THE AUSLANDER BUCHSBAUM FORMULA

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Abstract. This is the script for my talk about the Auslander-Buchsbaum formula [AB57, Theorem 3.7] at the Auslander Memorial Workshop, 15^{th}-18^{th} of November 2014 in Bielefeld.

0. Overview

This talk is about the *Auslander-Buchsbaum formula*:

Theorem (Auslander-Buchsbaum formula)**.** *If R is a commutative local Noetherian ring and M a finitely generated R-module of finite projective dimension, then*

$$\operatorname{depth}_R(R) - \operatorname{depth}_R(M) = \operatorname{pdim}_R(M).$$

The *plan of the talk* is the following:

- In §1 we recall some *basic notions* from commutative algebra that go into understanding, proving and using the Auslander-Buchsbaum formula.
- In §2 we present the *classical proof* given in [BH93].
- In §4 we ask how *new techniques* like derived categories may shed new light on the Auslander-Buchsbaum formula and its proof, and study some of its *generalizations*.

Conventions

In the following, we denote R a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$. Further, we denote R-mod the category of finitely generated left R-modules, and $M \in R$-mod unless otherwise stated.

1. Basic notions

1.1. **Regular sequences.** We begin by recalling the notion of a regular sequence.

Definition 1.1. Let M be an R-module, $x \in \mathfrak{m}$ and $\underline{x} = (x_1, ..., x_n) \in \mathfrak{m}^n$.

(i) x is called *M-regular* if $M \xrightarrow{\cdot x} M$ is injective.

(ii) A sequence $\underline{x} = (x_1, ..., x_n)$ is called M-regular if x_1 is M-regular and $(x_2, ..., x_n)$ is M/x_1M-regular.

If $M = R$, x resp. $\underline{x}$ are simply called *regular*.

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Definition 1.2. Let M be an R-module. The *depth of* M (as an R-module) is the maximal length of an M-regular sequence in $\mathfrak{m}$. It is denoted $\operatorname{depth}_R(M)$.

Remark 1.3. Note that a priori it is not clear that $\operatorname{depth}_R(M) < \infty$ or that any two maximal M-regular sequences have the same length. Both however is true and will be established below in Proposition 1.9.

By definition, the non-regular elements of M are those contained in the union

$$\bigcup_{m\in M\setminus\{0\}} \operatorname{Ann}_R(m) = \bigcup_{\substack{I\lhd R\\ R/I\hookrightarrow M}} I = \bigcup_{\substack{\mathfrak{p}\lhd R \text{ prime}\\ R/\mathfrak{p}\hookrightarrow M}} \mathfrak{p},$$

where for the last equality we used the fact that the ideals maximal among those of the form $\operatorname{Ann}_R(m)$ for $m \in M \setminus \{0\}$ are prime.

Definition 1.4. A prime ideal $\mathfrak{p} \lhd R$ is called *associated prime* of M if there exists an embedding $R/\mathfrak{p} \hookrightarrow M$, i.e. if there exists some $m \in M \setminus \{0\}$ such that $\mathfrak{p} = \operatorname{Ann}_R(m)$. The set of associated primes of M is denoted $\operatorname{Ass}_R(M)$.

Fact 1.5. *For a short exact sequence* $0 \to M' \to M \to M'' \to 0$ *of* R*-modules, we have* $\operatorname{Ass}_R(M) \subset \operatorname{Ass}_R(M') \cup \operatorname{Ass}_R(M'')$.

Proof. If $\mathfrak{p} \in \operatorname{Ass}_R(M)$ and $i : R/\mathfrak{p} \hookrightarrow M$ is an embedding, we distinguish between $\operatorname{im}(i) \cap M' = \{0\}$ and $\operatorname{im}(i) \cap M' \neq \{0\}$. In the first case, the $R/\mathfrak{p} \to M \to M''$ is injective, so $\mathfrak{p} \in \operatorname{Ass}_R(M'')$. In the second case, there exists some $x \in R \setminus \mathfrak{p}$ with $i(x) \in M'$, and then $R/\mathfrak{p} \xrightarrow{\cdot x} R/\mathfrak{p} \to M$ factors through M', so $\mathfrak{p} \in \operatorname{Ass}_R(M')$. □

Fact 1.6. *For any finitely generated* R*-module* M*,* $\operatorname{Ass}_R(M)$ *is finite.*

