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Morphisms $\overbrace{}^{\text{and functors (!)}}$ determined by objects – Gordana Todorov

M. Auslander introduced, developed and applied this concept
- for a given morphism, decide if there is an object which determines all the morphisms which factor through it
- Ringel: develops more theory to better understand mod Λ
- Krause: explores other categories and related notions (also Chen-Le)

<u>Definitions:</u> assume the category to be additive

<u>Def. 1:</u> A morphism $X \xrightarrow{f} Y$ is determined by an object C if given any $U \xrightarrow{g} Y$,

$$\left(g \text{ factors through } f \right) \iff \left(\begin{array}{c} C \xrightarrow{\alpha} U \xrightarrow{g} Y \\ \text{factors through } f \text{ for all } \alpha \end{array} \right)$$

"Enough to check maps from C"

<u>Def. 2:</u> A morphism $X \xrightarrow{f} Y$ is $\overset{\text{completely}}{\text{determined}}$ by an object C if given any $U \xrightarrow{g} Y$

$$\left(\mathrm{Im}(\,,g) \subset \mathrm{Im}(\,,f) \right) \iff \left(\mathrm{Im}(\,,g)(C) \subset \mathrm{Im}(\,,f)(C) \right),$$

where we have

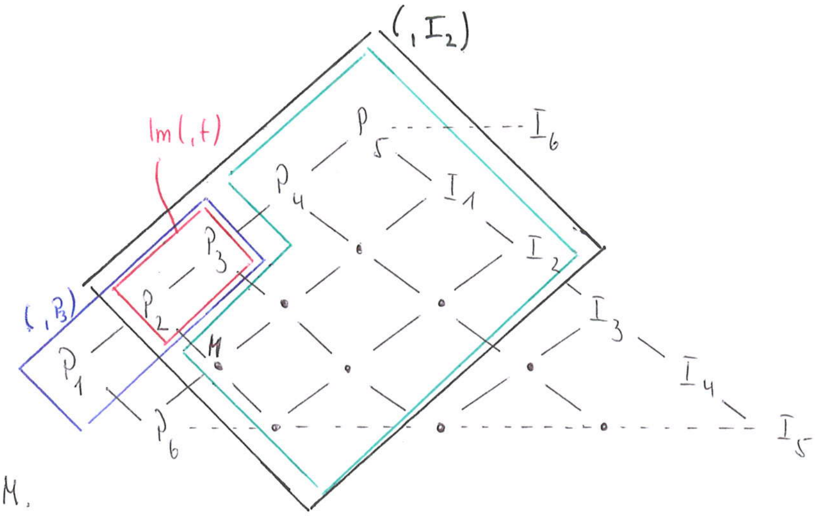
$$\begin{tikzcd}
 & & \mathrm{Im}(\,,g) \arrow[d, hook] & (\,,U) \arrow[l, two heads] \arrow[dl, "(\,,g)"] \\
(\,,X) \arrow[rr, "(\,,f)"] \arrow[dr, two heads] & & (\,,Y) & \\
 & \mathrm{Im}(\,,f) \arrow[ur, hook] & &
\end{tikzcd}$$

<u>Def. 3:</u> A morphism $X \xrightarrow{f} Y$ is determined by C if every non-zero subfunctor of F_f is non-zero on C where

$$\begin{tikzcd}
(\,,X) \arrow[rr] \arrow[dr, two heads] & & (\,,Y) \arrow[r] & F_f \arrow[r] & 0 \\
 & \mathrm{Im}(\,,f) \arrow[ur, hook] & & &
\end{tikzcd}$$

Example 1: $Q = 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5$, with $6 \to 1$

Then the AR-quiver looks as follows:



