Partial tilting complexes and beyond

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About PLACES

BIELEFELD Universititãt &

the **University of PADOVA** =

the most important places where

I studied, worked long time ago.

Results of this talk are

based of the hope and feeling that **INDECOMPOSABLE** complexes may play a big role [bigger than that of indecomp. modules]. Very often, the same holds for the complexes whose **INDECOMPOSABLE** summands have **INDECOMPOSABLE** non-zero components

(= **”well - behaved”** for short) .

The **trick** to **delete** many complicated properties and

to look only some of them, if possible of "**combinatorial** " type (concerning the **underlying vector spaces** of algebras,

modules, complexes) **=** the trick I used several times from the very beginning in quite different situations.

More or less old situations :

Abelian Groups & **finiteness conditions** showed up in my “tesi di laurea” on “ **Abelian Groups** whose **endomorphism ring** is locally compact in the **finite topology** ” (under the direction of **A. ORSATTI**)

Adalberto ORSATTI =

- supervisor of the master (master + PhD) thesis of many italian colleagues who studied or worked in Padova for some time.
- organganizer of algebra meetings in Italy.

Endomorphism rings of abelian groups -

equipped with the **finite topology** - and

Corner’ s type realization theorems =

subject of my first talk in BIELEFELD ......

....

and of my first conversation with

Claus Michael RINGEL

during a coffee break ...

Notation / Conventions

K = algebraically closed field

MODULE = left module over a **K** - algebra

COMPLEX = **right bounded** complex with projective components

MORPHISM of complexes = morphism / homotopy

M° = projective resolution of the module M

T partial n - tilting module = **. . . . . .**

With some hypothesis on the **Hom spaces of NON surjective morhisms** between …..

A **right bounded** string of integers > 0

[………. , m(2) , m(1)] stands for

the **indecomposable right bounded**

complex **C °** s.t. (proceeding from right

to left) the indecomp. projective modules

………. , P(m(2)) , P(m(1))

as the non - zero components of **C °** .

The meaning of **PARTIAL TILTING** (or n - tilting) module in this talk

T **PARTIAL TILTING** :

- proj dim **T** at most n

- $\mathrm{Ext}^{i}(\mathbf{T}, \oplus\, \mathbf{T}) = 0$ for all $i > 0$

The meaning of **TILTING (or** n-tilting) module **T** in this talk :

- the **projective** dimension of **T** is at most n .

- $\mathbf{Ext}^{i}\,(\mathbf{T},\ \Sigma)\ \mathbf{=\ 0}$ where Σ is any **direct sum** of copies of **T** and **i = 1, ….., n** .

- There is a **long** exact sequence of the form

$$\mathbf{0} \longrightarrow \mathbf{R} \longrightarrow * \ \ldots\ldots\ldots\ldots \ * \longrightarrow \mathbf{0}$$

where the **n + 1** symbols * stand for direct summands of **direct sums** of copies of **T** .

The meaning of **LARGE** partial tilting module in this talk

T partial tilting module **s.t.**

$\mathrm{Hom}(\mathbf{T}, X) = \mathrm{Ext}^*(\mathbf{T}, X) = 0$

implies $X = 0$.

FOR SHORT (in this talk)

C° is orthogonal to T° :

any morphism from T° to any shift

$C^\circ[\,i\,]$ of **C° is homotopic to zero**.

What is used to deal with complexes :

a characterization of tilting complexes given by **Y. MIYACHI** (in “Extensions of rings and tilting complexes “)

which replaces a condition on

triangulated categories

by a condition on

morphisms homotopic to zero .

Starting points :

- **BAZZONI** ` s question on the relationship between tilting modules and **large** partial tilting modules (i.e. with the functorial property described in the abstract).
- **MANTESE & TONOLO** ` s question on on the relationship between bounded and right bounded "real" complexes "orthogonal" to **the projective resolutions of** **……….**

Strategies used to deal with .....

RIGHT BOUNDED complexes of **PROJECTIVE modules** and their morphisms / homotopy :

Use as many as possible

NEW modules (with **“dual”** properties)

NEW directions (if possible)

(A)   Use as many as possible

- **INJECTIVE** modules
- **indecomposable** modules P , Q with

a **rigid** structure [i.e. Hom (P,Q) is a vector space of **dimension < 3** , and **< 2** if P and Q are not isomorphic].

FEW morphisms between …

… indecomposable projective modules

=

reason why non - zero morphisms of this form (which are not isomorphisms) are uniquely determined up to scalar, so that strings **[………. , m(2) , m(1)]** denote many useful complexes.

