Representations of $Sl_2(\mathbb{R})$ and derived Auslander – Reiten theory

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15–16h
(jt. with
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H compact real Lie group $\Longrightarrow$ ① $\mathrm{Rep}^{fd}(H)$ is semi-simple

② If $V \in \mathrm{Irr}\,\mathrm{Rep}(H) =: \check{H}$
$\Longrightarrow \dim_{\mathbb{C}} V < \infty$

③ Combinatorics of $\check{H}$ is well-understood.

G is real linear reductive Lie group ($\ni Sl_n(\mathbb{R}), Gl_n(\mathbb{R})$)

$G \supset K$ maximal compact subgroup (unique up to conjugation)

$$R(G) \xrightarrow{HC} (\mathfrak{g}, K)\text{-moduls} \xrightarrow{\cong} \bigoplus_{\chi \in \mathcal{U}(\mathfrak{g})^G} \Lambda_\chi\text{-fdmod}$$

(I) $R(G) = \{ V \in \mathrm{Rep}(G) \mid \dim_{\mathbb{C}}(U, V) < \infty \ \forall\, U \in \check{K}$, V has finite length$\}$

$\Downarrow$

V is called <u>admissible</u>.

(II) $\mathfrak{g} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$

$(\mathfrak{g}, K)\text{-mod} = \left\{ \begin{array}{l} \mathcal{U}(\mathfrak{g})\text{-mod } \underline{\text{and}} \\ \text{admissible} \\ \text{representation of } K \end{array} \middle| \begin{array}{l} \text{compact} \\ \text{actions} \end{array} \right\}$

Objects are called <u>Harish–Chandra modules</u>.

$HC(V) = \{ v \in V \mid \dim_{\mathbb{C}} Kv < \infty \}$

HC is exact, faithful with fully faithful right/left adjoints

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III $\chi \in \mathrm{Specm}\,(\mathcal{U}(\mathfrak{g})^G)$

$\Rightarrow \Lambda_\chi$ is a Cohen-Macaulay order of

$$\begin{aligned} \text{Krull dim} &= \mathrm{rk}(G) \\ \mathrm{gldim} &\leq c = \dim_{\mathbb{R}}(G/K) \\ \mathrm{gldim}(\Lambda_0) &= c \quad \text{for } \Lambda_0 \ni \text{trivial representation.} \end{aligned}$$

<u>Ex:</u> $G = SL_2(\mathbb{R}) \supset K = SO_2(\mathbb{R})$

$\dim_{\mathbb{R}}$: $\quad 3 \qquad\qquad 1$

$\mathfrak{g} = sl_2(\mathbb{C})$

$\mathrm{rk} = 1$

$c = 3 - 1 = 2$

$$\Lambda_\chi \in \left\{ \begin{tikzcd} \cdot \arrow[loop above] \end{tikzcd} \quad \begin{tikzcd} + \arrow[r, bend right, "\beta_+"'] & \cdot \arrow[l, bend right, "\alpha_+"'] \arrow[r, bend left, "\alpha_-"] & - \arrow[l, bend left, "\beta_-"] \end{tikzcd} \right\}$$

Gelfand quiver Γ

$$\Gamma = \begin{pmatrix} D & m & m \\ D & D & m \\ D & m & D \end{pmatrix} \qquad \begin{aligned} &D = \mathbb{C}[[t]] \supset m = (t) \\ &\mathrm{gldim}\,\Gamma = 2 \end{aligned}$$

I. Gelfand (ICM, 70) describe Ind (rep Γ)

1973 Nazarova and Roiter : tameness
1988 Bondarenko : exact combinatorics
1980 Khoroshkin : tameness
1988 Crawley-Boevey : precise combinatorics

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How to state the answer in "easy words"?

$$\mathrm{rep}(\Gamma) \ni \underbrace{S_*}_{\text{trivial representation}} = (0 \rightleftarrows \mathbb{C} \rightleftarrows 0)$$

$R(G)$

$$\mathrm{Ext}^p_\Gamma(S_*, S_*) = \begin{cases} \mathbb{C} & , p = 0,2 \\ 0 & , \text{otherwise} \end{cases}$$

$\Longrightarrow S_*$ is 2-spherical.

$$\Omega = R\mathrm{Hom}_D(\Gamma, D) = \begin{pmatrix} m & m & m \\ D & m & D \\ D & D & m \end{pmatrix}$$ dualizing module.

