamma}^{\Delta''})$.

Using this wma $\Delta = \Delta'$. So now want to decide when $(\Delta, d), (\Delta, d')$ are derived equiv.

Now 2) iff $[d-d'] = 0$ in $H^1(C^\bullet)$ iff d, d' are equivalent gradings.

Facts. • [DR-S] Λ has gl. dim 2 and is τ_2-finite.

• $\mathcal{C}_2(\Lambda) \simeq \mathcal{C}_{(S,M)}$ cluster categories.

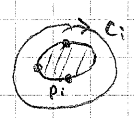
• [Brüstle-Zhang]. $\{$cluster-tilting objs in $\mathcal{C}\} \xleftrightarrow{1-1} \{$ideal triang. in $(S,M)\}$

$\{$CT subcat of $\mathcal{D}^b(\Lambda)\} \xleftrightarrow{1-1} \{$graded ideal triang$\}/$gr. equiv.

Thm [A-Oppermann] Derived equiv between τ_2-finite algs of gl. dim 2 iff graded equiv between CT subcats.

Applications and Interpretations

Rem. 1. Actually, to check $[d-d'] = 0$ enough to check condition 2) on a set of simple closed curves that generate $H_1(S, \mathbb{Z})$, so on $2g+b-1$ closed curves, g = genus, b = # bdry components.

2.  p_i = # marked pts on bdry component i. Then $\Phi(c_i) = \Phi(c_j)$ if $p_i = p_j$.

Cor. If $g = 0$, $\mathcal{D}^b\Lambda \simeq \mathcal{D}^b\Lambda'$ iff $\exists$ permutation σ st. $p_{\sigma(i)} = p_i$ and $d(\bar{c}_i^\Delta) = d'(\bar{c}_{\sigma(i)}^{\Delta'})$.

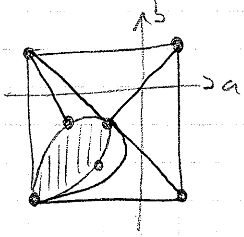
[Avella-Alaminos-Geiß] $\Rightarrow$ If $d(\bar{c}_i) = 0$, then in the AR-quiver of $\mathcal{D}^b\Lambda$ $\exists$ $\mathbb{Z}$-family of tubes of rank p_i. (using shift functor)

In example
$$\begin{tikzcd}[row sep=small] & 2 \arrow[dr] & \\ 1 \arrow[dr] & & 3 \arrow[ll] \arrow[dl] \\ & 4 & \end{tikzcd}$$
$d(\bar{c}_1) = d(\bar{c}_2) = 0$.

If $d(\bar{c}_i) \neq 0$, then there is a family of $|d(\bar{c}_i)|$ components of type $\mathbb{Z}A_\infty$. (so shift by $|d(\bar{c}_i)|$ fixes components)

In example $1 \to 2 \to 3$, 4 $\quad d(\bar{c}_1) = 1$.

$g=1, b=1$.



Lemma If γ simple closed, $[\gamma] = \lambda[a] + \mu[b] \neq 0$, then $d'(\bar{\gamma}^\Delta) = \lambda d(\bar{a}^\Delta) + \mu d(\bar{b}^\Delta)$.

If $[\gamma] = [\gamma']$, then γ not nec htpc to γ', but $\bar{\gamma} - \bar{\gamma}' = \partial\left(\sum_{\tau \in Q_2} \tau\right)$.

Action of $SL_2(\mathbb{Z})$ on $\mathbb{Z}^2 = \langle [a], [b] \rangle$.

Cor. $D^b\Lambda \simeq D^b\Lambda'$ iff $\gcd(d(\bar{a}^\Delta), d(\bar{b}^\Delta)) = \gcd(d'(\bar{a}^{\Delta'}), d'(\bar{b}^{\Delta'}))$.

Application to:

[Bobinski - Malicki] Qn on gentle 2-cycle algs.