Auslander McKay correspondence in dimension 3.

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(OR: Maurice $\Rightarrow$ Mori?)

§1. Auslander - McKay in dim 2.

§2. 3dimensional geometry background

§3. Auslander - McKay in dim 3.

§1. Input $G \underset{\text{finite}}{\leq} SL(2, \mathbb{C})$

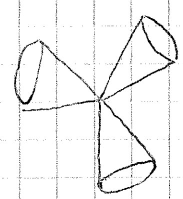
Example: $G = \left\langle \underset{g}{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}, \underset{h}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \right\rangle \circlearrowright \mathbb{C}^2$

Interested in

$$\mathbb{C}[\![x,y]\!]^G = \{ f \in \mathbb{C}[\![x,y]\!] \mid g \cdot f = f \ \ \forall g \in G \}$$

g sends: $x \mapsto ix$, $y \longrightarrow -iy$ h sends: $x \mapsto y$, $y \mapsto -x$

$x^4 \xmapsto{g} (ix)^4 = x^4$, $x^4 \xmapsto{h} y^4$ $y^4 \xmapsto{g} (-iy)^4 = y^4$, $y^4 \xmapsto{h} (-x^4) = x^4$



$A := x^4 + y^4$ is invariant

so are $B := (xy)^2$

$C := (xy)(x^4 - y^4)$ $\mathbb{C}[\![x,y]\!]^G \cong \mathbb{C}[\![A,B,C]\!] / C^2 = B(A^2 - 4B^2)$

singular at the origin

Geometry: want to understand minimal resolution of $\mathbb{C}^2/G$

A resolution of $\mathbb{C}^2/G$ is a morphism

$X \xrightarrow{f} \mathbb{C}^2/G$ s.t. (1) X is smooth

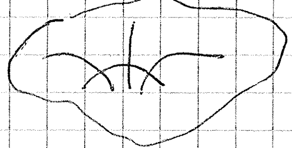
(2) f is blow up of ideal

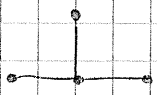
(3) $f^{-1}(0) \xrightarrow{\sim} (\mathbb{C}^2/G) \setminus 0$

Since $\dim(\mathbb{C}^2/G) = 2$ a <u>minimal resolution</u> exists

<u>Def</u>: Consider minimal resolution $X \longrightarrow \mathbb{C}^2/G$ above 0 consists of a tree of curves. Define <u>dual graph</u>:

vertices $\longleftrightarrow$ irred. curves join two vert. $\Longleftrightarrow$ curves intersect.





du Val: resolutions of $\mathbb{C}^2/G$ $(G \underset{f}{\leq} SL_2(\mathbb{C}))$ are all

ADE Dynkin diagrams.

cyclic groups $\longleftrightarrow$ Type A

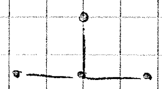
binary dihedral $\longleftrightarrow$ Type D ...

<u>THM</u> (Auslander) Suppose R is complete local du Val sing (i.e. $R \simeq \mathbb{C}[\![x,y]\!]^G$ for some $G \underset{f}{\leq} SL(2,\mathbb{C})$)

then

AR quiver of <u>CM</u> $\mathbb{C}[\![x,y]\!]^G$ = double of dual graph of minimal resolution

Proved: (AR quiver = McKay quiver)

Example:  $\rightsquigarrow$ 

<u>Fundamental insight</u>: homological algebra on the base singularity "knows" the resolution

§2. <u>Some 3d geometry</u>

What are we aiming for?

Reinterpret minimal resolution means (minimal model program MMP)

<u>Key</u>: for minimal resolutions $f^* \omega_R = \omega_X$ (i.e. crepant)