$\tau = \Omega \overset{L}{\otimes} - : D^b(\mathrm{rep}\,\Gamma) \xrightarrow{\sim} D^b(\mathrm{rep}\,\Gamma)$ is a Serre functor:

$\mathrm{Hom}(X, Y) \cong \mathrm{Ext}^1(Y, \tau X)^*$ [vd Bergh 2003]

$\tau(S_*) = S_*[1]$

Lemma: $T_{S_*} \cong i^* \circ \tau$ Seidel–Thomas twist
"can be written in terms of tensoring with a bimodule"
[Iyama-Reiten].

Def: $X \in D^b(\mathrm{rep}\,\Gamma)$

$$\partial(X) = \sum_{p \in \mathbb{Z}} \dim \mathrm{Ext}^p(S_*, X)$$ defect of X

Lemma: $$\partial(X) = \partial(\tau X) = \sum_{p \in \mathbb{Z}} \dim \mathrm{Ext}^p(X, S_*)$$ = total relative cohomology dimension.

Rem: $M \in (\mathfrak{g}, K)$-mod

$\mathrm{Ext}^p_{(\mathfrak{g}, K)}(\mathbb{C}, M)$ relative Lie cohomology of M

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$$\underbrace{\begin{pmatrix} D & m & m \\ D & m & m \\ D & m & m \end{pmatrix}}_{I} \subset \underbrace{\begin{pmatrix} D & m & m \\ D & D & m \\ D & m & D \end{pmatrix}}_{\Gamma} \subset \underbrace{\left(\begin{array}{cc|c} D & m & m \\ D & D & D \\ \hline D & D & D \end{array}\right)}_{\Lambda}$$

$$I = \Gamma I \Gamma = \Lambda I \Lambda \qquad = \begin{pmatrix} D & m \\ D & D \end{pmatrix}$$

$$\bar{\Gamma} := \Gamma / I = k \times k$$

$$\downarrow$$

$$\bar{\Lambda} := \Lambda / I = \mathrm{Mat}_2(k)$$

$$\begin{tikzcd}
D^b(\operatorname{rep}\Gamma) \arrow[r, "\Lambda \overset{L}{\otimes}_{\Gamma} -"] \arrow[d] & D^b(\operatorname{rep}\Lambda) \arrow[d, "\bar{\Lambda} \overset{L}{\otimes}_{\Lambda} -"] & \ni \mathcal{F} \\
D^b(\operatorname{rep}\bar{\Gamma}) \arrow[r, "\bar{\Lambda} \overset{L}{\otimes}_{\bar{\Gamma}} -"'] & D^b(\operatorname{rep}\bar{\Lambda}) &
\end{tikzcd}$$

$$\Psi$$

$$V$$

category of triples

$\mathrm{Tri}(\Gamma) \ni (\mathcal{F}, V, \theta)$ objects,

morph.: (ϕ, φ)

$(\mathcal{F}', V', \theta')$

$$\begin{tikzcd}
\bar{\Lambda} \otimes_{\bar{\Gamma}} V \arrow[r, "\theta", "\cong"'] \arrow[d, "1 \otimes \varphi"'] & \bar{\Lambda} \otimes_{\Lambda} \mathcal{F} \arrow[d, "1 \otimes \phi"] \\
\bar{\Lambda} \otimes_{\bar{\Gamma}} V' \arrow[r, "\theta'", "\cong"'] & \bar{\Lambda} \otimes_{\Lambda} \mathcal{F}'
\end{tikzcd}$$

Theorem I (Burban – Drozd)

The functors

$$D^b(\operatorname{rep}\Gamma) \xrightarrow{\mathbb{F}} \mathrm{Tri}(\Gamma) \xrightarrow{\mathbb{E}} \mathrm{Rep}(\mathcal{B}_\Gamma)$$

$$\mathcal{G} \longmapsto (\Lambda \otimes \mathcal{G}, \bar{\Gamma} \otimes \mathcal{G}, \theta_{\mathcal{G}})$$

$$(\mathcal{F}, V, \theta) \longmapsto \theta$$

are representation equivalences.

bimodule category "matrix problem" $\longleftrightarrow$ {collection of matrices}

