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Morphisms determined by objects - Johan Steen

Outline:
§1. Auslander's theorems (mod Λ)
- basic definitions
- lattices following Ringel
- the theorems
- relation to AR-theory

§2. Existence and non-existence in additive categories

Let Λ be an artin algebra. We work in mod Λ.

Definition [Auslander]: $\alpha : X \to Y$ is right C-determined ($C \in \operatorname{mod}\Lambda$) if for $\alpha' : X' \to Y$,
α' factors through $\alpha \iff \operatorname{im} \operatorname{Hom}_\Lambda(C, \alpha') \subseteq \operatorname{im} \operatorname{Hom}_\Lambda(C,\alpha)$

As a diagram:

$$\begin{tikzcd} & X \arrow[r] & Y \\ C \arrow[ur, dashed] \arrow[r, "\forall\varphi"'] & X' \arrow[u, dashed, "\Longleftrightarrow"'] \arrow[ur] & \end{tikzcd}$$

Recall:
- $f : A \to B$ is right minimal if $fh = f \Longrightarrow h$ isom.
- for any f, there is a right minimal version $f_1 : A_1 \to B$ by writing $A = A_0 \oplus A_1$, with $A_0 \subseteq \ker f$ maximal

Observations:
- α right C-determined $\Longrightarrow$ α right $C \oplus C'$-determined for all C'
- α right C-determined $\iff$ α_1 right C-determined (α_1 as in Recall above)
- α right minimal. α right Λ-determined $\iff$ α monom.

Proof: "$\Longrightarrow$" let $\alpha : X \to Y$, $\alpha\psi = 0$ with $\psi : \Lambda \to X$

Then we have
$$\begin{tikzcd} & X \arrow[r, "\alpha"] & Y \\ \Lambda \arrow[ur, "\psi"] \arrow[rr, "0"'] & & \operatorname{im}\alpha \arrow[ul, dashed, "s"] \arrow[u] \end{tikzcd}$$
and $\alpha s \alpha = \alpha$, thus $s\alpha$ is isom.
and $s\alpha\psi = 0 \Longrightarrow \psi = 0$

"$\Longleftarrow$" follows by def.

Theorem: Let $\alpha : X \to Y$ be any morphism. Then α is right $\tau^-(\ker\alpha) \oplus \Lambda$ - determined.
This result by Auslander was sharpened by Ringel: —— " —— $\tau^-(\ker\alpha) \oplus P(\operatorname{soc} \operatorname{coker}\alpha)$ - det.

Remark: [ARS] there is a minimal determining object, $C(\alpha)$.

Lattices

Fix $Y \in \operatorname{mod} \Lambda$; $\alpha : X \longrightarrow Y$, $\beta : W \longrightarrow Y$ are called right equivalent if α factors through β and β factors through α.

$[\longrightarrow Y\rangle := \{[\alpha : X \longrightarrow Y]\}$ is the right factorization lattice:

- poset: $[\alpha'] \leq [\alpha]$ if α' factors through α.
- join: $[\alpha : X \longrightarrow Y] \vee [\alpha' : X' \longrightarrow Y] = [(\alpha\, \alpha') : X \oplus X' \longrightarrow Y]$
- meet: $[\alpha] \wedge [\alpha'] = [PB \longrightarrow Y]$,

$$\begin{tikzcd} PB \arrow[r, dashed] \arrow[d] & X' \arrow[d, "\alpha'"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

(PullBack - construction)

${}^{C}[\longrightarrow Y\rangle := \{[\alpha : X \longrightarrow Y] \mid \alpha \text{ right } C\text{-determined}\}$; this is a subposet, it is closed under meets, but not joints.

Theorem 1 [Auslander]: $[\to Y\rangle = \bigcup_C {}^{C}[\to Y\rangle$.

On the other hand, we have $\quad \operatorname{SHom}_\Lambda(C, Y)$ submodule lattice (right $\operatorname{End}(C)$-mod)
$\operatorname{im} \operatorname{Hom}_\Lambda(C, \alpha : X \longrightarrow Y)$ is a submodule of $\operatorname{Hom}(C, Y)$.

So one defines $\eta_{CY} : {}^{C}[\longrightarrow Y\rangle \longrightarrow \operatorname{SHom}_\Lambda(C, Y)$, where $[\alpha] \mapsto \operatorname{im} \operatorname{Hom}_\Lambda(C, \alpha)$.

