Functorial Methods in Representation Theory

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The College of New Jersey

Maurice Auslander Memorial Workshop
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Mindset of the Functorial Approach

The Big Bang Theory

Stephanie So, how was your day?

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Mindset of the Functorial Approach

The Big Bang Theory

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Leonard	You know, I'm a physicist, so I thought about stuff.
Stephanie	That's it?
Leonard	I wrote some of it down.

Thesis of the Talk

The Functorial Approach

Functors are Modules.

Notation

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Ab - category of abelian groups.

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$\mathcal{A}$ - skeletally small abelian category.

The Functor Category

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Definition

A functor $F\colon \mathcal{A} \to \mathsf{Ab}$ is called representable if

$$F \cong \mathsf{Hom}_{\mathcal{A}}(X, \_\,)$$

for some $X \in \mathcal{A}$.

Notation For Representable Functors

Notation:

We will abbreviate the representable functor $\mathsf{Hom}_{\mathcal{A}}(X, \_\,)$ by

$$(X, \_\,)$$

and abbreviate $\mathsf{Hom}_{(\mathcal{A},\mathsf{Ab})}(F, \_\,)$ by

$$\mathsf{Nat}(F, \_\,)$$

or

$$(F, \_\,)$$

depending on the situation.

Properties of $(\mathcal{A}, \mathrm{Ab})$

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  - a cokernel of β $\iff$ α_X is a cokernel of β_X for all $X \in \mathcal{A}$.

Exactness in $(\mathcal{A}, \mathsf{Ab})$

- A sequence in $(\mathcal{A}, \mathsf{Ab})$:

$$F \longrightarrow G \longrightarrow H$$

is exact if and only if

$$F(X) \longrightarrow G(X) \longrightarrow H(X)$$

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- All universal objects such as the kernel, cokernel, pullback, pushout, etc. are constructed componentwise.
- For each $X \in \mathcal{A}$, the evaluation functor

$$\mathsf{ev}_X \colon (\mathcal{A}, \mathsf{Ab}) \to \mathsf{Ab}$$

  is exact.

Yoneda's Lemma

Lemma (Yoneda)

For any $X \in \mathcal{A}$ and any $F \in (\mathcal{A}, \mathsf{Ab})$,

$$\mathsf{Nat}\big((X, \_\ ), F\big) \cong F(X)$$

1. The isomorphism is defined by $\alpha \mapsto \alpha_X(1_X)$.
2. The isomorphism is natural in both X and F.

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According to Martsinkovsky, Auslander used to say that if you cannot prove something using Yoneda's lemma, then it isn't true.

Consequence of Yoneda's Lemma

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Consequence of Yoneda's Lemma

Take an exact sequence in $(\mathcal{A}, \mathsf{Ab})$:

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Apply $\mathsf{Nat}\big((X, \_\,), \_\,\big)$:

$$\big((X, \_\,), F\big) \longrightarrow \big((X, \_\,), G\big) \longrightarrow \big((X, \_\,), H\big)$$

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\big((X, _\,), F\big) \arrow[r] \arrow[d, "\cong"] & \big((X, _\,), G\big) \arrow[r] \arrow[d, "\cong"] & \big((X, _\,), H\big) \arrow[d, "\cong"] \\
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The bottom row is exact resulting in exactness of the top row.

Consequences of Yoneda's Lemma

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Proposition

The **Yoneda embedding** $\mathsf{Y}\colon \mathcal{A} \to (\mathcal{A}, \mathsf{Ab})$ defined by

$$\mathsf{Y}(X) = (X, \_\,)$$

is

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3. left exact

Left Exactness of the Yoneda Embedding

Start with exact sequence

$$0 \to A \to B \to C \to 0$$

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Since the exactness of this sequence holds for all $X \in \mathcal{A}$, the sequence

$$0 \to (C, \_\,) \to (B, \_\,) \to (A, \_\,)$$

is exact in $(\mathcal{A}, \mathsf{Ab})$.

Consequences of Yoneda's Lemma

Definition

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Consequences of Yoneda's Lemma

Definition

Recall that an object A is called finitely generated if whenever $A = \coprod_I A_i$, this sum may be taken to be finite.

