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Minimal approximations

§1 Existence of approximations

§2 Minimal approximations

§3 Examples

R ring, $\mathrm{Mod}\,R$, $\mathrm{mod}\,R$: fin. pres.

$\mathcal{A} \subset \mathrm{Mod}\,R$

$$\begin{tikzcd} A \arrow[r, "g"] & M \\ \mathcal{A} \ni A' \arrow[u, dotted] \arrow[ur, "k"'] & \end{tikzcd}$$

right $\mathcal{A}$-approx if $A \in \mathcal{A}$ and

g is also min. right $\mathcal{A}$-approx if also

$$\begin{tikzcd} A \arrow[r, "g"] & M \\ A \arrow[u, dotted, "h"] \arrow[ur, "g"'] & \end{tikzcd} \Rightarrow h \text{ iso}$$

Auslander-Smalø 1980
Auslander-Buchweitz 1989
Enochs 1981

<u>Definition</u> [Salce 1979]
$\mathcal{A}, \mathcal{B} \subset \mathrm{Mod}\,R$ form a <u>cotorsion pair</u> $(\mathcal{A}, \mathcal{B})$ if
$\mathcal{A} = \mathrm{Ker}\,\mathrm{Ext}^1_R(-, \mathcal{B})$
$\mathcal{B} = \mathrm{Ker}\,\mathrm{Ext}^1_R(\mathcal{A}, -)$
$(\mathcal{A}, \mathcal{B})$ is complete if $\forall M \in \mathrm{Mod}\,R$
$\exists\ 0 \to B \to A \to M \to 0$
$0 \to M \to B' \to A' \to 0$
$B, B' \in \mathcal{B}$
$A, A' \in \mathcal{A}$

<u>Theorem</u> (Eklof-Trlifaj 2001)
If $\mathcal{S} \subset \mathrm{Mod}\,R$ is a <u>set</u>, then the cotorsion pair generated by $\mathcal{S}$, i.e. $\mathcal{B} = \mathrm{Ker}\,\mathrm{Ext}^1_R(\mathcal{S}, -)$, $\mathcal{A} = \mathrm{Ker}\,\mathrm{Ext}^1_R(-, \mathcal{B})$ is complete

<u>Deconstruction of cotorsion pairs</u>: $(\mathcal{A}, \mathcal{B})$ is complete if every $A \in \mathcal{A}$ is transfinite extension of "small" modules in $\mathcal{A}$:
$\exists\ \kappa$ and a continuous chain of submodules $(A_\alpha \mid \alpha \leq \lambda)$ of A s.t. $A_0 = 0$, $A_{\alpha+1}/A_\alpha \in \mathcal{A}$ $< \kappa$-presented and $\bigcup_{\alpha \leq \lambda} A_\alpha = A$
Take $\mathcal{S} = \{A \in \mathcal{A} \mid < \kappa\text{-presented}\}$

<u>Applications</u>
- Bican-El Bashir-Enochs 2001: All modules have a flat cover
- Many more cotorsion pairs, e.g. all modules have right $\mathcal{P}_n$-approx $\mathcal{P}_n = \{M \mid \mathrm{pd}\,M \leq n\}$
- Bazzoni-Herbera-Stovicek (2007/2008)
  Every tilting cotorsion pair is generated by a set $\mathcal{S} \subseteq \mathrm{mod}\,R$
  ($\Rightarrow \mathcal{B}$ closed under $\varinjlim$, even definable)

§2 Minimal approximations

Theorem (Enochs–Xu 1996)

If $\mathcal{A}$ is closed under $\varinjlim$, then

(1) every M having right $\mathcal{A}$-approx. has a min. right $\mathcal{A}$-approx.

(2) if $(\mathcal{A},\mathcal{B})$ complete cot. pair, then $\exists$ min. right $\mathcal{A}$-approx. and min. left $\mathcal{B}$-approx.

Question (Enochs): If every module has min. right $\mathcal{A}$-approx.

$\Rightarrow$ $\mathcal{A}$ closed under $\varinjlim$?

Theorem P (Bass) Case $\mathcal{A} = \operatorname{add} R$: TFAE

(1) All flat modules are proj. ($\operatorname{Add} R$ closed under $\varinjlim$)

(2) All modules have proj. cover.

(3) R satisfies dcc for principal left ideals

Answer: yes if 1) $\mathcal{A} = \operatorname{add} M$ where $M = \bigoplus$ f.p.

