The Auslander-Reiten Components in the Rhombic Picture

Helene R. Tyler

Manhattan College

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This talk reflects joint work with Markus Schmidmeier.

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Lofty Goals

Our ultimate aim: To understand the relationship between A-R classification and G-R classification. (Ringel)

Our current focus: $k\widetilde{\mathbb{A}}_n$, $k = \overline{k}$. Modules and homomorphisms are controlled by the combinatorics of strings and bands. (Butler-Ringel, Krause)

Bonus: A nice general result.

The Gabriel-Roiter Measure

Let $\mathcal{F} = \{\text{all sequences } (a_i) \text{ with each } a_i \in \mathbb{N}\}$.

Endow $\mathcal{F}$ with the negative lexicographic ordering. Denote the ordering by $<_{\mathcal{F}}$.

e.g. $(1, 1, 2) <_{\mathcal{F}} (1, 1, 1)$.

Dually, $\mathcal{F}^*$ denotes the set $\mathcal{F}$ with the opposite ordering, $<^*_{\mathcal{F}}$.

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The Gabriel-Roiter Measure

Definition (Modification of Ringel's)

Λ an artin algebra, M a Λ-module. A chain $0 = M_0 \subsetneq M_1 \subsetneq \cdots$ of submodules of M, with M_i indecomposable ($i > 0$) and of finite length is called an **indecomposable filtration for** M. The **Gabriel-Roiter measure** $\mu(M) = \sup\{(|M_i/M_{i-1}|)\}$ w.r.t $<_{\mathcal{F}}$.

Dually, a chain $\cdots M_2 \subsetneq M_1 \subsetneq M_0 = M$, with each M/M_i indecomposable is called a **cofiltration**. The **Gabriel-Roiter comeasure** $\mu^*(M) = \inf\{(|M_{i-1}/M_i|)\}$ w.r.t $<^*_{\mathcal{F}}$.

Note: If M is fin. dim., then $\mu^*(M) = \mu(DM)$.

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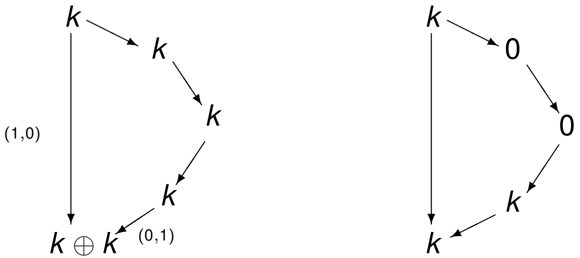
An example



$\mu = (1,1,1,1,2) \qquad <_{\mathcal{F}} \qquad \mu = (1,1,1)$

Such a computation is, in general, quite difficult.

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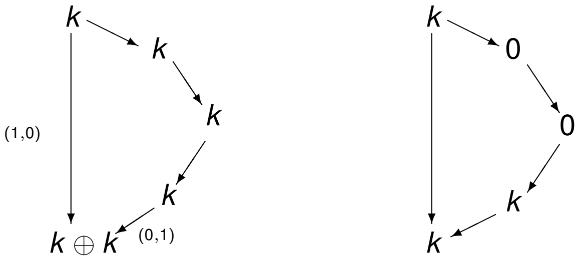

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String Me Along

Our focus: Tame hereditary algebras of type $\widetilde{\mathbb{A}}_n$.

Correspondence between string modules over $k\widetilde{\mathbb{A}}_n \leftrightarrow$ intervals in $\mathbb{A}_\infty^\infty$. Band modules lie in homogeneous tubes.

Maps between string modules are formed by "adding hooks" (monomorphisms) or "deleting cohooks" (epimorphisms).

The aforementioned combinatorics allow us to construct an algorithm for computing the G-R measure of a $k\widetilde{\mathbb{A}}_n$-module. After computing lots of examples, a pattern emerged.