- Representable functors are finitely generated projectives in $(\mathcal{A}, \mathsf{Ab})$.
- Representable functors generate $(\mathcal{A}, \mathsf{Ab})$ in the sense that given $F\colon \mathcal{A} \to \mathsf{Ab}$ there exists $X_i \in \mathcal{A}$ and exact sequence

$$\coprod_i (X_i, \_\,) \to F \to 0$$

Finitely Presented Functors

Definition (Auslander)

A functor $F \colon \mathcal{A} \to \mathsf{Ab}$ is called **finitely presented** if there exists exact sequence

$$(Y, \_\,) \to (X, \_\,) \to F \to 0$$

In other words, F is a cokernel of a representable transformation.

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Definition

$\mathsf{fp}(\mathcal{A}, \mathsf{Ab})$ = category of finitely presented functors.

Properites of Finitely Presented Functors

$$\mathsf{fp}(\mathcal{A}, \mathsf{Ab}) \hookrightarrow (\mathcal{A}, \mathsf{Ab})$$

- $\mathsf{fp}(\mathcal{A}, \mathsf{Ab})$ is abelian, the inclusion functor is exact, and reflects exact sequences.

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- $\mathsf{fp}(\mathcal{A}, \mathsf{Ab})$ is abelian, the inclusion functor is exact, and reflects exact sequences.
- $\mathsf{fp}(\mathcal{A}, \mathsf{Ab})$ has enough projectives and they are precisely the representable functors.
- All finitely presented functors have projective dimension at most 2:

$$0 \to (Z, \_\ ) \to (Y, \_\ ) \to (X, \_\ ) \to F \to 0$$

Examples of Finitely Presented Functors

Proposition (Auslander)

1. $\mathsf{Ext}^n(X, \_\ ) \in \mathsf{fp}(\mathsf{Mod}(R^{op}), \mathsf{Ab})$.

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1. $\mathsf{Ext}^n(X, \_\ ) \in \mathsf{fp}(\mathsf{Mod}(R^{op}), \mathsf{Ab})$.
2. $\_\ \otimes X \in \mathsf{fp}(\mathsf{Mod}(R^{op}), \mathsf{Ab})$ if and only if $X \in \mathsf{mod}(R^{op})$.

Construction of $w\colon \mathsf{fp}(\mathcal{A}, \mathsf{Ab}) \to \mathcal{A}$

Auslander constructed a contravariant functor

$$w\colon \mathsf{fp}(\mathcal{A}, \mathsf{Ab}) \to \mathcal{A}$$

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Step 1: Start with presentation

$$(Y, \_\,) \to (X, \_\,) \to F \to 0$$

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Step 3: The exact sequence

$$0 \to w(F) \to X \to Y$$

completely determines w.

Properties

Proposition (Auslander)

1. w does not depend on any choices of presentation.

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3. $w(X, \_\,) = X$.

What w Measures

Take presentation of F:

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Apply w:

$$0 \to w(F) \to X \to Y \to Z \to 0$$

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Proposition (Auslander)

For $F \in \mathsf{fp}(\mathcal{A}, \mathsf{Ab})$ the following are equivalent:

1. $w(F) = 0$
2. All presentations of F arise from short exact sequences.
3. There exists short exact sequence in $\mathcal{A}$

$$0 \to X \to Y \to Z \to 0$$

such that the following is a presentation of F:

$$0 \to (Z, \_\,) \to (Y, \_\,) \to (X, \_\,) \to F \to 0$$

Zeroth Derived Functors

Proposition (Auslander)

For any finitely presented functor F there is an exact sequence:

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Zeroth Derived Functors

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- If $\mathcal{A}$ has enough injectives, then $\big(w(F), \_\,\big) = R^0 F$.
- In this case F vanishes on injectives if and only if $w(F) = 0$.

Finite Dimensional k-Algebras

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Finite Dimensional k-Algebras

- For a finite dimensional k- algebra Λ, the category mod(Λ^{op}) is abelian and has enough injectives.
- Every finitely presented left Λ-module M is a finite direct sum of indecomposables
$$M = \coprod_{i=1}^{n} X_i$$

Projective Covers

Definition

Recall that an epimorphism $f\colon P \to X$ from a projective P to object X is called a projective cover if

$$fh = f$$

implies that h is an isomorphism.