Proof. Any finitely generated R-module admits a finite filtration $0 = M_0 \subset M_1 \subset ... \subset M_{n-1} \subset M_n = M$ such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for prime ideals $\mathfrak{p}_i \lhd R$, $i = 1, 2, ..., n$. Then $\operatorname{Ass}_R(M) \subset \bigcup_i \operatorname{Ass}_R(R/\mathfrak{p}_i) = \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$. □

Fact 1.7. *Let* M *be a finitely generated* R*-module and* $I \lhd R$ *an ideal containing no* M*-regular element. Then* $I \subset \mathfrak{p}$ *for some associated prime* $\mathfrak{p} \in \operatorname{Ass}_R(M)$ *of* M*.*

Proof. By Fact 1.6 there are only finitely many associated primes, and by assumption I is contained in their union. By prime avoidance, the claim follows. □

In the special case $I = \mathfrak{m}$ we obtain:

Corollary 1.8. *Let* M *be an* R*-module. Then* $\operatorname{depth}_R(M) = 0$ *if and only if* $\mathfrak{m} \in \operatorname{Ass}_R(M)$*, i.e. if and only if* $\operatorname{Hom}_R(k, M) \neq 0$.

In fact, this Corollary admits the following very useful generalization giving an alternative description of the depth of a module:

Proposition 1.9. *For any finitely generated R-module M, we have*

$$\operatorname{depth}_R(M) \;=\; \min\{i \in \mathbb{N}_{\geq 0} \mid \operatorname{Ext}^i_R(k, M) \neq 0\} < \infty.$$

Moreover, any maximal M-regular sequence has length $\operatorname{depth}_R(M)$.

This follows from Corollary 1.8 and the following Lemma:

Lemma 1.10. *Let M, N be R-modules and $x_1, ..., x_n$ an M-regular sequence in M contained in* $\operatorname{Ann}_R(N)$. *Then there is a canonical isomorphism*

$$\operatorname{Ext}^n_R(N, M) \cong \operatorname{Hom}_R(N, M/(x_1, ..., x_k)M).$$

1.2. **Geometric considerations.** We want to provide some geometric intuition for regularity and associated primes. For this, recall first that any commutative ring R can be viewed as the ring of functions on its spectrum $\operatorname{Spec}(R)$, with ideals (resp. prime ideals) of R corresponding to closed (resp. closed and irreducible) subsets of $\operatorname{Spec}(R)$; in particular, the minimal prime ideals $\mathfrak{p}_1, ..., \mathfrak{p}_n$ of R correspond to the irreducible components $Z_1, ..., Z_n$ of $\operatorname{Spec}(R)$. Now, if $x \in \mathfrak{p}_1 \cap ... \cap \mathfrak{p}_{n-1} \setminus \mathfrak{p}_n$, then intuitively x is nonzero on Z_n but vanishes on all the $Z_1, ..., Z_{n-1}$, so should be annihilated by any $y \in \mathfrak{p}_n$. It turns out, however, that $\mathfrak{p}_1 \cap ... \cap \mathfrak{p}_n$ is not necessarily $\{0\}$, but consists precisely of the nilpotent elements of R – nevertheless, one can construct *some* element $x \in \mathfrak{p}_1 \cap ... \cap \mathfrak{p}_{n-1}$ such that $\operatorname{Ann}_R(x) = \mathfrak{p}_n$, and we have:

Fact 1.11. *Any minimal prime ideal is associated.*

The presence of nilpotent elements is a bit challenging from the elements-as-functions viewpoint, but one might imagine them as formal Taylor approximations in some "virtual" direction of $\operatorname{Spec}(R)$. The actual subset of $\operatorname{Spec}(R)$ they are "virtually extending" is then given by the zero set of their annihilator, so that we obtain the following geometric interpretation of associated primes:

Intuition. *Any associated prime corresponds either to an irreducible component of* $\operatorname{Spec}(R)$ *or to an irreducible subset of* $\operatorname{Spec}(R)$ *along which there is some "infinitesimal extension", which one might think of as a "virtual irreducible component".*

Building on this intuition, the regular elements of R can be thought of as those functions on $\operatorname{Spec}(R)$ that do not vanish on any actual or "virtual" irreducible component.

Example 1.12. Consider $R := k[\![x, y]\!]/(xy)$, the functions on the union of the two coordinate axis in the plane. Then R is reduced, and its associated primes are precisely the minimal primes (x) and (y) corresponding to the y-axis and x-axis, respectively. Now, passing to the quotient $R' := R/(y^2)$, geometrically the y-axis has vanished, but an infinitesimal part of it survives as witnessed by the nilpotent function $y \in R'$, based at $\operatorname{Ann}_{R'}(y) = (x)$. In other words, R' is the coordinate ring of the x-axis with an additional, infinitesimal y-axis attached to it at the origin, and $\operatorname{Ass}_{R'}(R') = \{(x, y), (y)\}$.

