Kanda : Auslander–Buchweitz approximations and cotilting modules

(based on [Auslander – Buchweitz] and [Auslander – Reiten])

1) Cotorsion pairs

$\mathcal{A}$: abelian cat., $\mathcal{B} \subseteq \mathcal{A}$ add.

Def. $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{B}$ add.

$(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair in $\mathcal{B}$ $\iff$

- $\mathrm{Ext}^{\geq 1}(\mathcal{X}, \mathcal{Y}) = 0$
- $\forall B \in \mathcal{B}$, $0 \to \underset{\in \mathcal{Y}}{Y_B} \to \underset{\in \mathcal{X}}{X_B} \to B \to 0$,

  $0 \to B \to \underset{\in \mathcal{Y}}{Y^B} \to \underset{\in \mathcal{X}}{X^B} \to 0$

  in $\mathcal{A}$

$(\mathcal{X}, \mathcal{Y})$ cotorsion pair.

$$\mathcal{X}^{\perp} = \{ M \in \mathcal{A} \mid \mathrm{Ext}^{\geq 1}(\mathcal{X}, M) = 0 \}$$

$$^{\perp}\mathcal{Y} = \{ M \in \mathcal{A} \mid \mathrm{Ext}^{\geq 1}(M, \mathcal{Y}) = 0 \}$$

Prop (1) $\mathcal{X} = {}^{\perp}\mathcal{Y} \cap \mathcal{B}$. $\mathcal{Y} = \mathcal{X}^{\perp} \cap \mathcal{B}$

(2) $\mathcal{X} \subseteq \mathcal{B}$: contravariantly finite

$\mathcal{Y} \subseteq \mathcal{B}$: covariantly finite

$\omega := \mathcal{X} \cap \mathcal{Y}$.

Prop (1) $\omega \subseteq \mathcal{X}$: injective (i.e., $\mathrm{Ext}^{\geq 1}(\mathcal{X}, \omega) = 0$)

$\omega \subseteq \mathcal{Y}$: projective

$\mathrm{Ext}^{\geq 1}(\omega, \omega) = 0$

(2) $\omega \subseteq \mathcal{X}$: cogenerator (i.e., $\forall X \in \mathcal{X}$, $\exists\, 0 \to X \to \underset{\omega}{\overset{\cup\!\!\shortmid}{W}} \to \underset{\mathcal{X}}{\overset{\cup\!\!\shortmid}{X'}} \to 0$)

(3) X_B and Y^B are unique in $\mathcal{A}/\omega$.

$$\begin{array}{ccccc}
 & & \mathcal{A} & & \\
 & & \cup\!\! \mid & & \\
 & \overset{\text{gen.}}{\subseteq} & \mathcal{B} & \overset{\text{cogen.}}{\supseteq} & \\
\mathcal{X} & & & & \mathcal{Y} \\
\cup\!\!\mid & & & & \subseteq \\
\text{inj. cogen} & & \omega & & \text{proj. gen}
\end{array}$$

2) AB-approximations

$\mathcal{A}$: abelian, $\mathcal{X} \subseteq \mathcal{A}$ additive, ext. closed, kernels of epis, ("epiker")

$\omega \subseteq \mathcal{X}$: add., inj. cogen.

$$\mathcal{B} := \hat{\mathcal{X}} := \{ M \in \mathcal{A} \mid 0 \to \underbrace{X_n \to \cdots \to X_0}_{\in \mathcal{X}} \to M \to 0 \}$$

$$\mathcal{Y} := \hat{\omega}$$

<u>Thm</u> $(\mathcal{X}, \hat{\omega})$ is a cotorsion pair in $\hat{\mathcal{X}}$. Moreover,

$$\hat{\mathcal{X}} = \{ M \in \mathcal{A} \mid 0 \to \overset{\in \mathcal{Y}}{Y} \to \overset{\in \mathcal{X}}{X} \to M \to 0 \}$$

$$= \{ M \in \mathcal{A} \mid 0 \to M \to Y \to X \to 0 \}$$

$\hat{\mathcal{X}} \subseteq \mathcal{A}$: add, ext., epiker, cokernels of monomorphisms ("monocok")

$$\omega = \mathcal{X} \cap \hat{\omega} = \mathcal{X} \cap \mathcal{X}^{\perp}.$$

<u>Proof of theorem</u> (2nd part) Let $M \in \hat{\mathcal{X}}$. Thus we have

$$0 \to \underbrace{X_n \to X_{n-1} \to \cdots \to X_0}_{\in \mathcal{X}} \to M \to 0$$

<u>Show</u> via induction: M has AB-approx.

