

Lidia Angeleri-Hügel:

Minimal approximations

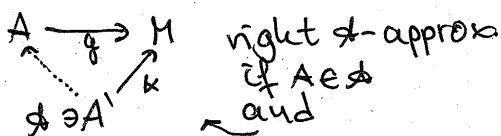
§1 Existence of approximations

§2 Minimal approximations

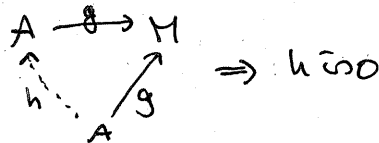
§3 Examples

Ring, Mod R, mod R: fun. pres.

$\mathcal{A} \subset \text{Mod } R$



g is also min. right \mathcal{A} -approx if also



Auslander-Smalø 1980

Auslander-Buchweitz 1989

Enochs 1981

Definition [Salce 1979]

$\mathcal{A}, \mathcal{B} \subset \text{Mod } R$ form a cotorsion pair $(\mathcal{A}, \mathcal{B})$ if

$$\mathcal{A} = \text{Ker Ext}_R^1(-, \mathcal{B})$$

$$\mathcal{B} = \text{Ker Ext}_R^1(\mathcal{A}, -)$$

$(\mathcal{A}, \mathcal{B})$ is complete if $\forall M \in \text{Mod } R$

$$\exists 0 \rightarrow \mathcal{B} \rightarrow A \rightarrow M \rightarrow 0$$

$$0 \rightarrow M \rightarrow \mathcal{B}' \rightarrow A' \rightarrow 0$$

$\mathcal{B}, \mathcal{B}' \in \mathcal{B}$
 $A, A' \in \mathcal{A}$

Theorem (Eklof-Trifaj 2001)

If $\mathcal{S} \subset \text{Mod } R$ is a set, then the cotorsion pair generated by \mathcal{S} , i.e. $\mathcal{B} = \text{Ker Ext}_R^1(\mathcal{S}, -)$, $\mathcal{A} = \text{Ker Ext}_R^1(-, \mathcal{B})$ is complete

Deconstruction of cotorsion pairs: $(\mathcal{A}, \mathcal{B})$ is complete if every $A \in \mathcal{A}$ is transfinite extension of "small" modules in \mathcal{A} :

$\exists \kappa$ and a continuous chain of submodules $(A_\alpha | \alpha \leq \kappa)$ of A st. $A_0 = 0$, $A_{\alpha+1}/A_\alpha \in \mathcal{A} < \kappa$ -presented and $\bigcup_{\alpha < \kappa} A_\alpha = A$

Take $\mathcal{S} = \{A \in \mathcal{A} | < \kappa\text{-presented}\}$

Applications. Bican-El Bashir-Enochs 2001: All modules have a flat cover

- Many more cotorsion pairs, e.g. all modules have right \mathbb{P}_n -approx $\mathbb{P}_n = \{M | \text{pd } M \leq n\}$

- Buzzoni-Herbera-Stovicek (2007/2008)

Every Gihny cotorsion pair is generated by a set $\mathcal{S} \subset \text{mod } R$ ($\Rightarrow \mathcal{B}$ closed under \varinjlim , even definable)

§2 Minimal approximations

Theorem (Enochs-Xu 1996)

If \mathcal{A} is closed under \varinjlim , then

- (1) every M having right \mathcal{A} -approx. has a min. right \mathcal{A} -approx.
- (2) if $(\mathcal{A}, \mathcal{B})$ complete cot. pair, then \exists min. right \mathcal{A} -approx. and min. left \mathcal{B} -approx.

Question (Enochs): If every module has min. right \mathcal{A} -approx. $\Rightarrow \mathcal{A}$ closed under \varinjlim ?

Theorem P (Bass) Case $\mathcal{A} = \text{add } R$: TFAE

- (1) All flat modules are proj. ($\text{add } R$ closed under \varinjlim)
- (2) All modules have proj. cover.
- (3) R satisfies dcc for principal left ideals

Answer: yes if

- 1) $\mathcal{A} = \text{add } M$ where $M = \bigoplus \text{f.p.}$
- 2) $(\mathcal{A}, \mathcal{B})$ cot. pairs with \mathcal{B} closed under \varinjlim
- 3) $\mathcal{A} = \text{add } T$, T tilting

①

Theorem (A 2013)

Let $M = \bigoplus \text{f.p. modules}$. TFAE

- (1) $\text{add } M$ closed under \varinjlim
- (2) Every module has a min. right $\text{add } M$ -approx.
- (3) M has a perfect decomposition, i.e.
 - (i) $M = \bigoplus_{i \in I} X_i$, $\text{End } X_i$ local, and
 - (ii) M satisfies the descending chain condition for cyclic $\text{End } M$ -submodules.