The following fits well with the above intuition:

Proposition 1.13. *The following inequalities holds:*

$$\begin{aligned}(1.1)\qquad \operatorname{depth}_R(R) &\leq \min\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}(R)\} \\ &\leq \max\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}(R)\} = \dim(R).\end{aligned}$$

Proof. This follows from $\operatorname{Ext}^i_R(N, M) = 0$ for all finitely generated R-modules N, M with $i < \operatorname{depth}_R(M) - \dim(N)$ (Ischebeck's Theorem, see [Mat89, Theorem 17.1]); note that $\dim(N) = 0$ is essentially the definition of depths, and the general case is obtained by induction, wlog restricting to the case $N = R/\mathfrak{p}$ by the argument in the proof of Fact 1.6. □

Definition 1.14. R is called *Cohen-Macaulay* if $\operatorname{depth}_R(R) = \dim(R)$.

For example, it follows from Proposition 1.13 that any Cohen-Macaulay ring is *unmixed* in the sense that it has no non-minimal associated primes, and all irreducible components of $\operatorname{Spec}(R)$ have the same dimension.

2. Classical Proof

Theorem 2.1 (Auslander Buchsbaum Formula)**.** *Let R be a local commutative Noetherian ring and M a finitely generated R-module with $\operatorname{pdim}_R(M) < \infty$. Then*

$$\text{(AB)}\qquad \operatorname{depth}_R(R) - \operatorname{depth}_R(M) = \operatorname{pdim}_R(M).$$

Proof. We present the proof given in [BH93, Theorem 1.3.3], which goes by ascending induction on $\operatorname{depth}_R(R)$.

If $\operatorname{depth}_R(R) = 0$, then Lemma 2.2 tells us that $\operatorname{pdim}_R(M) = 0$, hence M is free. In particular, $\operatorname{depth}_R(M) = \operatorname{depth}_R(R) = 0$, and (AB) holds.

Suppose now that $\operatorname{depth}_R(R) > 0$. If $\operatorname{pdim}_R(M) = 0$, again (AB) is trivial. If not, and if $\operatorname{depth}_R(M) = 0$, then denoting ΩM a first syzygy of M we have $\operatorname{pdim}_R(\Omega M) = \operatorname{pdim}_R(M) - 1$ while $\operatorname{depth}_R(\Omega M) = \operatorname{depth}_R(M) + 1$ by the Depth lemma 2.3. Hence, the Auslander-Buchsbaum formulas for M and ΩM are equivalent, and we may consequently assume $\operatorname{depth}_R(M) > 0$. In this case, we have $\mathfrak{m} \notin \operatorname{Ass}_R(M)$, and since also $\mathfrak{m} \notin \operatorname{Ass}_R(R)$, prime avoidance implies that there exists $x \in \mathfrak{m}$ which is both M- and R-regular. Then

$$\begin{aligned}\operatorname{depth}_{R/xR}(M/xM) &= \operatorname{depth}_R(M/xM) = \operatorname{depth}_R(M) - 1, \\ \operatorname{depth}_{R/xR}(R/xR) &= \operatorname{depth}_R(R/xR) = \operatorname{depth}_R(R) - 1, \quad \text{and} \\ \operatorname{pdim}_{R/xR}(M/xM) &= \operatorname{pdim}_R(M),\end{aligned}$$

and the Auslander Buchsbaum formula for R and M follows by induction from the Auslander Buchsbaum formula for R/xR and M/xM. □

Lemma 2.2. *If* $\operatorname{depth}_R(R) = 0$ *and* $\operatorname{pdim}_R(M) < \infty$*, then* $\operatorname{pdim}_R(M) = 0$.

Proof. From our assumption $\operatorname{depth}_R(R) = 0$ we infer $\mathfrak{m} \in \operatorname{Ass}_R(R)$, i.e. we have an embedding $\iota : k \hookrightarrow R$. Then, if $\varphi : F \to G$ is a homomorphism between nonzero free R-modules F and G, we have a commutative diagram of R-modules

$$\begin{array}{ccccc} F \otimes_R k & \xrightarrow{\mathrm{id}_F \otimes \iota} & F \otimes_R R & \xrightarrow{\cong} & F \\ \downarrow{\scriptstyle \varphi \otimes \mathrm{id}_k} & & \downarrow{\scriptstyle \varphi \otimes \mathrm{id}_R} & & \downarrow{\scriptstyle \varphi} \\ G \otimes_R k & \xrightarrow{\mathrm{id}_F \otimes \iota} & G \otimes_R R & \xrightarrow{\cong} & G \end{array}$$