Minimal Resolutions

Definition

A projective resolution

$$\cdots P_k \longrightarrow P_{k-1} \longrightarrow \cdots P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

is a minimal projective resolution if each

$$P_n \longrightarrow \Omega^n X$$

is a projective cover.

$\mathrm{fp}(\mathrm{mod}(\Lambda^{op})$ Has Minimal Projective Resolutions

Proposition (Auslander)

All finitely presented functors $F\colon \mathrm{mod}(\Lambda^{op}) \to \mathrm{Ab}$ have minimal projective resolutions.

fp(mod(Λ^{op})) Has Minimal Projective Resolutions

Proposition (Auslander)

All finitely presented functors $F\colon \text{mod}(\Lambda^{op}) \to \text{Ab}$ have minimal projective resolutions.

- Given $(X, \_\,) \to F \to 0$, one can take X to have smallest dimension. This will be a projective cover.

Simple Functors

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For a simple functor $S\colon \mathsf{mod}(\Lambda^{op}) \to \mathsf{Ab}$ there exists a unique indecomposable N such that

1. $S(N) \neq 0$
2. There is a projective cover $(N, \_\ ) \to S \to 0$.

Sketch of Proof

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- x determines $\varepsilon_x \colon (N, \_\ ) \to S$ where $\varepsilon_x(1_N) = x$.

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- Because S is simple, ε_x is an epimorphism.

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- N will be indecomposable.

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- $x \neq 0$ implies $\varepsilon_x \neq 0$
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- Otherwise $N \cong A \coprod B$.

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Step 2: Choose N from above to have smallest dimension.

- N will be indecomposable.
- Otherwise $N \cong A \coprod B$.
- $S(N) = S(A) \coprod S(B) \neq 0$.

Sketch of Proof

Step 1: Find exact sequence $(N, \_\ ) \to S \to 0$.

- Since $S \neq 0$, there exists $N \in \text{mod}(\Lambda^{op})$ such that $S(N) \neq 0$.
- Since $S(N) \neq 0$, there exists non-zero $x \in S(N)$.
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$\therefore$ $(N, \_\ ) \to S$ is a projective cover.

- Uniqueness of N follows from uniqueness of projective covers.

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S_N denotes simple functor S such that $S(N) \neq 0$ for indecomposable N.

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Theorem (Auslander)

The simple functors are finitely presented in $\big(\mathsf{mod}(\Lambda^{op}), \mathsf{Ab}\big)$.

Simple Functors Come From Exact Sequences

Recall that we are looking at the category

$$\mathsf{fp}(\mathsf{mod}(\Lambda^{op}), \mathsf{Ab})$$

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Proposition

Suppose that N is a non-injective indecomposable. Then

$$w(S_N) = 0.$$

Proof

Since N is not injective, $S_N(I) = 0$ for any indecomposable injective I.

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Because $\text{mod}(\Lambda^{op})$ has enough injectives, this is equivalent to

$$w(S_N) = 0$$

Hence there exists an exact sequence in $\text{mod}(\Lambda^{op})$:

$$0 \to N \to Y \to Z \to 0$$

such that

$$0 \to (Z, \_\,) \to (Y, \_\,) \to (N, \_\,) \to S_N \to 0$$

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Step 1: Take minimal projective presentation:

$$0 \to (Z, \_\,) \to (Y, \_\,) \to (N, \_\,) \to S_N \to 0$$

Step 2: Apply w to get the short exact sequence

$$0 \to N \to Y \to Z \to 0$$

What Are Almost Split Sequences?

Start with any morphism $u\colon N \to K$

$$
\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r] & Z \arrow[r] & 0 \\
& K & & &
\end{tikzcd}
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\end{tikzcd}$$

push out
diagram

Two Possibilities for u

$$
\begin{tikzcd}
0 \arrow[r] & (Z, \_\,) \arrow[r] \arrow[d, "1"'] & (E, \_\,) \arrow[r] \arrow[d] & (K, \_\,) \arrow[r] \arrow[d, "{(u, \_\,)}"] & F \arrow[r] & 0 \\
0 \arrow[r] & (Z, \_\,) \arrow[r] & (Y, \_\,) \arrow[r, "{(f, \_\,)}"'] & (N, \_\,) \arrow[r] \arrow[d] & S_N \arrow[r] & 0 \\
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Case 1: $\alpha = 0$

$$
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$(u,\ \_\ )$ factors through $(f,\ \_\ )$

