Existence and non-existence of AR-sequences

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Plan

§1 Setting and definitions
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§4 Hereditary case and AR-quivers.

§1 Setting and definitions

k field, $k = \overline{k}$

Q infinite, connected quiver, locally finite.

kQ : path category
  objects: vertices
  morphisms: Linear combinations of path between two fixed vertices.

$I \subset kQ$ ideal s.t.

- kQ/I is hom-finite.
- $I \subseteq (kQ^+)^2$
- Elements in kQ^+/I are nilpotent.

$\mathrm{rep}(Q,I) = \mathrm{Fun}(kQ/I, \mathrm{mod}\, k) \subset \mathrm{Rep}(Q,I) = \mathrm{Fun}(kQ/I, \mathrm{Mod}\, k)$

Remarks:
- $\mathrm{rep}(Q,I)$ abelian, has finite direct sums
- $\mathrm{Rep}(Q,I)$ abelian, has infinite direct sums.
- $I = 0$, Q is intervall-finite, that is $\forall x,y \in Q_0 : Q(x,y) < \infty$ finitely many path.

Definition/Proposition: Let $x \in Q_0$

1) P_x indec. projective, in $\mathrm{rep}(Q,I)$ (because of hom-finiteness)
   because of assumptions on I.
2) I_x indec. injective in $\mathrm{rep}(Q,I)$.

3) $M \in \operatorname{rep}(Q,I)$ is <u>finitely generated</u> if $\exists$ epim.

$$\bigoplus_{i=1}^{n} P_{x_i} \xrightarrow{f_M} M \longrightarrow 0$$

If $\operatorname{Ker} f_M$ is finitely generated, then M is called <u>finitely presented</u>.

4) $M \in \operatorname{rep}(Q,I)$ is finitly co-generated if $\exists$ mono

$$0 \longrightarrow M \xrightarrow{g_M} \bigoplus_{i=1}^{m} I_{y_i}$$

If $\operatorname{coker} g_M$ is finitely co-generated, then M is <u>finitely co-presented</u>

<u>Prop</u>: M finite dimensional

$\Longleftrightarrow$ M fin. gen. and fin. cogen.

<u>§2 AR-seq. in rep(Q,I)</u>

$$\begin{tikzcd} & & N' \arrow[dl, dashed] \arrow[d] & \text{nonsplit} \\ 0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[dl, dashed] & N \arrow[r] & 0 \\ & L' & & & \end{tikzcd}$$

ses, non-split

1) [Aus] If N is indec., non-projective and finitely presented, then $\exists$

$$0 \to L \to M \to N \to 0 \qquad \text{AR-seq.}$$

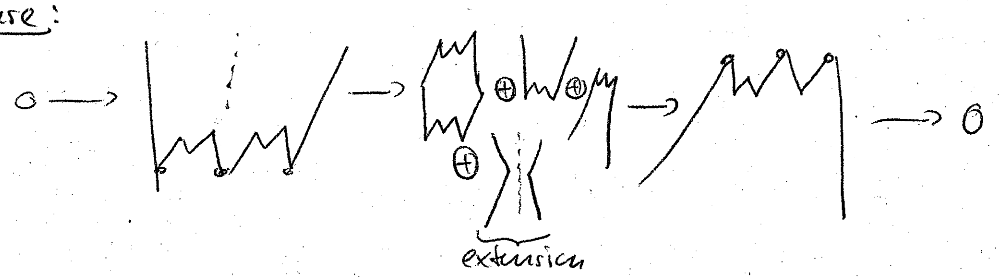
2) If L is indec, non-injective and finitely copresented, then $\exists$

$$0 \to L \to M \to N \to 0 \qquad \text{AR-seq.}$$

<u>Remark</u>: 1) fin dim $\underset{\not\Leftarrow}{\Longrightarrow}$ fin presented $\underset{\not\Leftarrow}{\Longrightarrow}$ fin. generated

2) If Q has no infinite paths, then they are all equivalent.