If here φ is chosen to be minimal in the sense that $\varphi \otimes_R \mathrm{id}_k = 0$ (i.e. if the coefficients of φ, when written as a matrix, all belong to $\mathfrak{m}$), we infer that

$$\operatorname{im}(F \otimes_R k \xrightarrow{\mathrm{id}_F \otimes \iota} F \otimes_R R \cong F) \subseteq \ker(\varphi),$$

hence in particular φ is not injective. However, in a bounded and minimal free resolution of a non-projective R-module the leftmost non-zero differential would be a minimal and injective homomorphism between free R-modules, so we infer that such a resolution cannot exist, as claimed. □

Lemma 2.3 (Depth lemma). *Let* $\operatorname{depth}_R(M) < \operatorname{depth}_R(R)$ *and let* ΩM *be a syzygy of* M*. Then* $\operatorname{depth}_R(\Omega M) = \operatorname{depth}_R(M) + 1$.

Proof. Pick a short exact sequence $0 \to \Omega M \to P \to M \to 0$ with P free. Then the long exact Ext-sequence shows that for all $i \leq \operatorname{depth}_R(R)$ there is a monomorphism $\operatorname{Ext}_R^{i-1}(k, M) \hookrightarrow \operatorname{Ext}_R^i(k, \Omega M)$, which for $i < \operatorname{depth}_R(R)$ is even an isomorphism. The claim follows. □

3. An application

Definition 3.1. The *finitistic global dimension* $\operatorname{f.gl.dim}(R\text{-mod})$ is defined as

$$\operatorname{gl.dim}(R\text{-mod}) \ := \ \sup\{\operatorname{gl.dim}_R M \mid M \in R\text{-mod}, \operatorname{gl.dim}_R M < \infty\}.$$

This definition also makes sense if R is not commutative, and an important first question is to ask whether the finitistic global dimension of some ring is finite. While for finite-dimensional algebras over fields this seems open, the Auslander-Buchsbaum formula AB settles the question affirmatively:

Corollary 3.2. *For a local (commutative) Noetherian ring* R*, we have*

$$\operatorname{f.gl.dim}(R\text{-mod}) \ \leq \ \operatorname{depth}_R(R) < \infty.$$

In particular, the finitistic global dimension of R *is finite.*

Proof. If M is a finitely generated R-module with $\operatorname{pdim}_R M < \infty$, the Auslander-Buchsbaum formula tells us $\operatorname{pdim}_R M = \operatorname{depth}_R R - \operatorname{depth}_R M \leq \operatorname{depth}_R R$. □

4. New techniques & Generalizations

The key arguments for the results described in this section are contained in [FI03], in particular [FI03, Theorem 2.4].

4.1. A proof using derived categories. Using derived categories, we can provide an alternative proof of the Auslander-Buchsbaum formula. We have the following sequence of isomorphisms in $\mathbf{D}(R\text{-Mod})$:

$$\begin{aligned}\mathbf{R}\operatorname{Hom}_R(k,R)\otimes^{\mathbf{L}}_k(k\otimes^{\mathbf{L}}_R M) &\cong \mathbf{R}\operatorname{Hom}_R(k,R)\otimes^{\mathbf{L}}_R M\\ &\xrightarrow[\cong]{\Theta} \mathbf{R}\operatorname{Hom}_R(k,R\otimes^{\mathbf{L}}_R M)\\ &\cong \mathbf{R}\operatorname{Hom}_R(k,M).\end{aligned}\tag{4.1}$$

Here, the first isomorphism is an instance of the projection formula, and for the second isomorphism note that there is always an arrow as indicated, which is an isomorphism for $M = R$ and hence also for any complex quasi-isomorphic to a bounded complex of finitely generated projective R-modules.

The Auslander-Buchsbaum formula now follows by looking at the highest degree in which one has cohomology on both sides of (4.1): for the left hand side, it is $\operatorname{depth}_R(R)+\operatorname{pdim}_R(M)$, while for the right hand side it is $\operatorname{depth}_R(M)$.