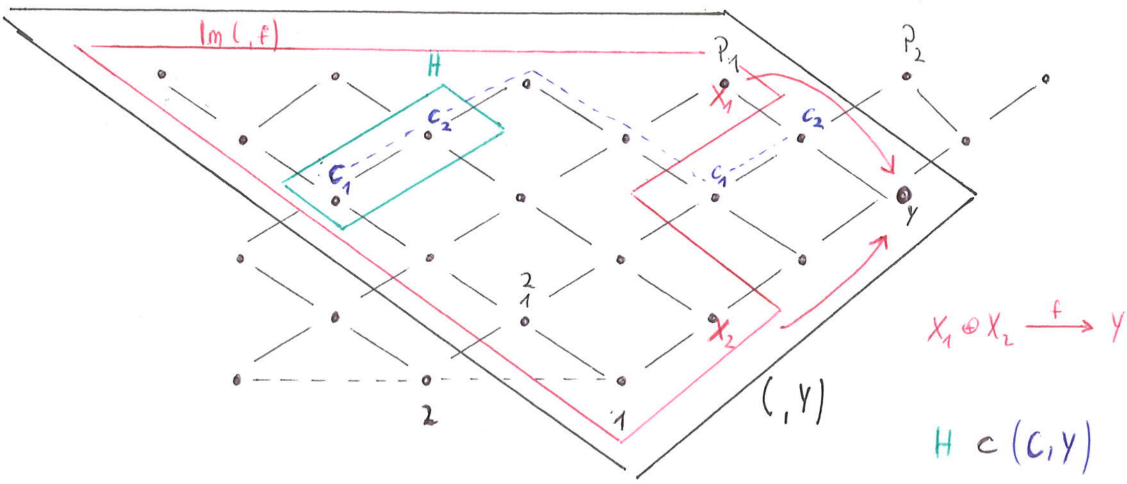
Take $P_3 \xrightarrow{f} I_2$, then

$$(\ ,P_3) \xrightarrow{(\ ,f)} (\ ,I_2)$$
$$(\ ,P_3) \to \mathrm{Im}(\ ,f) \to (\ ,I_2)$$

Every subfunctor of F_f is non-zero on $P_4 \oplus M$.

Remark: AR-quiver is the opposite quiver of the Auslander algebra Γ.

Example 2: $Q = 1 \rightleftarrows 2$. Then the AR-quiver is given as



This is an illustration of the construction as it appears in Thm. 2 below.

Questions (Auslander):

(A1) Given a morphism $X \xrightarrow{f} Y$, is there an object C so that f is determined by C?

(A2) Given $Y, C, H \subset (C, Y)$ as $\mathrm{End}(C)^{op}$-module, is there a morphism $X \xrightarrow{f} Y$ so that
- f is determined by C
- $(\mathrm{Im}(, f))(C) = H$

?

Theorem 1 [Auslander]: Λ artin algebra. Let $X \xrightarrow{f} Y$ be a Λ-homom. Then f is determined by $C = \mathrm{Tr}\, D(\mathrm{Ker}\, f) \oplus P_0(\mathrm{Soc}\, \mathrm{Coker}\, f)$.

Remark: (i) C is not "minimal determinator" in general.
(ii) Ringel has nice results about minimal determinators (criterion on Ext^2)
(iii) Many more questions

Theorem: Λ artin algebra and $X \xrightarrow{f} Y$. Consider again $(, X) \longrightarrow (, Y) \longrightarrow F_f \longrightarrow 0$ (in mod Λ). Let $\mathrm{Soc}\, F_f = \coprod^{n} (, C_i) / \mathrm{rad}(, C_i)$. Then $\coprod^{n} C_i$ determines f.

Theorem 2 [Auslander]: (mod Λ) Given $Y, C, H \subset (C, Y)$ as $\mathrm{End}(C)^{op}$-module.
There exists $X \xrightarrow{f} Y$ so that
- f is determined by C
- $\mathrm{Im}(, f)(C) = H$

Idea: Auslander constructed f by "constructing $\mathrm{Im}(, f)$"

Proposition: Setup as in Thm 2. $\mathrm{Im}(, f) \subset (, Y)$ is the maximal subfunctor of $(, Y)$ so that $\mathrm{Im}(, f)(C) = H$.

Remark: This "construction" can be generalised to subfunctors determined by objects.

Another kind of construction: $(\underline{\underline{A}}^{op}, Ab) \underset{ev}{\overset{\mathrm{coind}_C}{\rightleftarrows}} \mathrm{mod}\,\Gamma$, $\quad \Gamma = \mathrm{End}(C)^{op}$

$$F \longmapsto F(C),$$

right adjoint to evaluation: $\mathrm{coind}_C\left(I\left((C,Y)/H\right)\right)$ is an injective functor $A^{op} \longrightarrow Ab$.

$$\rightsquigarrow \quad (\ ,X) \xrightarrow{(\ ,f)} (\ ,Y) \longrightarrow \mathrm{coind}_C(I)$$

Theorem [Krause]: $\underline{\underline{A}}$ category with some conditions (e.g. Mod R), and let $\underline{\underline{C}} \subset \mathrm{fp}\,\underline{\underline{A}}$ (e.g. mod R). Given Y in $\underline{\underline{A}}$ and $H \subset (C,Y)|_{\underline{\underline{C}}}$ $\exists\ X \xrightarrow{f} Y$ such that f is determined by C and $\mathrm{Im}\,(\ ,f)|_{\underline{\underline{C}}} = H$.

3 main steps in the proof:

$$\begin{tikzcd}
\text{[flat cover]} & (\ ,X)|_{\mathrm{fp}\,\underline{\underline{A}}} \arrow[d] \arrow[dr, "(\ ,f)"] & \\
\text{[Auslander]} & F_H & (\ ,Y)|_{\mathrm{fp}\,\underline{\underline{A}}} \\
& H \arrow[r, hook] & (\ ,Y)|_{\underline{\underline{C}}}
\end{tikzcd}$$