(B) Use as many as possible " directions "

to investigate morphisms between

BOUNDED complexes (in the category of

right bounded ones) :

from RIGHT to LEFT (= **THE** obvious direction in the **WHOLE** category)

from LEFT to RIGHT (= **THE NEW** possible & less natural direction)

A few words on **different points of view**:

- **A. De Saint Exupery**

A. De Saint Exupery ‘s assertion :

“To see clearly it is often enough to **change** our viewing **direction**.”

sums up the strategy used to deal with complexes, and - more generally - to simplify complicated objects.

Part 1 (on modules)

A result on **CANCELLATIONS** of the **OBVIOUS** direct summands of tilting modules , used to obtain **LARGE** partial tilting modules.

The meaning of LARGE **PARTIAL TILTING** module in this talk

T partial tilting module **s.t.**

$$\mathbf{Hom}(T, X) = \mathbf{Ext}^*(T, X) = 0$$

implies $X = 0$

COLPI ` s result (the “classical” case)

LARGE **partial TILTING** modules

of projective dimension at most 1

=

TILTING modules **of ……………**

BAZZONI ' s result (the "general" case)

For any n , any **TILTING module**

of projective dimension **n** **is a**

LARGE PARTIAL TILTING module.

What I proved :

For any $n > 1$ (i.e. in all possible cases) there are **LARGE** partial tilting modules of projective dimension n which are **NOT** tilting modules .

Some properties of **LARGE partial tilting** modules T (of finite length)

These modules are **SINCERE** **but NOT** always **faithful**. They may be rather small, i.e. **indecomposable** injective, and their dimension / K may be equal to the **# of simples modules** (= least dimension for a **sincere** module).

No restriction on # :

 # runns over all n > 1

even under the additional hypotheses that

- **T** is **INJECTIVE & uniserial**
- **The class of all modules X s.t.**

Ext * (T , X) = 0 for all * > 0

is the class of INJECTIVE modules.

Consequence :

LARGE partial tilting modules

&

proper direct summands of tilting modules may **NOT** be **ALMOST COMPLETE** tilting modules.

My answer to the following question:

- **WHY LARGE** partial tilting modules which are **NOT** tilting modules **?**

is that

Among many other things

(classes of modules, functors, ...)
AUSLANDER - REITEN quivers
make these modules **visible** & give the idea to find the "minimal" ones.

Idea suggested by :

AUSLANDER - REITEN quivers :

SOMETIMES

every **SINCERE** summand **T** of a **LARGE** partial tilting module

M = T ⊕ P with **P projective**

inherits from **M** the property of being a **LARGE** partial tilting module.

The following property :

" The class of all modules X s.t. Ext * (T , X) = 0 for all * > 0 is the class of INJECTIVE modules " is satisfied by many **LARGE** partial tilting modules (and "explains" why may be rather small).

THEOREM (possible choice of SOMETIMES)

(a) M large partial n- tilting module

such that the orthogonal class

$$\boxed{M^{\perp_\infty} = \bigcap_{i \geq 1} \operatorname{Ker} \operatorname{Ext}^i (M , -)}$$

is the class of all injective modules

(b) T SINCERE summand of M

with a PROJECTIVE complement

(a) & (b) IMPLY T LARGE

partial n - tilting

Example

If n > 1 , there is an **A** - module **T s.t.**

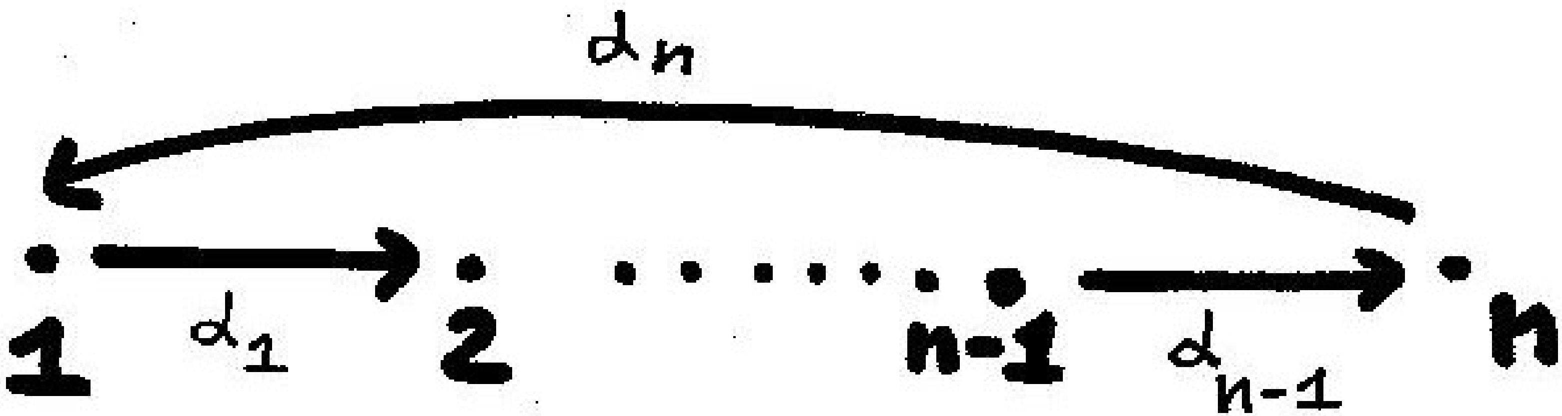
T (= unique indecomp. **injective** module which is **NOT projective**) is a **NON faithful large** partial tilting module obtained from   **D (A)** , **the K - dual of A** , after **CANCELLATION of** all its indecomposable projective summands &

