McKay correspondence for Reflection Groups? 1

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Auslander 1986
Auslander - Platzeck - Todorov 1992

Classical McKay Correspondence

$G \underset{\text{finite}}{\subseteq} Sl_2(\mathbb{C})$

Reflection Groups

$\exists G' \subseteq Gl_2(\mathbb{C})$
$[G':G]=2$, G' gen. by reflections
$G' \supset G$

("always known" ?!)

G small = no (pseudo-)reflections

<u>Def</u>: $g \in Gl_n(\mathbb{C})$ <u>pseudo-reflection</u> if Fixed(g) is a hyperplane (= "mirror")
$g \in G \subseteq Gl_n(\mathbb{C})$, g has finite order

$$g \longleftrightarrow \begin{pmatrix} \rho^{\pm 1} & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

ρ is a root of unity
$\det g = \rho \neq 1$ (i.e. $g \notin Sl_n(\mathbb{C})$)

<u>Notation</u>: $G \underset{\text{finite}}{\subseteq} Gl_n \cong Gl(V)$, V n-dim. v.s.

$S = \mathrm{Sym}_K(V) \cong K[x_1, \ldots, x_n] \supset G$, $R = S^G$

$A = S * G$ twisted group algebra

$\mathrm{End}_R(S) \longleftarrow S*G = A$

G small, this is an isomorphism.

If G contains pseudo-reflections, this is not an isomorphism.

$A \not\ni \frac{1-g}{x_1} \in \mathrm{End}_R(S)$

<u>Thm</u> (Kostant-Kumar 1988)
Q field of fractions of S
Q^G = field of fractions of R

$$\begin{array}{ccc} S*G = A & \hookrightarrow & \mathrm{End}_R(S) =: T \\ \downarrow & \circlearrowleft & \downarrow \\ A \otimes_S Q & \xrightarrow{\cong} & \mathrm{End}_{Q^G}(Q) \end{array}$$

$A = T \cap {}^{\tau}T$

$\tau: A \cong A$

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Goal: Consider $e = \frac{1}{|G|} \sum_{g \in G} g \in A$

Goal: (G reflection group)

$$\bar{A} = A/AeA$$

is a noncom. desingularization of the discriminant of the group action

(1) $\operatorname{gldim} \bar{A} < \infty$ (APT)

$\operatorname{gldim} \bar{A} = n$ except for trivial cases

(2) $\bar{A}$ is MCM as module on the discriminant (✓)

(3) $\bar{A} \cong \operatorname{End}_{R/(\Delta)}(M)$??

G acts on

$$\begin{tikzcd} \mathbb{C}^n \arrow[d] & \supseteq & \text{branch locus} = \bigcup_{g \text{ pseudo reflection}} H_g \arrow[d, "p"] \\ \mathbb{C}^n/G \cong \mathbb{C}^n & \supseteq & \Delta \end{tikzcd}$$

$H_g = \operatorname{Fix}(g)$

Ex: S_{n+1} acts on $V/_{x_0 = \dots = x_n}$ n-dim rep

invariants = elementary sym. fcts

discriminants = classical discription of polys of deg $= n+1$

Fact $\underset{z}{\underbrace{\operatorname{Jac}(p_1, \dots, p_n)}} = \det\left(\frac{\partial p_i}{\partial x_j}\right) = \prod_{g \text{ pseudo r.}} x_g$ $x_g = 0$ equation of the mirror

$= \prod_k \ell_k^{\alpha_k - 1}$ α_k = order of p-refl. with mirror $\ell_k = 0$

$\underset{\Delta}{\underbrace{\operatorname{disc}}} = \prod_k \ell_k^{\alpha_k} \in R$

If G is generated by (true) reflections, then $\Delta = z^2$

$G \subseteq Sl_2(\mathbb{C})$

$G' \subset Gl_2(\mathbb{C})$ gen. by reflections

$\tilde{R} = k[x, y, z] = S^G$

$\cup$

$R = k[x, y] = S^{G'}$

$z^2 \in R$, $z^2 - g(x,y) = 0$ the simple plane singularities occur as the discriminants of these reflection groups

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Case: D_4 $\quad S = \mathbb{C}[u, v]$

$x = u^4 + v^4$

$y = (uv)^2$

$|z| = 6$

K. Saito: These discriminants are free divisors (singular in codim one)

$A = S * G \longleftarrow S$ ring homomorphism

Consider A as S- bimodule

$0 \to AeA \to A \to \bar{A} \to 0$

$A = \bigoplus_{g \in G} S_g$ each S_g is an S-bimodule

$S \otimes_k S = k[\underline{x}, \underline{y}]$ $\quad S_g \cong S \otimes_k S / (y_i - g(x_i))_i \cong S$

$X = \operatorname{Spec} S \cong$ affine space

$X \times X = \operatorname{Spec} S \otimes S$

S_g is supported on $\{x, gx \in X \times X\}$

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$\mathrm{Spec}(S \otimes_R S) = X \times_{X/G} X$

$A e A$ cyclic $S \otimes_R S$-module

$S \otimes_R S$ is CM (even ci)

generically $S \otimes_R S \longrightarrow AeA$ injective

$\Rightarrow$ it is an isom

$\mathrm{Spec}(S \otimes_R S)$ = graph of group action

$\uparrow$ normalization

$\mathrm{Spec}(\bigoplus \mathcal{O}(1 \ast g)(X))$ $\qquad$ $\bar{A}$ CM of depth $= n-1$

G finite reflection group

$\bar{A}$ as ~~$\bar{R}$~~ $\bar{R} = R/(\Delta)$ - module

has rank $\left(\frac{|G|}{2}\right)^2$ if the discriminant is irreducible

proof: $\quad H_{\bar{A}}(t) = H_A(t) - H_{AeA}(t)$

$$= \frac{|G|}{(1-t)^n} - \frac{\prod_{i=1}^{n}(1-t^{d_i})}{(1-t)^{2n}}$$

$$\Rightarrow \lim \frac{H_{\bar{A}}(t)}{H_{\bar{R}}(t)} = \left(\frac{|G|}{2}\right)^2.$$

$H_{\bar{R}}(t) = \dfrac{1-t^{2|z|}}{\prod_{i=1}^{n}(1-t^{d_i})}$ $\qquad |z| = \Sigma(d_i - 1)$

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Ex: $S = k[x] \circlearrowleft \mu_{d+1}$ through multiplication

$R = k[x^{d+1}]$

$z = x^d$

$\Delta = x^{d+1}$

$$R/(\Delta) \cong k$$

$\overline{R} = k(\vec{\mathbb{A}}_d)$ path alg of Dynkin quiver.

Guo - Martinez - Villa 2002

$G \subseteq Gl_n$ finite

$$A = S * G \underset{\text{Morita eq}}{\sim} k(\text{McKay graph})/\text{relations}$$