Bielefeld, 15 Nov 2014

Jan Stovicek, Higher triangulations via the calculus of homotopy (co)limits

arXiv: 1401.6451, 1402.6984, 1409.5003 (jt. M. Groth)

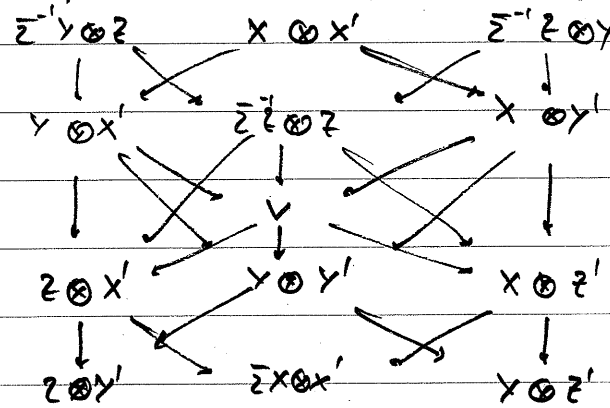
I. Motivation

- pick $(\mathcal{T}, \otimes)$ triangulated with tensor product $\otimes$
  - R comm. noetherian $(D^b(\text{proj } R), \otimes^{\mathbb{L}}_R)$
  - G finite group, k field $(\underline{\text{mod}}\, kG, \otimes_k)$
- take $f: X \to Y$, $f': X' \to Y'$ in $\mathcal{T}$
  - inspect $f \otimes f'$

$$\begin{tikzcd}
X \otimes X' \arrow[r, "f\otimes id"] \arrow[d, "id\otimes f'"'] & Y \otimes X' \arrow[r] \arrow[d, "id\otimes f'"] & Z \otimes X' \arrow[r] \arrow[d] & \Sigma X \otimes X' \arrow[d] \\
X \otimes Y' \arrow[r] \arrow[d] & Y \otimes Y' \arrow[r] \arrow[d] & Z \otimes Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ominus"] & \Sigma X \otimes Y' \arrow[d] \\
\Sigma(X \otimes X') \arrow[r] & \Sigma(Y \otimes X') \arrow[r] & \Sigma(Z \otimes X') \arrow[r] & \Sigma^2(X \otimes X')
\end{tikzcd}$$

- we have 9 objects but up to $\cong$ and Σ only 3 new U, V, W
- together 12 objects
- suggestion of Keller-Neeman

May, The additivity of traces in triang. categories



(TC3)

$$V \to (Z \otimes X') \oplus (Y \otimes Y') \oplus (X \otimes Z') \to W \to \Sigma V$$

(TC4)

II. Triangles and mapping cones

- $\mathcal{A}$ ess. small abelian
  - if Q is a quiver: $Rep_{\mathcal{A}}(Q)$
  - if $\mathcal{C}$ is a small cat.: $Rep_{\mathcal{A}}(\mathcal{C}) = Fun(\mathcal{C} \to \mathcal{A})$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

- consider $D^b(\mathcal{A})$, $D^b(rep_{\mathcal{A}}(\mathcal{C}))$
- for instance $\mathcal{C}: \bullet \to \bullet$

  $Coker: rep_{\mathcal{A}}(\bullet \to \bullet) \to \mathcal{A}$

  $\Rightarrow$ left derive $\underset{\underset{Cone}{\|}}{\mathbb{L}Coker}: D^b(rep_{\mathcal{A}}(\bullet \to \bullet)) \to D^b(\mathcal{A})$

- $\mathbb{L}Coker$ satisfies a universal property

$$\begin{tikzcd} C^b(rep_{\mathcal{A}}(\cdot \to \cdot)) \arrow[r] \arrow[d, "Q"'] & C^b(\mathcal{A}) \arrow[d, "Q"] \\ D^b(rep_{\mathcal{A}}(\cdot \to \cdot)) \arrow[r, dashed] \arrow[ur, dashed] & D^b(\mathcal{A}) \end{tikzcd}$$

$\Rightarrow$ $\mathbb{L}Coker$ unique, up to unique iso $\Rightarrow$ $\mathbb{L}Coker \cong Cone$

$$rep_{D^b(\mathcal{A})}(\cdot \to \cdot) \xleftarrow[(*)]{dia} D^b(rep_{\mathcal{A}}(\cdot \to \cdot)) \xrightarrow{cone} D^b(\mathcal{A})$$

Verdier localization, both adjoints

(*) full, ess. surjective, reflects isos

associate this $\rightsquigarrow$ triangulated categories

III. Bicartesian squares

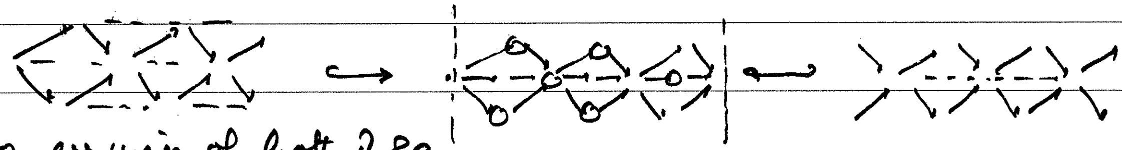
- consider

$$\begin{tikzcd} D^b(rep_{\mathcal{A}}(\ulcorner)) \arrow[r, bend left, "\lambda \text{ fully faithful}"] & D^b(rep_{\mathcal{A}}(\square)) \arrow[l, hook', "forget"'] \arrow[r, "forget"] & D^b(rep_{\mathcal{A}}(\lrcorner)) \arrow[l, hook', bend left, "q \text{ fully faithful}"] \end{tikzcd}$$