2 (n - 1) = projective dimension of   T =

= global dimension of A

Example

A K- algebra given by the following quiver with $n \geq 2$ vertices



with relation $\alpha_n \cdot \alpha_{n-1} \cdot \cdots \cdot \alpha_1 = 0$

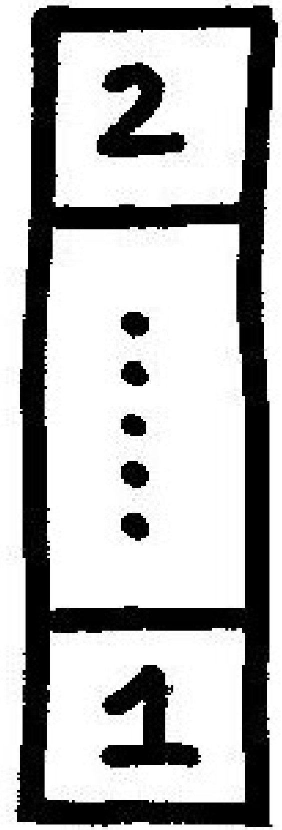
$\boxed{n \geqslant 2 \ , \ m = 2n - 2}$

The module **T** :

(unique indecomposable

injective module which is

NOT projective)



is a **NON FAITHFUL**
large partial m - tilting module

Part 2 (on complexes)

By **RICKARD + MIYACHI**`s results:

the projective resolution **T°** of a **LARGE** partial tilting module **T** (which is **NOT** a tilting module) is a partial tilting complex **T°** **s.t.** for every non - zero module **M** there is a morphism from **T°** to shift **M°[i]** of **M°** which is **NOT** homotopic to zero , **BUT**

…… BUT

there is a non - zero complex **C °**

s.t.

any morphism from **T °** to **C ° [i]**

is homotopic to zero for any integer i

[i.e. “ C ° orthogonal to T ° “] .

Reasonable “conjectures” (more or LESS correct) :

- The indecomposable right bounded complexes **C °** (of projective modules) orthogonal to **T °** are as different as possible from “concealed” complexes, that is projective resolutions of indecomposable modules.

Natural question:

How many choices, up to shifts , for a **well - behaved** indecomp. complex $\mathbf{C}^{\circ}$ orthogonal to $\mathbf{T}^{\circ}$ **?**

[**well - behaved** complex : the non-zero components of its **indecomposable** summands are **indecomposable**] .

ANSWER to the natural question :

With the special hypothesis that

T ° is a **well- behaved complex**,

the answer to the above question may

be ....

....

0

1

finitely many but **> 1**

$\aleph_0$ **(and only 1 left unbounded)**

$2^{\aleph_0}$ (and only $\aleph_0$ bounded)

3 possible constructions used to find $C°$ orthogonal to $T°$:

CANCELLATIONS [**2 or 3** different types]

ADDITIONS [**2** different types]

LEGO - TYPE CONSTRUCTIONS

[oo - many types] to get more complicated (even **LEFT unbounded**) complexes from the minimal ones.

FOR ME

CANCELLATION = the **best & easy** construction

LEGO - TYPE construction = the **best & more complicated** construction

RIGHT ADDITION = the **less natural & oldest** construction (sometimes the unique possible one)

2 (quite different) examples

where the choice of **C°** is unique and **T°** has at most 2 indecomp. summands :

Ex. A = example with only 1

choice for **C °** , obtained by means of **RIGHT addition** from the **indecomposable** complex **T °** :

C ° : [1 2 2 2]

T ° : [1 2 2]

Ex. B = example with only 1

choice for **C °** , obtained by means of **LEFT cancellation & addition** from the unique **indecomposable** non stalk summand **X °** of **T °** :

C ° : **[………… 2 2 2 2 2 2 1 3]**

X ° : **[1 2 1 3]**

Ex. C = example with **oo - many (but countably many)**

choices for **$C°$** , where **$T°$** has

2 indecomposable summands, i.e.