Example: Λ artin algebra, $M = \bigoplus$ all ind. preprojectives

(3) \Leftrightarrow Harada $\# M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \dots$ non isos with M_i ind. preproj. $\exists n$ s.t. $f_n \circ f_{n-1} \circ \dots \circ f_1 = 0$

\Leftrightarrow Auslander Λ finite repr. type

② $(\mathcal{A}, \mathcal{B})$ cot. pair with \mathcal{B} closed under \varinjlim

\Rightarrow • $(\mathcal{A}, \mathcal{B})$ complete, \mathcal{B} definable

• $\mathcal{A} \cap \mathcal{B} = \text{add } M$ for some module M

• An inverse system $(M_i, M_i \xrightarrow{h_{ij}} M_j)_{i \geq j}$ is Mittag-Leffler if $\forall k \in I \exists j \geq k$ s.t. $\text{Im } h_{kj} = \text{Im } h_{ki}$ for all $i \geq j$

If $M \in \mathcal{A}$, then $M = \varinjlim (M_i, f_{ij})_{i \geq j \in I}$ s.t. M_i f.p. and the inverse system $(\text{Hom}(M_i, \mathcal{B}), \text{Hom}(f_{ij}, \mathcal{B}))$ is $M \in \mathcal{B}$

Theorem (A-Saroch-Trlifaj 2014)

TFAE

- (1) \mathcal{A} closed under (countable) lims
- (2) Every module has min. right \mathcal{A} -approx.
- (3) $\text{Add } M$ closed under (countable) lims
- (4) Every module has min. right $\text{Add } M$ -approx.
- (5) M has perfect decomp.

③ Let T be tilting, i.e.

(T1) $\text{pdim } T < \infty$

(T2) $\text{Ext}_R^i(T, T^{(I)}) = 0 \quad \forall i > 0, \forall \text{ sets } I$

(T3) $\exists 0 \rightarrow R \rightarrow T_0 \rightarrow T_1 \rightarrow \dots \rightarrow T_n \rightarrow 0, T_i \in \text{Add } T$

Then $\mathcal{B} = T^\perp = \{X \mid \text{Ext}_R^i(T, X) = 0 \quad \forall i > 0\}$

$\mathcal{A} = \text{Ker Ext}^1(-, \mathcal{B})$

form a cotorsion pair $(\mathcal{A}, \mathcal{B})$ as in ②

Corollary TFAE

(1) $\text{Add } T$ closed for lims

(2) Every module has min. right $\text{Add } T$ -approx.

(3) T has perf. decomp.

If $\mathcal{A} \cap \text{mod } R \subset \text{mod } R$ is covar. finite (e.g. R left noeth., $\text{pdim } T \leq 1$) then further equiv.

(4) T Σ -pure-inj.

(5) $\text{Add } T$ closed for products

§3 Examples

(1) 1-hered.

[A-Sanchez] T Σ -pure-inj.:

- T fin. dim.
- T inj. dim. with $R \in \text{Add } T$

(e.g. 1 Kronecker: only one $T = G \oplus \oplus$ all Prüfer)

(2) 1-hered. artin algebra, let \mathcal{L} Lukas tilting module, i.e.
 $\text{Gen } \mathcal{L} = \{\text{modules without preproj. summands}\}$

$\mathcal{L} \Sigma$ -pure-inj. $\Leftrightarrow 1$ f.r.t.

(3) R left pure-semisimple if all left R -modules are \oplus finite generated.

Theorem (Auslander, Ringel-Tachikawa, Auslander 1974-1976)

If 1 artin algebra, then 1 f.r.t. $\Leftrightarrow 1$ is left pure-semisimple.

Herzog (1994) - also true for PI-rings, rings with selfduality.
- enough to consider hered. R

Corollary Let R be left pure-semisimple local.

TFAE

(1) R has finite top-type

(2) There are only finitely many tilting left R -modules

(3) ————— " ————— right "

(4) [A-Herbert] L is Σ -pure-inj.