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0 \arrow[r] & (Z,\ \_\ ) \arrow[r] \arrow[d, "1"'] & (E,\ \_\ ) \arrow[r] \arrow[d] & (K,\ \_\ ) \arrow[r] \arrow[d, "{(u,\ \_\ )}"] \arrow[dl] & F \arrow[r] \arrow[d, "\alpha = 0"] & 0 \\
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$$

$$
\begin{tikzcd}
N \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[dl] \\
K &
\end{tikzcd}
$$

Case 2: $\alpha =$ epimorphism

$$
\begin{tikzcd}
0 \arrow[r] & (Z, \_\,) \arrow[r] \arrow[d, "1"'] & (E, \_\,) \arrow[r] \arrow[d] & (K, \_\,) \arrow[r] \arrow[d, "(u, \_\,)"] & F \arrow[r] \arrow[d, "\alpha"] & 0 \\
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 & & & G \arrow[d] & 0 & \\
 & & & 0 & &
\end{tikzcd}
$$

$(u, \_\,) =$ epimorphism and hence $u =$ section.

$$
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0 \arrow[r] & (Z, \_\,) \arrow[r] \arrow[d, "1"'] & (E, \_\,) \arrow[r] \arrow[d] & (K, \_\,) \arrow[r] \arrow[d, "(u, \_\,)"] & F \arrow[r] \arrow[d, "\alpha"] & 0 \\
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In this case u must be a section.

What Are Almost Split Sequences?

The fact that f is left minimal follows from the minimality of the presentation

$$(Y, \_\ ) \to (N, \_\ ) \to S_N \to 0$$

Almost Split Sequences

Definition (Auslander, Reiten)

An exact sequence

$$0 \longrightarrow N \xrightarrow{f} Y \longrightarrow Z \longrightarrow 0$$

is **almost split** if

1. f is left minimal, so $hf = f$ implies h is an isomorphism.
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Note: These are simply the properties of the sequences obtained by applying w to minimal projective resolutions of simple functors S_N.

A Quote from Buchsbaum

David Buchsbaum in a personal communication to Robin Hartshorne:

“It was always a little difficult to know just what Maurice had in mind when he started on something. Certainly in the case of coherent functors, the choice of the term “coherent” indicates that he was onto the notion of finite presentation... when he first spoke to me about coherent functors, he didn’t speak about them in any way in connection with the applications he finally came up with. He was playing; his representable functors were his finitely generated projectives, and so his coherent functors generalized existing notions of the time (this is what he told me).”

$\mathcal{A}$-modules

$\mathcal{A}$ - any small pre-additive category.

Definition

$$\mathsf{Mod}(\mathcal{A}^{op}) = (\mathcal{A}, \mathsf{Ab})$$

A left $\mathcal{A}$-module is a functor $F \colon \mathcal{A} \to \mathsf{Ab}$.

In the general approach, functors are viewed as generalizations of modules.

Left Modules Over a Ring

For the ring R, one recovers all left R-modules as follows:

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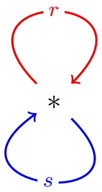

General Tensor Product

Definition (Kan, Lawvere, Freyd, Ulmer)

Fisher-Palmquist in her dissertation studied the general tensor product

$$\otimes_{\mathcal{A}} \colon \mathsf{Mod}(\mathcal{A}) \times \mathsf{Mod}(\mathcal{A}^{op}) \to \mathsf{Ab}$$

This bifunctor is completely determined by the following criterion:

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- $\_ \otimes_{\mathcal{A}} G$ is right continuous.

What about Bi-modules?