- the stalk complex **[2]**

and

- the complex **$X° =$ [3 1 2 1]**

and ……

..... $X^\circ = [\,3\;1\;2\;1\,]$

and the following strings describe all the possible choices of C° :

$[\,1\;\;2\;\;1\,]$,

$[\,1\;2\;\;2\;\;1\,]$,

$[\,1\;2\;\;2\;\;2\;\;1\,]$,

....

....

$[\,\ldots\ldots\ldots\ldots\ldots\;2\;2\;2\;2\;\;2\;\;1\,]$

Remarks

- Only in one case (= Lego - type case) one proceeds in the most obvious direction (**from RIGHT to LEFT**) , but the ingredients (building blocks) are complexes and **NOT** “isolated” modules.
- The less obvious construction (**= RIGHT addition**) may be the unique possible one (Example A).

Ex. D = CANCELLATIONS

For any **$m > 1$** , there is a large partial tilting module T s.t. **pdim T = $2m > 2$** and T is **injective & uniserial** .

If P & Q are indec. projective and C° is an indecomp. complex of the form

$0 \longrightarrow P \longrightarrow Q \longrightarrow 0$, then **TFAE** :

TFAE :

1) C° is orthogonal to T° .

2) P & Q **injective**, not isomorphic and we obtain C° from T° by means of **cancellations (LEFT , RIGHT ,....)** .

3) P & Q injective, not isomorphic and the morphism from P to Q is a composition of **IRREDUCIBLE** maps $X \longrightarrow Y$ with either X or Y injective.

Continuation of **TFAE :**

(4) (**reduction to an easy case**) :
for any morphism of complexes from T° to a shift of C° of the form $(\mathbf{f}, \mathbf{0})$ where $f : X \longrightarrow P$ and X is the last non-zero components of T°, we have $\mathbf{f} = \mathbf{0}$.

The complex T ° in Ex. D is

the projective resolution of the uniserial module T , considered in the first **Example** of this talk , of the form :

2
3
.
.
n
1

Ex. E = example with uncountably many choices for $C°$:

T = direct sum of 2 indecomp. modules:

P uniserial & projective

S simple and pdim S = 5

4 = number of simple modules

5 = dimension of T over K

Hom (P , S) = 0 = Hom (S , P) .....

The shape of the proj. resol. S° of S is

$[\ 1\ 3\ 1\ 2\ 1\ 4\ ]$

and the indecomp. projectives look like as follows:

$$P(1) = \begin{matrix}1\\2\\3\\ \end{matrix}\ ,\quad P(2) = \begin{matrix}2\\3\\1\\2\end{matrix}\ ,\quad P(3) = \begin{matrix}3\\1\\2\\3\end{matrix}\ ,\quad P(4) = \begin{matrix}4\\1\\ \\ \end{matrix}$$

Two types of indecomp. complexes orthogonal to the proj. resol. of T

(1) the " simple ones " (no indecomposable subcomplexes has the same property) .

(2) the "non simple" indecomp. complexes, obtained from complexes of type **(1)** by means of a Lego - type construction.

The shape of some complexes of type **(1)**

[3 2 2 3], **[3 2 2 2 3]** , **[3 2 2 2 2 3]** **,**

………………………………… and their "limit"

[…………… 2 2 2 2 …… 2 3] .

[1 2 1 4] subcomplex of **S °** obtained after left cancellation of 2 components.

Remark

All possibile first (resp. last) non-zero components, that is **P(1) & P(3)** (resp. **P(3) & P(4)**) show up.

The shape of some complexes of type (2)

- complexes with **injective** components

[3 2 ... 2 3 3 2 ...... 2 3] , ...

[3 2 .... 2 3 3 2 .... 2 3 3 2 ..... 2 3],

....

- complexes **with NON injective components**

[3 2 .... 2 3 1 2 1 4] ,

[3 2 .... 2 3 3 2 .... 2 3 1 2 1 4],

[............2 .... 2 3 1 2 1 4]

THANKS & picture(s)

Thank you very much to

ALL the ORGANIZERS

for the great hospitality & the informal atmosphere during

- this big & international Conference

- many other ------------ meetings

- ----------- less official ---------

Possible meanings for me of less official … :

- Seminar Darstellungstheorie

- Vorlesung

Picture taken by

ANNETTE HOEWELMANN

and

MONIKA HAENSCH

somewhere in this building in the last millennium

2 (unofficial) pictures

Niagara Falls (ICRA X)

On the left: **BAROT , BRENNER , BUTLER , KRAUSE , SCHROER**

On the right: young people attending **WYD2002** (**W**orld **Y**outh **D**ay **2002**)