<u>picture</u>:



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§3 Non-existence of AR-sequences

Ex:

$$\begin{tikzcd}[row sep=small, column sep=small] & 2 \arrow[dl] \arrow[dr] & & 4 \arrow[dl] \arrow[dr] & \\ 1 & & 3 & & \cdots \end{tikzcd}$$

$I = 0$

$$M = \begin{tikzcd}[row sep=small, column sep=small] & K \arrow[dl, "1"'] \arrow[dr, "1"] & & K \arrow[dl, "1"'] \arrow[dr, "1"] & \\ K & & K & & K \cdots \end{tikzcd}$$

$End(M) = K$

Is there an AR-sequence ending in M?

Indec representations $M(i,j)$ $1 \leq i \leq j \leq \infty$

$$\begin{tikzcd}[row sep=small, column sep=small] \overset{i}{K} \arrow[dr] & & K \arrow[dl] \arrow[dr] & \cdots \arrow[dr] & \\ & K & & \cdots & \overset{j}{K} \end{tikzcd}$$

$$M(3,\infty) \oplus \bigoplus_{i=1}^{n\,\infty} \underbrace{M(1,j_i)}_{\text{fin.dim.}} \longrightarrow M(1,\infty) \longrightarrow 0$$

$\Rightarrow n = 1$ $\quad 0 \to L \to M(1,m) \oplus M(3,\infty) \to M(1,\infty) \to 0$ $\lightning$ ($\Leftarrow \dim L < \infty$)

$(\dim L = \infty)$

<u>Thm</u> [P. 2012] The only AR-seq in $rep(Q,I)$ are the ones mentioned previously.

<u>Conjecture</u> [Krause, 14]

R ring. If $0 \to L \to M \to N \to 0$ AR-sequence $\Longrightarrow N$ finitely presented.

§4 $rep(Q, I=0)$

[Bautista, Liu, –] AR-quiver of $rep^{lP}(Q)$

AR-quiver of rep(Q) : Γ_Q

vertices $\longleftrightarrow$ iso classes of indec reps

arrows : $M \to N$

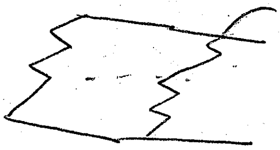
$\quad \# = \dim_k \left(\dfrac{\mathrm{rad}(M,N)}{\mathrm{rad}^2(M,N)} \right)$

rep(Q) is a hereditary category.

<u>Thm</u>: AR-components of Γ_Q are as follows

- Preprojective component

$\mathbb{N}Q^{op}$

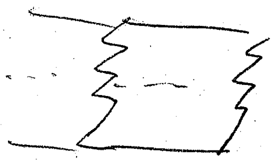


same section

either all of $\mathbb{N}Q^{op}$ or a piece of it as in the picture

- Preinjective component

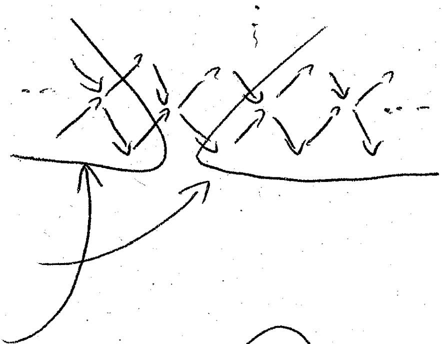
$\mathbb{N}^- Q^{op}$



either all of it or a piece of it.

- other "regular" components:

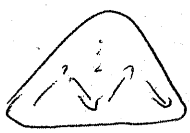
$\mathbb{Z}A_\infty$



$\mathbb{N}A_\infty$

$\mathbb{N}^- A_\infty$

finite wing



other components that do not contain meshes.

<u>Conjecture</u>: Components of the last type are all full subquiver of $\cdots \to \to \to \cdots$

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<u>Thm</u>: Conjecture holds if Q is noetherian and artinian

(fin. gen. $\Leftrightarrow$ fin pres.
fin cogen $\Leftrightarrow$ fin copres)