The role of a bi-module is played by a bifunctor

$$b\colon \mathcal{A}^{op} \times \mathcal{A} \to \mathsf{Ab}$$

for which

$$b(A, \_\ ) \in \mathsf{Mod}(\mathcal{A}^{op})$$

$$b(\_\ , A) \in \mathsf{Mod}(\mathcal{A})$$

Definition

$$\big[F \otimes_{\mathcal{A}} b\big](A) = F \otimes_{\mathcal{A}} b(A, \_\ )$$

$$\big[b \otimes_{\mathcal{A}} G\big](A) = b(\_\ , A) \otimes_{\mathcal{A}} G$$

Bifunctors and Nat

Definition (Fisher-Palmquist and Newell)

For $F \in \mathsf{Mod}(\mathcal{A}^{op})$:

$$\mathsf{Nat}(b, F)(A) = \mathsf{Nat}(b(A, \_\ ), F)$$

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For $G \in \mathsf{Mod}(\mathcal{A})$:

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Adjunction of $\otimes_{\mathcal{A}}$ and Nat

Theorem (Fisher-Palmquist and Newell)

Let $\mathcal{A}, \mathcal{B}$ be pre-additive categories. For any bifunctor

$$b\colon \mathcal{A}^{op} \times \mathcal{B}$$

The functor $\_ \otimes b\colon \mathrm{Mod}(\mathcal{A}) \to \mathrm{Mod}(\mathcal{B}^{op})$ is the left adjoint to the functor $\mathrm{Nat}(b, \_\,)$.

Theorem (Fisher-Palmquist and Newell)

All adjunctions between $\mathrm{Mod}(\mathcal{A})$ and $\mathrm{Mod}(\mathcal{B}^{op})$ arise in this way.

Duals

The role of the ring as a bi-module over itself is played by the bifunctor $\mathrm{Hom}\colon \mathcal{A}^{op} \times \mathcal{A} \to \mathsf{Ab}$.

Definition (Fisher-Palmquist and Newell)

For $F \in \mathsf{Mod}(\mathcal{A}^{op})$, define $F^* \in \mathsf{Mod}(\mathcal{A})$ by

$$F^* := \mathsf{Nat}(F, \mathsf{Hom})$$

that is on any object $A \in \mathcal{A}$

$$F^*(A) := \mathsf{Nat}\big(F, \mathsf{Hom}(A, \_\,)\big)$$

Proposition (Fisher-Palmquist and Newell)

For $F \in \mathsf{Mod}(\mathcal{A})$ there is a natural transformation $\beta\colon \_ \otimes_{\mathcal{A}} F^* \to (F, \_)$ such that the following are equivalent:

1. $\beta\colon \_ \otimes_{\mathcal{A}} F^* \to (F, \_)$ is an isomorphism.
2. β is an epimorphism.
3. β_F is an epimorphism.
4. F is a small projective in $\mathsf{Mod}(\mathcal{A}^{op})$.

Finitely Presented $\mathcal{A}$-modules

Definition

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- $\text{mod}(\mathcal{A}^{op})$ is an additive category with cokernels.
- $\text{mod}(\mathcal{A}^{op})$ will not be abelian in general.
- $\text{fp}\big(\text{mod}(\mathcal{A}^{op}), \text{Ab}\big)$ is abelian and satisfies some very nice properties.

Examples of Finitely Presented Functors

Proposition

For a small pre-additive category $\mathcal{A}$, the functor $\_ \otimes F \colon \mathsf{Mod}(\mathcal{A}) \to \mathsf{Ab}$ is finitely presented if and only if $F \in \mathsf{mod}(\mathcal{A}^{op})$. In this case $\_ \otimes F \in \mathsf{fp}(\mathsf{mod}(\mathcal{A}^{op}), \mathsf{Ab})$.

Summary

Module Theoretic Concept	**Functor Theoretic Concept**
R - Ring	$\mathcal{A}$ - Preadditive Category
Mod(R)	Mod$(\mathcal{A})$
mod(R)	mod$(\mathcal{A})$
Bi-Module	Bifunctor
Finitely Generated Projectives	Direct Summands of $\coprod_{i=1}^{n}(X_i, \_\ )$
$\otimes_R$	$\otimes_{\mathcal{A}}$
Hom$(_\ , R)$	Nat$(_\ ,$ Hom$)$

Auslander-Gruson-Jensen Duality

Theorem (Auslander)

Let R be a Noetherian ring. There is a duality

$$\begin{tikzcd} \mathsf{fp}(\mathsf{mod}(R), \mathsf{Ab}) \arrow[r, bend left, "D"] & \mathsf{fp}(\mathsf{mod}(R^{op}), \mathsf{Ab}) \arrow[l, bend left, "D"] \end{tikzcd}$$

given by

$$DF(X) := \mathsf{Nat}(F, \_ \otimes X)$$

$$DF(X) := \mathsf{Nat}(F, X \otimes \_\,)$$

Properties of D

Proposition (Auslander)

