Higher Auslander correspondence and d-Cohen-Macaulay finiteness

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16:30-17:30

1. Auslander correspondence
2. Cohen-Macaulay representations
3. Geigle-Lenzing complete intersections

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1. Λ, Γ finite dimensional algebra / field k

mod Λ

Functorial method: Regard mod Λ as a ring, then study their homological properties

$\boxed{\text{Additive category} = \text{rings with several objects}}$

Prototype: Λ : representation-finite

mod Λ = add M

(mod Λ as a ring) $= \mathrm{End}_\Lambda(M) = \Gamma$

Auslander [71]

$$\{\Lambda \mid \text{rep.-fin}\} \underset{\substack{\text{(up to}\\ \text{Morita-eq)}}}{\overset{1-1}{\longleftrightarrow}} \underbrace{\left\{\Gamma \,\middle|\, \begin{array}{l} \mathrm{gldim}\,\Gamma \le 2 \\ 2 \le \mathrm{domdim}\,\Gamma \end{array}\right\}}_{\text{Auslander algebra}}$$

$$\text{mod }\Lambda = \text{add}M \longmapsto \mathrm{End}_\Lambda(M)$$

Def:

$0 \to {}_\Gamma\Gamma \to I^0 \to I^1 \to I^2 \to \cdots$ min inj res.

$\ell \le \mathrm{domdim}\,\Gamma \iff I^0, \ldots, I^{\ell-1} \in \mathrm{proj}\,\Gamma$

So $2 \le \mathrm{domdim}\,\Gamma$, $\mathrm{gldim}\,\Gamma \le 2$

$$0 \to \Gamma \to \underbrace{I^0 \to I^1}_{\in \mathrm{proj}\,\Gamma} \to I^2 \to 0$$

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Gorenstein condition

Γ Auslander algebra

$$\left\{\begin{matrix}\text{simple } \Gamma\text{-module with} \\ \text{proj dim} = 2\end{matrix}\right\} \xleftrightarrow{\quad} \left\{\begin{matrix}\text{simple } \Gamma^{op}\text{-module} \\ \text{with proj dim} = 2\end{matrix}\right\}$$

$$\xrightarrow[\operatorname{Ext}^2_\Gamma(-,\Gamma)]{}$$

R Gorenstein ring with $\dim = d$.

$$\operatorname{Ext}^i(k,R) = \begin{cases} 0 & , i \neq d \\ k & , i = d \end{cases}$$

This duality means

"selfduality of AR-sequences"

$$\left\{ \mathcal{C} \;\middle|\; \begin{matrix}\text{nice finite} \\ \text{additive category}\end{matrix} \right\} \longleftrightarrow \left\{ \Gamma \;\middle|\; \begin{matrix}\text{homologically} \\ \text{nice algebra}\end{matrix} \right\}$$

$$\mathcal{C} = \operatorname{add} M \longmapsto \operatorname{End}(M)$$

Ex:

(1) $$\left\{ \mathcal{T} \;\middle|\; \begin{matrix}\text{finite torsion} \\ \text{classes in } \operatorname{mod}\Lambda\end{matrix} \right\} \xleftrightarrow{1-1} \left\{ \Gamma \;\middle|\; \begin{matrix} \cdot \operatorname{gldim}\Gamma \leq 2 \\ \cdot\ 0 \to {}_\Gamma\Gamma \to \underbrace{I^0 \to I^1}_{\text{projdim} \leq 1} \to I^2 \to 0 \\ \text{same} \\ \text{conditions for } \Gamma^{op} \end{matrix} \right\}$$

(2) $$\{ \Lambda \mid \operatorname{repdim}\Lambda \leq n \} \twoheadleftarrow \left\{ \Gamma \;\middle|\; \begin{matrix}\operatorname{gldim}\Gamma \leq n \\ 2 \leq \operatorname{domdim}\Gamma\end{matrix} \right\}$$

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Def: $\mathcal{C} \subset \operatorname{mod}\Lambda$ n − CT ("n-cluster tilting")

$n \geq 1$;

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \bullet\ \mathcal{C} = \{ X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(\mathcal{C}, X) = 0 \quad i = 1, \dots, n-1 \} \\ \quad = \{ X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, \mathcal{C}) = 0 \quad i = 1, \dots, n-1 \} \\ \bullet\ \mathcal{C} \text{ is functorially finite} \end{cases}$$

n − Auslander correspondence

$$\left\{ \mathcal{C} \;\middle|\; \begin{array}{l} \text{finite } n\text{-CT} \\ \text{in } \operatorname{mod}\Lambda \\ \exists \text{ additive generator} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \Gamma \;\middle|\; \begin{array}{l} \operatorname{gldim}\Gamma \leq n+1 \\ n+1 \leq \operatorname{domdim}\Gamma \end{array} \right\}$$

n − Auslander algebra

2. CM − representations [Functor and morph. 78]

(R, m) complete regular local ring of $\dim = d+1$

(e.g. $R = k[[x_0, \dots, x_d]]$)

$$\Lambda \quad \begin{cases} ① \text{ R-order } (\Lambda: R\text{-algebra}, \ {}_R\Lambda \text{ fin. gen. proj.}) \\ ② \text{ isolated singularity } (\operatorname{gldim} \Lambda \otimes_R R_p = \operatorname{krulldim} R_p \quad \forall p \in \operatorname{Spec} R \setminus \{m\}) \end{cases}$$

$$\operatorname{CM}(\Lambda) = \{ X \in \operatorname{mod}\Lambda \mid {}_R X \text{ fin. gen. proj.} \}$$