This is well-defined and injective: $[\alpha] = [\beta] \iff \operatorname{im} \operatorname{Hom}(C, \alpha) = \operatorname{im} \operatorname{Hom}(C, \beta)$.

It also preserves meets

Theorem 2 [Auslander, Ringel]: η_{CY} is surjective, in fact it is a lattice isom.

Remark: • α C-determined, is actually determined by both C and $\operatorname{im} \operatorname{Hom}_\Lambda(C, \alpha)$.

Relation to AR-theory

Let $C = Y$ be indecomposable. Then ${}^{Y}[\longrightarrow Y\rangle \simeq \operatorname{End}_\Lambda(Y) \geq \underline{m}$ (local)

Let α be the right minimal morphism determined by Y and $\underline{m}$. α is not split epim.

and:

$$\begin{tikzcd} X \arrow[rr, "\alpha"] & & Y \\ & Y \arrow[ul, dashed] \arrow[ur, "h"'] & \end{tikzcd}$$

factors $\iff \operatorname{im}(Y, h) \subseteq \operatorname{im}(Y, \alpha) = \underline{m} \iff h$ not isom.

Hence α is right minimal almost split.

If Y is non-projective, then $0 \longrightarrow \underset{\underset{\tau Y}{\|}}{\ker \alpha} \longrightarrow X \longrightarrow Y \longrightarrow 0$ is a short ex. seq. and almost split exact.

§2. Existence and non-existence in additive categories [Krause, Chen-Le]

Let $\mathcal{C}$ be an additive category

Definition: $\mathcal{C}$ has right determined (epi)morphisms if for all $Y \in \mathcal{C}$:

(1) every $\alpha : X \longrightarrow Y$ (epi)morphism is right C-determined by some $C \in \mathcal{C}$

(2) for any $C \in \mathcal{C}$ and $H \subseteq \mathrm{Hom}_{\mathcal{C}}(C, Y)$ there is a (epi)morphism $\alpha : X \longrightarrow Y$ which is right C-determined and $\mathrm{im}\, \mathrm{Hom}_{\mathcal{C}}(C, \alpha) = H$.

From now let $\mathcal{C}$ be
- essentially small
- k-linear
- Hom-finite
- idempotent complete

Proposition [Krause]: $\mathcal{C}$ has determined morphisms $\Longleftrightarrow$ $\mathcal{C}$ is a dualizing k-variety [AR '74].

Triangulated categories

Theorem [Krause]: $\mathcal{C}$ in addition triangulated. $\mathcal{C}$ admits a Serre functor S

$\Longleftrightarrow$ $\mathcal{C}$ has determined morphisms. α is right determined $S^{-1}(\mathrm{cone}(\alpha))$.

Abelian categories

$\mathcal{C}$ abelian has Serre duality if $\exists\ \tau : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ such that

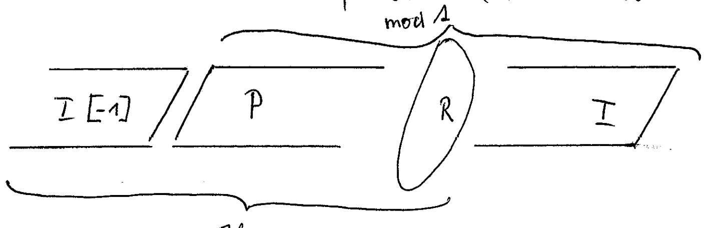
$$D\mathrm{Ext}^1(X, Y) \cong \mathrm{Hom}(Y, \tau X) \qquad (*)$$

Theorem [Chen-Le]: $\mathcal{C}$ has Serre duality $\Longleftrightarrow$ $\mathcal{C}$ has right determined epim. and left determined monom.

Observe: Let $\alpha : X \longrightarrow Y$ be right determined by some $C \in \mathcal{C}$, where $\mathcal{C}$ is as in the thm. Choose $\beta : Z \twoheadrightarrow Y$ right C-determined st. $\mathrm{im}\, \mathrm{Hom}(C, \alpha) = \mathrm{im}\, \mathrm{Hom}(C, \beta)$. Then $[\alpha] = [\beta]$, so α is epim.

<u>Consequence</u>: $\Lambda = kQ$, Q not dynkin.

(*) holds for AR-translate, but $\tau(P) = 0$. So τ is not an equivalence.



Then $\mathcal{H}$ is a hereditary category with Serre duality. There are morphisms in $\mathcal{H}$ that are not epim. and are therefore not right C-determined by any C.