(1) $n=0$. $M \in \mathcal{X}$, thus $0 \to M \to \underset{\hat{\omega}}{\overset{\cap}{W}} \to \underset{\hat{\mathcal{X}}}{\overset{\cap}{X}} \to 0$

(2) $n \geq 1$. $\quad 0 \to K \to X_0 \to M \to 0$.

By ind. hyp. $\exists \quad 0 \to K \to Y^K \to X^K \to 0$

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & K \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y^K \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & M \arrow[r] & 0 \\ & X^K \arrow[r, equal] \arrow[d] & X^K \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$

i.e. $(E \in \mathcal{X})$

$$0 \to E \to W \to X \to 0, \quad W \in \omega, \quad X \in \hat{\mathcal{X}}$$

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & Y^K \arrow[r] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y^K \arrow[r] & W \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ & & X \arrow[r, equal] \arrow[d] & X \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

i.e. $(C \in \hat{\omega})$

$\square$

Ex. (Iwanaga-Gorenstein rings)

R: left and right noeth. ring with $\mathrm{id}_R R < \infty$, $\mathrm{id}\, R_R < \infty$.

$\mathcal{A} = \mathrm{mod}\, R$; $\quad \mathcal{X} = CM(R)$, the category of max. CM-modules, i.e, all $M \in \mathrm{mod}\, R$ with $\mathrm{Ext}^{\geq 1}(M,R) = 0$, thus $\mathcal{X} = {}^{\perp}R$.

$\omega = \mathrm{proj}(R)$, $\quad (-)^* = \mathrm{Hom}_R(-,R)$.

$$\begin{array}{ccc} CM(R) & \xleftrightarrow{(-)^*} & CM(R^{op}) \\ \cup\!| & & \cup\!| \\ \mathrm{proj}\, R & \xleftrightarrow{\sim} & \mathrm{proj}\, R^{op} \end{array}$$

$\mathrm{proj}(R) \subseteq CM(R)$ inj. cogen.

$\forall M \in CM(R)$, $M^* \in CM(R^{op})$

$$\begin{array}{ccccccccc} 0 & \to & L & \to & P & \to & M^* & \to & 0 \\ & & \cap & & \cap & & & & \\ & & CM(R^{op}) & & \mathrm{proj}\,R & & & & \end{array}$$

$$\begin{array}{ccccccc} 0 & \to & M^{**} & \to & P^* & \to & L^* & \to & \mathrm{Ext}^1(M^*, R) \\ & & \| & & \cap & & \cap & & \| \\ & & M & & \mathrm{proj}\,R & & CM(R) & & 0 \end{array}$$

$\square$

$\hat{\mathcal{X}} = \{M \in \mathrm{mod}\,R \mid \mathrm{Ext}^{\geq n}(M,R) = 0 \text{ for some } n \geq 0\} = \mathrm{mod}\,R$

$\hat{\omega} = \{M \in \mathrm{mod}\,R \mid \text{fin. proj. dim.}\}$

$(CM(R), \{\mathrm{pd}\,M < \infty\})$ cotorsion pair in $\mathrm{mod}\,R$.

Ex (R-orders) R: comm. noetherian ring with a canonical module ω_R, i.e.

$$\left\{\begin{array}{l} \omega_R \in \mathrm{mod}\,R \\ \forall \mathfrak{p} \in \mathrm{Spec}\,R,\ (\omega_R)_{\mathfrak{p}} \text{ fin. inj. dim.} \\ R \xrightarrow{\sim} \mathrm{End}_R(\omega_R) \\ \mathrm{Ext}^{\geq 1}(\omega_R, \omega_R) = 0 \end{array}\right. \quad \Rightarrow \quad R \text{ is CM.}$$

Λ R-alg. st. $\Lambda_R \in CM(R) = \{M \in \mathrm{mod}\,R \mid \mathrm{Ext}^{\geq 1}(M, \omega_R) = 0\}$

$(-)^* = \mathrm{Hom}_R(-, \omega_R)$

$\mathcal{A} = \mathrm{mod}\,\Lambda$, $\mathcal{X} = CM(\Lambda) = \{M \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^{\geq 1}(M, \omega_R) = 0\}$