λ = htpy pushout

q = htpy pullback

$\mathcal{A} = \mathrm{mod}\, k$



o ess. image of both R & Q

— terminology

~~rep $D^b_{\mathcal{A}}$(…)~~ $\mathrm{rep}_{D^b(\mathcal{A})}(\square)$ uncolored squares

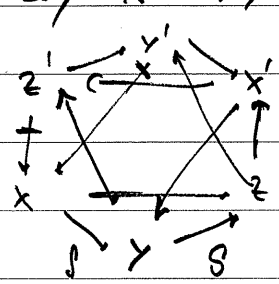
$D^b(\mathrm{rep}_{\mathcal{A}}(\square)) \to$ colored squares

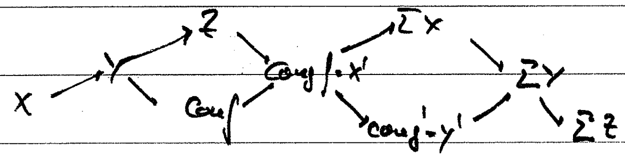
$\cup\!|$

$\mathrm{Im}\, R = \mathrm{Im}\, Q$ bicartesian squares

IV. Octahedron, higher triangles

— say $X \xrightarrow{f} Y \xrightarrow{g} Z \to$ in $D^b(\mathcal{A})$





— Q: If we have

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[d] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \end{tikzcd}$$

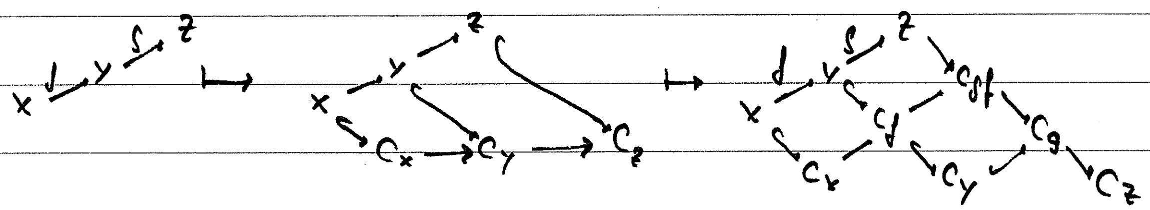
Does this extend to a map of octahedra?

— No! (Künzer)

— yes: if we consider only distinguished octahedra (3-triangles)

— definition by construction

· down to earth: in $C^b(\mathrm{rep}_{\mathcal{A}}(\cdot\to\cdot\to\cdot)) \cong \mathrm{rep}_{C^b(\mathcal{A})}(\cdot\to\cdot\to\cdot)$



- more formally
  - $\mathcal{G} = \mathrm{Ind}\,\mathcal{A} = \mathrm{Fct}(\mathcal{A}^{op}, Ab)$, Grothendieck, $\mathcal{A} = fp\,\mathcal{G}$
  - $D^b \rightsquigarrow D$
  - notation: $D_{\mathcal{U}}(\mathcal{C}) = D(\mathrm{Rep}_{\mathrm{Ind}\mathcal{A}}(\mathcal{C}))$

$$D_{\mathcal{A}} = D(\mathrm{Ind}\,\mathcal{A})$$

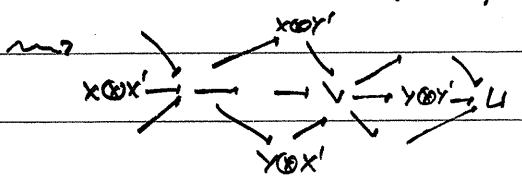
$$\mathrm{rep}_{D_{\mathcal{U}}}(\cdot\to\cdot\to\cdot) \xleftarrow[\text{full, ess. surj, reflects isos}]{\text{diag}} D_{\mathcal{A}}(\cdot\to\cdot\to\cdot) \underset{q}{\overset{\text{forget}}{\rightleftarrows}} D_{\mathcal{U}}(\quad) \xleftarrow{\text{forget}} D(\quad)$$

- objects in the ess. image of the chain: standard octahedra (3-triangles)
- distinguished octahedra: closure under $\cong$
- repres. of A_n $\rightsquigarrow$ N-triangles
- axioms (Maltsiniotis, Balmer, Künzer)

V. May's axioms, n-angulations, other

- May: $f, f' \in \mathrm{rep}_{D_{\mathcal{U}}}(\cdot\to\cdot)$
  $F, F' \in$ ~~rep~~ $D_{\mathcal{U}}(\cdot\to\cdot)$
  $F \otimes F' \in D_{\mathcal{U}}(\square)$

$\rightsquigarrow$



VI. Conceptual approach

$F \in D_{\mathcal{U}}(\cdot\to\cdot\to\cdot)$, $\mathcal{A}$ k-linear

$$D^b(\mathrm{mod}\, k(\cdot\to\cdot\to\cdot)) \xrightarrow{\text{octahedron}_F} D_{\mathcal{A}}$$

- k a field, $k = \mathbb{Z}$

↳ what should this be, if general $\mathcal{T}$ instead of $D_{\mathcal{U}}$

- answer (Heller, Grothendieck, Cisinski, Tabuada, ...)

$\mathrm{Ho}(\mathrm{rep}_{\mathcal{G}}(\cdot\leftarrow\cdot\leftarrow\cdot))$