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The Krull-Gabriel dimension of the derived discrete algebras – Grzegorz Bobiński

$k = \bar{k}$. [joint with Krause]

§1. Derived discrete algebras

Definition [Vossieck]: A is derived discrete if for all $\underline{h} \in \mathbb{N}^{\mathbb{Z}}$

$$\# \{ X \in \operatorname{ind} \mathcal{D}^b(\operatorname{mod} A) \mid \dim H^n(X) = h_n \} /_{\simeq} < \infty .$$

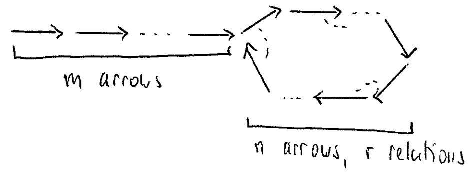
Theorem [Vossieck]: A is derived discrete iff either

(1) A piecewise hereditary Dynkin (finite type) or

(2) A gentle one-cycle satisfying no-clock condition.

Theorem [B-Geiss-Skowroński]: A derived discrete non-Dynkin. Then $\exists\, n, r, m \in \mathbb{N}$, $0 < r \leq n$ such that $A \sim_{der} \Lambda(n,r,m)$ where

$\Lambda(n,r,m)$:



Here, $\Lambda(n,r,m) \sim_{der} \Lambda(n',r',m')$ iff $(n,r,m) = (n',r',m')$

Remark: $\operatorname{inv}_{AG} \Lambda(n,r,m) = \{ (n+r, m), (n-r, n) \}$

A, A' der. dis. non-Dynkin : $A \sim_{der} A' \iff \operatorname{inv}_{AG} A = \operatorname{inv}_{AG} A'$.

Some results: (1) [BGS] AR-quiver

(2) [B] Graded centers

(3) [Han] Homotopy category of inj. modules

(4) [Broomhead-Pauksztello-Ploog] Autoequivalences, t-structures, ...

§2. Krull-Gabriel dimension

Let $\mathcal{A}$ be an abelian category, $\mathcal{C} \subseteq \mathcal{A}$ Serre subcategory (full subcat., closed under ext., subobj. and quotient obj.)

Definition: $\mathcal{A}/\mathcal{C}$:
$$\mathrm{Ob}(\mathcal{A}/\mathcal{C}) := \mathrm{Ob}(\mathcal{A})$$
$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(X,Y) := \varinjlim_{\substack{X'\subseteq X,\ X/X'\in\mathcal{C} \\ Y'\subseteq Y,\ Y'\in\mathcal{C}}} \mathrm{Hom}(X', Y/Y')$$

$\Pi_{\mathcal{C}} : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{C}$.

KG-dimension [Geigle, Gabriel] : $\mathcal{A}_{-1} = 0$, $\mathcal{A}_n = \Pi^{-1}_{\mathcal{A}_{n-1}}\left(\mathrm{fin}\left(\mathcal{A}/\mathcal{A}_{n-1}\right)\right)$, [e.g. $\mathcal{A}_0 = \mathrm{fin}\,\mathcal{A}$]

then $KG(\mathcal{A}) = \min\{ n \in \mathbb{N} \mid \mathcal{A}_n = \mathcal{A}\}$

Module category case: $\mathcal{A} := \mathrm{fp}(\mathrm{mod}\, A) := \{F : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, k \mid F \simeq \mathrm{Coker}\, H_f\}$,
where $H_* := \mathrm{Hom}(*, -)$.

$$\{\text{simple objects}\} \longleftrightarrow \{\text{AR-sequences}\} \supset \left(0\to 0 \to \mathrm{rad}(P) \to P \to 0\right)$$
[Auslander]

- [Auslander] : $KG(\mathcal{A}) = 0 \iff A$ f.r.t.
- [Krause, Herzog] : $KG(\mathcal{A}) \neq 1$.
  $\exists$ (good) generic modules
- [Burke-Prest, Schröer] : $\forall\, n \geq 2\ \exists A$ such that $KG(\mathcal{A}) = n$.

Triangulated case

$\mathcal{T}$ triangulated, $\mathcal{A} := \mathrm{fp}(\mathcal{T})$.

Remark: $\mathcal{A}$ is the abelianization of $\mathcal{T}$, i.e. $\mathcal{T} \xrightarrow{i} \mathcal{A}$, $X \longmapsto H_X$, cohomological.
$\forall\ \varphi : \mathcal{T} \longrightarrow \mathcal{A}'$ cohomological, $\exists!\ \varphi' : \mathcal{A} \longrightarrow \mathcal{A}'$ s.t. $\varphi = \varphi' \circ i$ and φ' is exact.