$\omega_\Lambda := \Lambda^*$

$$\begin{array}{ccc} CM(\Lambda) & \xleftrightarrow[\sim]{(-)^*} & CM(\Lambda^{op}) \\ \cup\!| & & \cup\!| \\ \mathrm{add}\,\omega_\Lambda & \xleftrightarrow[\sim]{} & \mathrm{proj}\,\Lambda^{op} \\ \mathrm{proj}\,\Lambda & \xleftrightarrow[\sim]{} & \mathrm{add}\,\omega_{\Lambda^{op}} \end{array}$$

$\mathrm{add}\, \omega_\Lambda \in \mathrm{CM}(\Lambda)$ inj. cogen

$\widehat{\mathrm{CM}}(\Lambda) = \{ M \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^{\geq n}(M, \omega_R) = 0 \text{ for some } n \}$

$(\mathrm{CM}(\Lambda), \widehat{\mathrm{add}\,\omega_\Lambda})$: cotorsion pair in $\widehat{\mathrm{CM}}(\Lambda)$.

$\mathrm{Kdim}\, R < \infty \iff \mathrm{id}\, \omega_R < \infty \implies \widehat{\mathrm{CM}}(\Lambda) = \mathrm{mod}\,\Lambda$,

$\widehat{\mathrm{add}}(\omega_\Lambda) \subseteq \{\mathrm{id}\, M < \infty\}$

3) Cotilting modules [AR]

Λ artin alg. (mod-fin. alg. / comm. artin ring)

Def. $T \in \mathrm{mod}\,\Lambda$: cotilting

$$:\iff \begin{cases} \cdot\ \mathrm{Ext}^{\geq 1}(T,T) = 0 \\ \cdot\ \mathrm{id}_\Lambda T < \infty \\ \cdot\ \mathrm{inj}\,\Lambda \subseteq \widehat{\mathrm{add}\,T} \end{cases}$$

Thm Let $T \in \mathrm{mod}\,\Lambda$: cotilting.

$\Rightarrow$ $({}^\perp T, \widehat{\mathrm{add}\,T})$: cotorsion pair in $\mathrm{mod}\,\Lambda$.

${}^\perp T \cap \widehat{\mathrm{add}\,T} = \mathrm{add}\,T$

Def. $\mathcal{X} \subseteq \mathrm{mod}\,\Lambda$ resolving $\iff$ $\mathcal{X}$ add, ext, epiker, $\mathrm{proj}\,\Lambda \subseteq \mathcal{X}$.

Thm

$$\begin{array}{cc}
\{ \mathcal{X} \subseteq \mathrm{mod}\,\Lambda \mid \mathcal{X} \text{ contrav.-fin. resolving}, \hat{\mathcal{X}} = \mathrm{mod}\,\Lambda \} & {}^\perp T \\
\updownarrow 1\text{-}1 & \uparrow \\
\{ T \in \mathrm{mod}\,\Lambda \mid \text{basic cotilting} \} / \cong & T \\
\updownarrow & \downarrow \\
\{ \mathcal{Y} \subseteq \mathrm{mod}\,\Lambda \mid \text{cov. fin. coresolving}, \mathcal{Y} \subseteq \{\mathrm{id}\, M < \infty\} \} & \widehat{\mathrm{add}\,T}
\end{array}$$

Thm $\{\text{Cotorsion pairs}\} \xleftrightarrow{1-1} \{\text{basic cotilting}\}$ if gldim $\Lambda < \infty$.

4) Application to quasi-hereditary algebras [Ringel]

Λ quasi-hereditary alg [Cline - Parshall - Scott]

$\Delta = \{\Delta_1, \dots, \Delta_n\}$ standard

$\nabla = \{\nabla_1, \dots, \nabla_n\}$ costandard

$\mathcal{F}(\Delta) := \langle \Delta \rangle_{ext}$

$\mathcal{F}(\nabla) = \langle \nabla \rangle_{ext}$

Thm $(\mathcal{F}(\Delta), \mathcal{F}(\nabla))$ cotorsion pair in mod Λ

$\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \operatorname{add} T$

$\operatorname{End}(T) = \Lambda'$, Ringel: $\Lambda'' \underset{\text{Morita}}{\sim} \Lambda$