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Thm II (Bondarenko)

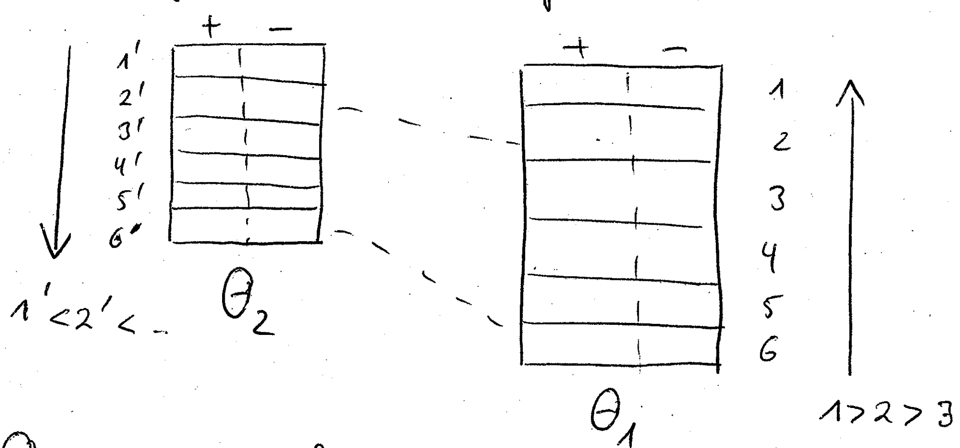
$\mathrm{Rep}(B_\Gamma)$ is tame and indecomposables are

1. bands (continuous series)
2. $\left.\begin{array}{l}\text{bispecial}\\ \text{special}\\ \text{usual}\end{array}\right\}$ strings (discrete series)

$$\begin{array}{ccccccccc}
R(G) & \xrightarrow{HC} & (\mathfrak{g},K)\text{-mod} & \longrightarrow & \mathrm{rep}(\Gamma) & \hookrightarrow & D^b(\mathrm{rep}\,\Gamma) & \longrightarrow & \mathrm{Rep}(B_\Gamma) \\
 & & & & \cup & & & & \cup \\
 & & & & \mathrm{rep}_0(\Gamma) & & \longrightarrow & & \mathrm{rep}_0(B_\Gamma) \\
 & & & & \| & & & & \\
 & & & & \{X \mid \mathrm{proj\,dim}\, X = 1\} & & & &
\end{array}$$

In $\mathrm{rep}_0(B_\Gamma)$:

$\Theta_1 = (\Theta_1^+ \mid \Theta_1^-)$ and $\Theta_2 = (\Theta_2^+ \mid \Theta_2^-)$ are square matrices
conjugate horizontal stripes have the same of rows



1. One may perform arbitrary transformations of columns inside of $\Theta_i^\pm$, $i = 1,2$
2. One may perform simultaneous transformations of rows inside conjugated (i.e. connected by dotted line) horizontal stripes
3. One may add any multiple of any row of Θ_i with lower weight to any row of Θ_i of higher weight.

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Main Thm

Let $X \in \operatorname{Ind} \mathcal{D}^b(\operatorname{rep} \Gamma)$

(I) $\partial X \in \{0, 1, 2\}$

(II) X band $\iff \begin{cases} \partial X = 0 \\ i^* X = X \end{cases}$

$\iff \tau X \cong X$

X bispecial string $\iff \begin{cases} \partial X = 0 \\ i^* X \not\cong X \end{cases}$

$\iff \begin{cases} \tau X \not\cong X \\ \tau^2 X \cong X \end{cases}$

X special string $\iff \partial X = 1$

X usual string $\iff \partial X = 2$

(III) All 4 types of indecomposables are stable wrt τ, i^*, Matlis duality ($\leftrightarrow$ Lie dual)

(IV) $X \in \operatorname{Rep}(\Gamma)$

$\operatorname{projdim} X = 2 \iff S_* \in \operatorname{add}(\operatorname{soc}(X))$

$\iff H^0_{(\mathfrak{g},K)}(X) \neq 0$

$\operatorname{injdim} X = 2 \iff S_* \in \operatorname{add}(\operatorname{top}(X))$

$\iff H^2_{(\mathfrak{g},K)}(X) \neq 0.$