[A86] iso sing. $\Longleftrightarrow$ existence of AR-sequences in $\operatorname{CM}(\Lambda)$

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n-Auslander correspondence

Def: $\mathcal{C} \subset CM(\Lambda)$ n-CT

$$\overset{\Longleftrightarrow}{\text{def}} \begin{cases} \mathcal{C} = \{X \in CM(\Lambda) \mid Ext^i_\Lambda(\mathcal{C}, X) = 0,\ i = 1, \dots, n-1\} \\ \quad = \{X \in CM(\Lambda) \mid Ext^i_\Lambda(X, \mathcal{C}) = 0,\ i = 1, \dots, n-1\} \\ \mathcal{C} \text{ is functorially finite} \end{cases}$$

R; $\dim R = d+1$

$n \geq 1$

- For $n = d$

$$\left\{ \mathcal{C} \;\middle|\; \begin{array}{l} \text{finite} \\ n\text{-CT in} \\ CM(\Lambda) \end{array} \right\} \longrightarrow\!\!\!\!\!\to \left\{ \Gamma \;\middle|\; \begin{array}{l} R\text{-order}, \\ \operatorname{gldim} \Gamma = d+1 \end{array} \right\}$$

"non-singular"

- For $n > d$:

$$\left\{ \mathcal{C} \;\middle|\; \begin{array}{l} \text{finite} \\ n\text{-CT in} \\ CM(\Lambda) \end{array} \right\} \xleftrightarrow{1-1} \left\{ \Gamma \;\middle|\; \begin{array}{l} ⓪\ \Gamma : R\text{-order} \\ \quad \text{isolated singularity} \\ ①\ \operatorname{gldim} \Gamma \leq n+1 \end{array} \right\}$$

② $0 \to {}_R\Gamma \to \underbrace{I^0 \to \cdots \to I^{n-d-1}}_{\in \operatorname{proj} \Gamma} \to I^{n-d} \to 0$

min. inj. res. in $CM(\Lambda)$

- For $n < d$

$\nexists$ simple characterization of n-Auslander algebra

Γ is not R-order

3. Geigle-Lenzing complete intersection

(jt with Herschend, Minamoto, Oppermann)

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k field, $d \geq 1$.

$C = k[T_0, \dots, T_d]$

$\cup$

$l_1, \dots, l_n$ linear forms in a general position.

$p_1, \dots, p_n \geq 2$

$$R = C[X_1, \dots, X_n] \big/ (X_i^{p_i} - l_i)_{1 \leq i \leq n}$$

- Krull dim $R = d+1$
- Gorenstein
- $\mathbb{L}$-graded, $\mathbb{L} = \langle \vec{c}, \vec{x}_1, \dots, \vec{x}_n \rangle \big/ \langle p_i \vec{x}_i - \vec{c} \rangle_{1 \leq i \leq n}$ ab. group

  $\deg X_i = \vec{x}_i$

  $\deg T_j = \vec{c}$

$(R, \mathbb{L})$: CM-finite

$\overset{}{\underset{\text{def}}{\iff}}$ $\exists$ only finitely many indec. objects in $\mathrm{CM}^{\mathbb{L}} R$ up to degree shift

Theorem [HIMO]

① $(R, \mathbb{L})$ CM-finite

$$\iff \begin{cases} \bullet\ n \leq d+1 \\ \text{or} \\ \bullet\ n = d+2 \text{ and } (p_1, \dots, p_n) = (2, \dots, 2, 2, *) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad (2, \dots, 2, 3, \substack{3\\4\\5}) \end{cases}$$

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$(R, \mathbb{L})$ d-CM-finite

$\underset{\text{def}}{\iff}$ $\exists\, \mathcal{C} \subset \mathrm{CM}^{\mathbb{L}} R$: d-CT subcategory
s.t. $\mathcal{C}$ contains only finitely many
indec. objects in $\mathrm{CM}^{\mathbb{L}} R$ up to degree shift

Theorem 2 [HIMO]

- $n \leq d+1$

  or

- $n = d+2$ and $(p_1, \dots, p_n) = (2, 2, *, \dots, *)$
  $\cdot\ (2, 3, \{3,4,5\}, *, \dots, *)$

$\implies (R, \mathbb{L})$ is d-CM finite.

Conjecture:

$(R, \mathbb{L})$ d-CM finite $\iff$ Fano
$\left(\underset{\text{def}}{\iff} \sum_{i=1}^{n} \frac{1}{p_i} > n-d-1\right)$

for $d = 1$, this is the domestic case.

Theorem [HIMO]

$(R, \mathbb{L})$ GL-CI.

$\underline{\mathrm{CM}}^{\mathbb{L}} R \simeq D^b(\mathrm{mod}\, A)$ for some f.d. algebra A.

for $n = d+2$, $A = \bigotimes_{i=1}^{n} kA_{p_i - 1}$
[Kussin–Lenzing–Meltzer, Futaki–Ueda]