The duality D satisfies the following. For $X \in \text{mod}(R^{op})$

1. $D(X, \_\ ) = \_ \otimes X$
2. $D(\_ \otimes X) = (X, \_\ )$

Appearances of D

- It was first discovered by Auslander
- It was independently discovered by Gruson and Jensen.
- Hartshorne found D using an approach similar to Auslander.
- Krause showed how to obtain D from a universal property.
- It was discovered model theoretically through work of Mike Prest, Ivo Herzog, and Kevin Burke.
- Russell recovered D in a different way while extending the concept of linkage of modules to finitely presented functors.

Free Abelian Category

Let $\mathcal{A}$ denote any pre-additive category. A **free abelian category on** $\mathcal{A}$ is an abelian category $\mathsf{Ab}(\mathcal{A})$ together with an additive functor $j\colon \mathcal{A} \to \mathsf{Ab}(\mathcal{A})$ satisfying the following universal property:

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Theorem (Gruson, Krause)

The double Yoneda embedding $\mathsf{Y}^2 \colon \mathcal{A} \to \mathsf{fp}(\mathsf{mod}(\mathcal{A}^{op}), \mathsf{Ab})$ is the free abelian category on $\mathcal{A}$.

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The Functorial Approach

The category $\mathsf{fp}(\mathsf{mod}(\mathcal{A}^{op}), \mathsf{Ab})$ is a universal solution to the abelianization problem.

ppf's

Notation: (Last Time)

R - ring

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A, B - matrices with entries from R.

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Definition

A **pp-formula**, abbreviated ppf, is a formula of the form

$$\varphi(\overline{x}) \iff \exists \overline{y}\ A\overline{x} + B\overline{y} = 0$$

Examples

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Example

Take A to be an $n \times n$ matrix with entries in R and again take $B = 0$. Then the matrix equation

$$A\overline{x} = 0$$

is a ppf.

A Very Concrete Example

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Example

Take $R = \mathbb{Z}$ and

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Then the following is a ppf:

$$\exists y \quad \text{such that}$$

$$\begin{aligned} 2x_1 + 2x_2 + 2y &= 0 \\ x_1 + 2x_2 + y &= 0 \\ 3x_1 + 4x_2 + 2y &= 0 \end{aligned}$$

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$$F_\varphi(M) = \left\{ \overline{x} \in M^{l(\overline{x})} \mid \exists \overline{y} \in M^{l(\overline{y})} \; A\overline{x} + B\overline{y} = 0 \right\}$$

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- Given a morphism $f\colon M \to N$,

$$F_\varphi(f)\colon F_\varphi(M) \to F_\psi(N)$$

is defined to be the restriction of f to these subgroups.

The Functor Determined by a ppf

Proposition (Prest)

For each ppf φ, the functor $F_\varphi\colon \mathsf{mod}(R^{op}) \to \mathsf{Ab}$ is a finitely presented functor.

pp-pairs

Definition

A **pp-pair** φ/ψ is a pair (φ, ψ) of ppf's in the same number of variables such that ψ implies φ.

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Theorem (Herzog)

1. The pp-pairs form a category denoted $\mathbb{L}_R^{\mathsf{eq}+}$
2. The category $\mathbb{L}_R^{\mathsf{eq}+}$ is abelian.

An Equivalence of Categories

Theorem (Burke)

The categories $\mathbb{L}_R^{\mathsf{eq}+}$ and $\mathsf{fp}(\mathsf{mod}(R^{op}), \mathsf{Ab})$ are equivalent.

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$$\mathbb{L}_R^{\mathsf{eq}+} \cong \mathsf{fp}(\mathsf{mod}(R^{op}), \mathsf{Ab})$$

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Theorem

The duality D on $\mathbb{L}_R^{\mathsf{eq}+}$ is indeed the Auslander-Gruson-Jensen duality.

The Merit of Auslander's Approach

I. The Functorial Approach has applications to representation theory. (e.g. Almost Split Sequences.)
II. The Functorial Approach generalizes module theoretic concepts.
III. The Functorial Approach produces a universal solution to the abelianization problem.
IV. The Functorial Approach establishes connections with other fields which are not at all obvious. (e.g. Model Theory)

Thank You!