2) $(\mathcal{A},\mathcal{B})$ cot. pairs with $\mathcal{B}$ closed under $\varinjlim$

3) $\mathcal{A} = \operatorname{add} T$, T tilting

① **Theorem** (A 2013)

Let $M = \bigoplus$ f.p. modules. TFAE

(1) $\operatorname{Add} M$ closed under $\varinjlim$

(2) Every module has a min. right $\operatorname{Add} M$-approx.

(3) M has a perfect decomposition, i.e.

(i) $M = \bigoplus_{i\in I} X_i$, $\operatorname{End} X_i$ local, and

(ii) M satisfies the descending chain condition for cyclic $\operatorname{End} M$-submodules.

Example: Λ artin algebra, $M = \bigoplus$ all ind. preprojectives

(3) $\underset{\text{Harada}}{\Longleftrightarrow}$ $\forall$ $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \dots$ nonisos with M_i ind. preproj. $\exists n$ s.t. $f_n \circ f_{n-1} \circ \dots \circ f_1 = 0$

$\underset{\text{Auslander}}{\Longleftrightarrow}$ Λ finite repr. type

② $(\mathcal{A},\mathcal{B})$ cot. pair with $\mathcal{B}$ closed under $\varinjlim$

$\Longrightarrow$ • $(\mathcal{A},\mathcal{B})$ complete, $\mathcal{B}$ definable

• $\mathcal{A} \cap \mathcal{B} = \operatorname{Add} M$ for some module M

• An inverse system $(H_i, H_i \xleftarrow{h_{ij}} H_j)_{i\le j}$ is Mittag-Leffler if $\forall k \in I$ $\exists j \geq k$ st. $\operatorname{Im} h_{kj} = \operatorname{Im} h_{ki}$ for all $i \geq j$

If $M \in \mathcal{A}$, then $M = \varinjlim (M_i, f_{ji})_{i\le j \in I}$ s.t. M_i f.p. and the inverse system $(\operatorname{Hom}(M_i, B), \operatorname{Hom}(f_{ji}, B))$ is ML $\forall B \in \mathcal{B}$

Theorem (A-Saroch-Trlifaj 2014)

TFAE

(1) $\mathcal{A}$ closed under (countable) $\varinjlim$

(2) Every module has min. right $\mathcal{A}$-approx.

(3) Add M closed under (countable) $\varinjlim$

(4) Every module has min. right - Add M-approx.

(5) M has perfect decomp.

(3) Let T be tilting, i.e.

(T1) $\mathrm{pdim}\, T < \infty$

(T2) $\mathrm{Ext}^i_R(T, T^{(I)}) = 0 \quad \forall i > 0, \ \forall$ sets I

(T3) $\exists\ 0 \to R \to T_0 \to T_1 \to \dots \to T_n \to 0, \quad T_i \in \mathrm{Add}\, T$

Then $\mathcal{B} = T^{\perp} = \{\mathrm{Ext}^i(T, X) = 0 \ \forall i > 0\}$

$\mathcal{A} = \mathrm{Ker}\, \mathrm{Ext}^1(-, \mathcal{B})$

form a cotorsion pair $(\mathcal{A}, \mathcal{B})$ as in (2)

Corollary TFAE

(1) Add T closed for $\varinjlim$

(2) Every module has min. right Add T-approx.

(3) T has perf. decomp.

If $\mathcal{A} \cap \mathrm{mod} R \subset \mathrm{mod} R$ is covar. finite (e.g. R left noeth, $\mathrm{pdim}\, T \le 1$)
then further equiv.

(4) T Σ-pure-inj.

(5) Add T closed for products

§3 Examples

(1) Λ tame hered.

[A-Sanchez] T Σ-pure-inj. : • T fin.dim.
• T inf.dim. with $G \in \mathrm{Add}\, T$

(e.g. Λ Kronecker : only one $T = G \oplus \bigoplus$ all Prüfer)

(2) Λ hered. artin algebra, let L Lukas tilting module, i.e.
Gen L = {modules without preproj. summands}
L Σ-pure-inj. $\Leftrightarrow$ Λ f.r.t.

(3) R left <u>pure-semisimple</u> if all left R-modules are $\bigoplus$ finite generated.

Theorem (Auslander, Ringel-Tachikawa, Auslander 1974-1976)
If Λ artin algebra, then Λ f.r.t. iff Λ is left pure-semisimple.

Herzog (1994): also true for PI-rings, rings with selfduality.
- enough to consider hered. R

<u>Corollary</u> Let R be left pure-semisimple hered.
TFAE
(1) R has finite repr. type
(2) There are only finitely many tilting left R-modules
(3) — " — right "
(4) [A-Herbera] L is Σ-pure-inj.