4.2. Generalizations. It is instructive to study the proof (4.1) further: The main point in it is the isomorphism Θ, which is essentially the reflexivity of M with respect to $(-)^\vee := \mathbf{R}\operatorname{Hom}_R(-,R)$ and the fact that $\mathbf{R}\operatorname{Hom}_R(M,N)\cong M^\vee\otimes^{\mathbf{L}}_R N$ for any perfect M. Namely, we can rewrite Θ as a sequence of isomorphisms

$$\begin{aligned}\mathbf{R}\operatorname{Hom}_R(k,R)\otimes^{\mathbf{L}}_R M &\cong \mathbf{R}\operatorname{Hom}_R(k,R)\otimes^{\mathbf{L}}_R M^{\vee\vee}\\ &\cong \mathbf{R}\operatorname{Hom}_R(M^\vee,\mathbf{R}\operatorname{Hom}_R(k,R))\\ &\cong \mathbf{R}\operatorname{Hom}_R(M^\vee\otimes^{\mathbf{L}}_R k,R)\\ &\cong \mathbf{R}\operatorname{Hom}_R(k,\mathbf{R}\operatorname{Hom}_R(M^\vee,R))\\ &\cong \mathbf{R}\operatorname{Hom}_R(k,M^{\vee\vee})\\ &\cong \mathbf{R}\operatorname{Hom}_R(k,M);\end{aligned}$$

here, it is only the second isomorphism where the perfectness of M actually plays a role, and only the first and last isomorphism where the reflexivity of M is important. Moreover, we see that we use nothing particular about R in the above sequence of isomorphisms, only the assumption that M is reflexive with respect to $\mathbf{R}\operatorname{Hom}_R(-,R)$. Hence, we may summarize:

Proposition 4.1. *Let M,ω be complexes of R-modules such that M is reflexive w.r.t. $\mathbb{D}_\omega := \mathbf{R}\operatorname{Hom}_R(-,\omega)$. Then there are canonical isomorphisms in $\mathbf{D}(R\text{-Mod})$:*

$$\mathbf{R}\operatorname{Hom}_R(k,M)\cong\mathbf{R}\operatorname{Hom}_R(\mathbb{D}_\omega M,\mathbb{D}_\omega k)\cong\mathbf{R}\operatorname{Hom}_k(k\otimes^{\mathbf{L}}_R\mathbb{D}_\omega M,\mathbb{D}_\omega k)\tag{4.2}$$

If in the situation of Proposition 4.1 the complex $\mathbb{D}_\omega M$ is bounded with finitely generated cohomology, then looking at the lowest degree of cohomology in both sides of (4.2), we obtain the equality

$$\operatorname{depth}_R(M) =: \inf \mathbf{R}\operatorname{Hom}_R(k, M) = \inf \mathbb{D}_\omega k - \sup \mathbb{D}_\omega M. \tag{4.3}$$

Example 4.2. We recover the classical Auslander-Buchsbaum formula if $\omega := R$ and if M is a finitely generated R-module of finite projective dimension, since $\mathbb{D}_\omega k = \mathbf{R}\operatorname{Hom}_k(k, R)$ computes $\operatorname{depth}_R(R)$ and $\mathbb{D}_\omega M = \mathbf{R}\operatorname{Hom}_R(M, R)$ computes $\operatorname{pdim}_R(M)$ in this case.

Example 4.3. More generally, we say that M is of *finite Gorenstein-projective dimension* if it is reflexive with respect to $\mathbf{R}\operatorname{Hom}_R(-, R)$ and if $\mathbf{R}\operatorname{Hom}_R(M, R)$ is cohomologically bounded. In this case, the largest degree of cohomology of $\mathbf{R}\operatorname{Hom}_R(M, R)$ is called the Gorenstein-projective dimension $\operatorname{gp-dim}_R(M)$ of M; see [Chr00], in particular [Chr00, Theorem 2.2.3]. Hence, (4.3) generalizes the Auslander-Buchsbaum formula to the *Auslander-Bridger formula*[AB69, Theorem 4.13]: For any finitely generated R-module M of finite Gorenstein-projective dimension, we have

$$\operatorname{gp-dim}_R(M) = \operatorname{depth}_R(R) - \operatorname{depth}_R(M).$$

The Gorenstein-projective dimension is finite for all finitely generated R-modules if and only if R is Gorenstein, i.e. of finite injective dimension over itself.

Finally, note there can be no concept of dimension which makes the analogue of the Auslander-Buchsbaum and Auslander-Bridger formulas valid without any assumptions on the module M, as in general $\operatorname{depth}_R(M) \not\leq \operatorname{depth}_R(R)$: for example, taking $R := k[\![x, y]\!]/(xy, y^2)$, we have $\operatorname{depth}_R R = 0$ but $\operatorname{depth}_R R/(y) = 1$.

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