for non-minimal resolutions this is false

<u>Guess 1</u>: aim for crepant resolutions

$X \xrightarrow{f} \operatorname{Spec} R$ X smooth $f^* \omega_R = \omega_X$

NO!! They don't always exist

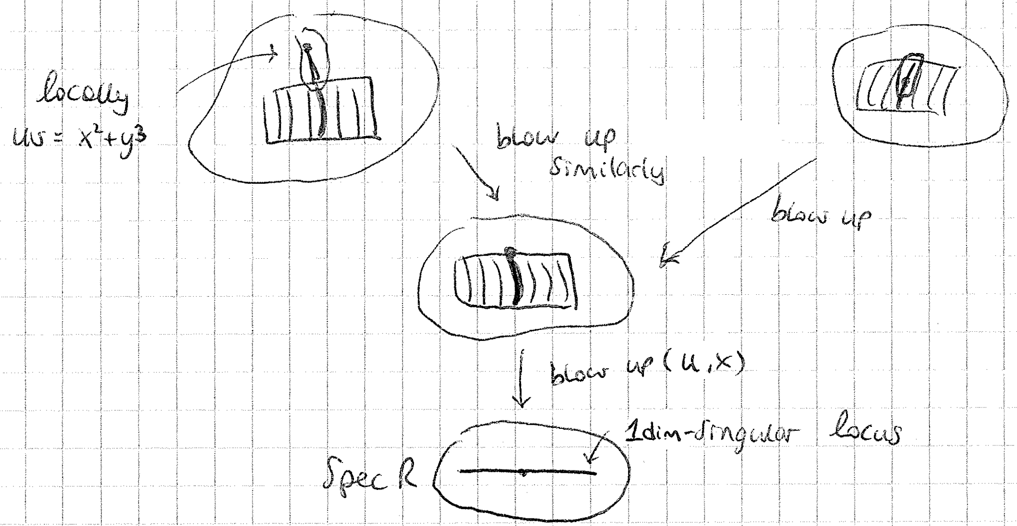
$\boxed{\text{INPUT: Spec } R}$ $\longrightarrow$ $\boxed{\text{BLACK BOX: MMP (involves choices)}}$ $\longrightarrow$ $\boxed{\text{OUTPUT: } X \to \text{Spec } R \text{ minimal model(s)}}$

Notes: 1. X may be singular! Saying exactly what singularities are allowed is quite technical.

→ "$\mathbb{Q}$ factorial terminal"

2. Minimal models are not unique, they are connected by flops.

Example: $R = \mathbb{C}[u,v,x,y] / uv = x^2(x^2+y^3)$ Non isolated singularity



Aim: Replace minimal resolution ⟶ minimal model

Input: those R Gorenstein normal 3-folds for which the minimal model(s) have fibres at most one dimensional

(= compound du Val singularities)
CDV

§3 Auslander-Mackay in dim 3

finite CM type ⟶ gl dim End(M) = 2

⟶ (better) Iyama finite type = 0-cluster tilting

Guess 1: aim of 1-cluster tilting objects in CM R

No!!

2 reasons

⟶ 1) gldim End(1-cluster tilting object) = 3

⟶ 2) Ext vanishing is too strong

Solves both at same time: R 3d Gorenstein normal (e.g. cdV)

Def: $M \in \mathrm{CM}R$ is called modifying if

$$\mathrm{fl}_R \, \mathrm{Ext}^1_R(M, M) = 0$$

(R isolated equiv to $\mathrm{Ext}^1_R(M,M) = 0$)

We say M is max modifying if it is modifying & maximal with respect to this property

$$\mathrm{add}\, M = \{ X \in \mathrm{CM}R \mid \mathrm{fl}\, \mathrm{Ext}^1(M \oplus X, M \oplus X) = 0 \}$$

(R isolated max mod $\Leftrightarrow$ max rigid)

Aim: max modifying objects

Theorem: If R is complete local cDV then

$$\left\{ \begin{array}{c} f_i : X_i \to \mathrm{Spec}\, R \\ \text{min. models} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{basic max mod} \\ \text{objects in } \underline{\mathrm{CM}}R \end{array} \right\}$$

$$X_i \longmapsto H^* \left(\begin{array}{c} \text{basic proj. gen.} \\ \text{of perverse sheaves} \end{array} \right)$$

certain Moduli space of reps of $\mathrm{End}_R(R \oplus M)$ $\longleftarrow$ M

Further: for fixed M max modifying

(a) Indecomp. summands of M $\xleftrightarrow{1:1}$ exceptional curves corresponding minimal model

(b) quiver of $\underline{\mathrm{End}}_R(M) =$ double of dual graph (+ new loops)

<u>In particular</u>: the number of max mod objects is finite

Further: mutation graph of max mod objects $\longleftrightarrow$ flops graph from geometry

If R isolated + minimal model(s) are smooth reduces to

$$\left\{ \begin{array}{c} \text{crepant} \\ \text{resolutions} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{cluster tilting objects} \\ \text{in } \underline{\mathrm{CM}}\, R \end{array} \right\}$$