In our case : $\mathcal{T} = K^b(\mathrm{proj}\, A)$ or $\mathcal{T} = \mathcal{D}^b(\mathrm{mod}\, A)$

0° : A piecewise hereditary Dynkin $\rightsquigarrow KG(\mathcal{T}) = 0$

1° : A derived discrete non Dynkin

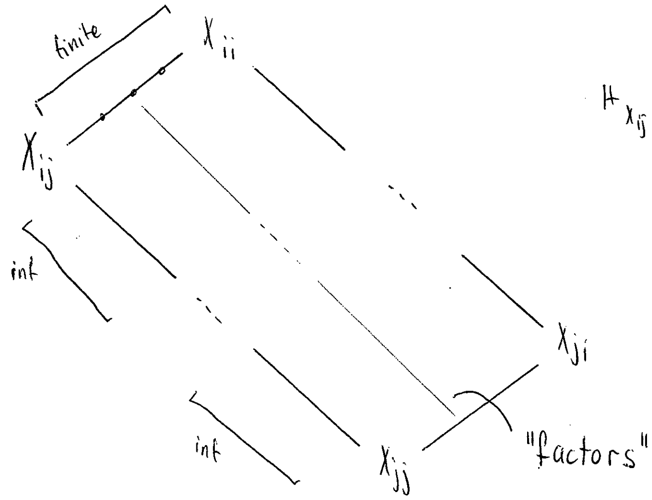
Prototypical example: $A = k[\varepsilon]/\varepsilon^2 = \Lambda(1,1,0)$. $\overbrace{\text{Ind. obj}}^{\mathcal{T} = K^b(\mathrm{proj}\, A)}$: $X_{ij} = \cdots \to 0 \to \underset{i}{A} \to \cdots \to \underset{j}{A} \to 0 \to \cdots$
for all $i \geq j$.

Morphisms :

$$\begin{array}{ccccccccccccccc}
\cdots \to 0 \to A \to \cdots \to & A & \to & \cdots \to & A & \to 0 \to \cdots & & \\
& \downarrow\varepsilon & & \downarrow 0 & \downarrow 0 & & & \\
\cdots \to 0 \to & A & \to & \cdots \to & A & \to \cdots \to A \to 0 \to \cdots & &
\end{array}$$

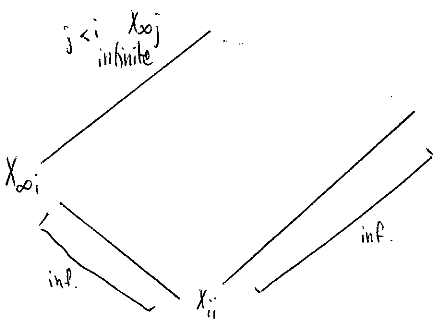
or

$$\begin{array}{ccccccccc}
& 0 \to & A & \to \cdots \to & A & \to \cdots \to A \to 0 \to \cdots \\
& & \downarrow 1 & & \downarrow 1 & \\
\cdots \to 0 \to A \to \cdots \to & & A & \to \cdots \to & A & \to 0 \to \cdots
\end{array}$$



$H_{X_{ij}} \notin \mathrm{fin}\, \mathcal{A}$

Now, $\mathcal{T} = \mathcal{D}^b(\mathrm{mod}\, A)$, Ind. obj. . . $\cdots \xrightarrow{\varepsilon} A \longrightarrow \cdots \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \longrightarrow 0 \longrightarrow \cdots = X_{\infty i}$



<u>Theorem [B-Krause]:</u> $$KG\left(\mathrm{fp}(K^b(\mathrm{proj}\, A))\right) = \begin{cases} 1 & \mathrm{gldim}\; A = \infty \quad (r = n) \\ 2 & \mathrm{gldim}\; A < \infty \end{cases}$$

$$KG\left(\mathrm{fp}(\mathcal{D}^b(\mathrm{mod}\, A))\right) = 2$$

<u>Theorem [B-Krause]:</u> A not derived discrete. Then $KG\left(\mathrm{fp}(\mathcal{T})\right) \geq 2$.

$k(t)$ embeds in $\mathrm{End}(X)$

<u>Crucial [Bautista]:</u> $\exists$ good generic object in $\mathcal{D}(\mathrm{Mod}\, A)$,
X ind. $\notin \mathcal{D}^b(\mathrm{mod}\, A)$, $H^d(X)$ fin. length $\mathrm{End}(X)$