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The IPF Decomposition

Proposition

Let M be a $k\widetilde{\mathbb{A}}_n$-module, let $h = n + 1$, and let $\mathcal{L}$ and $\mathcal{R}$ denote the minimally rotated left and right hook sequences for the quiver. If M is large enough, then

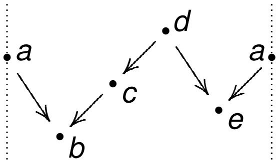
$$\mu(M) \;=\; \mathrm{init}(M) \cdot \mathrm{per}(M)^{\mathrm{mult}(M)} \cdot \mathrm{fin}(M),$$

where $\mathrm{per}(M)$, is of the form $\mathcal{L}$, $\mathcal{R}$ or (h). □

Minimally Rotated Sequences

Ex: Consider the two-source $\widetilde{\mathbb{A}}_4$ quiver.

Q :



Here's a regular module:

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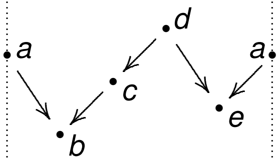

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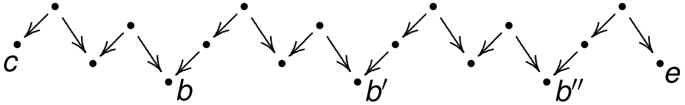

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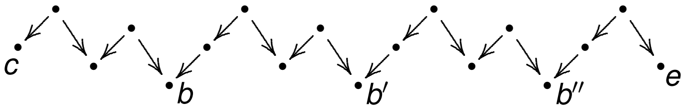

$$\mathcal{R} = (2,2,1) <_{\mathcal{F}} (2,1,2) <_{\mathcal{F}} (1,2,2)$$

$$\mathcal{L} = (3,2) <_{\mathcal{F}} (2,3)$$

$$\mu(ce_{18}) = (1,1,\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},2,2,2)$$

Link to Auslander-Reiten Theory

We extract representation-theoretic information from the IPF Decomposition.

Proposition

Let M be an indecomposable $k\widetilde{\mathbb{A}}_n$-module. Then

- *M is preprojective if and only if* $\mathrm{fin}(M) <_{\mathcal{F}} \mathrm{per}(M)$ *and* $\mathrm{fin}(DM) >_{\mathcal{F}} \mathrm{per}(DM)$
- *M is preinjective if and only if* $\mathrm{fin}(M) >_{\mathcal{F}} \mathrm{per}(M)$ *and* $\mathrm{fin}(DM) <_{\mathcal{F}} \mathrm{per}(DM)$
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Distinguishing the Tubes

Within the regular component, we can say more.

Recall that for all but one orientation of $\widetilde{\mathbb{A}}_n$, the regular component consists of two exceptional tubes and one homogenous tube. We call the two exceptional tubes the left tube and the right tube.

Proposition

An indecomposable regular module M is in the left tube, in the right tube, or in a homogeneous tube if and only if the periodic part $\mathrm{per}(M)$ *is* $\mathcal{L}$, $\mathcal{R}$, *or* (h), *respectively, where* $h = n + 1$.

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Module Families

Each $k\widetilde{\mathbb{A}}_n$-module M is the starting point and end point of at most two irreducible morphisms; hence, it lies on two rays, or two corays, or one of each.

Definition

Let M be an indecomposable $k\widetilde{\mathbb{A}}_n$-module. The **family** of M consists of the modules on the intersection of the two rays (M preprojective), the two corays (M preinjective) or the ray and the coray (M regular) defined by M. We order the modules in the family by dimension and denote the family of M by $(M_i)_{i=1}^{\infty}$.

Note: Two string modules lie in the same family if and only if they have the same string type.

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Limits and Colimits

Rhombic Picture (Ringel): Provides a visual organization of the module category. Each module M is given the coordinates $(\mu(M), \mu^*(M))$ in $\mathcal{F} \times \mathcal{F}^*$.

Definition

Let M be an indecomposable module.

- The **GR-limit** of M is $\overrightarrow{\mu}(M) = \lim_{i\to\infty} \mu(M_i)$
- The **GR-colimit** of M is $\overrightarrow{\mu^*}(M) = \lim_{i\to\infty} \mu^*(M_i)$
- The **rhombic limit** of M is the point $\overrightarrow{\rho}(M) = (\overrightarrow{\mu}(M), \overrightarrow{\mu^*}(M))$ in the rhombic picture.

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Limits and Colimits

We apply an earlier result to this definition to obtain

Proposition

Let M be an indecomposable $k\widetilde{\mathbb{A}}_n$-module.

- *M is preprojective iff the points $\rho(M_i)$ approach $\vec{\rho}(M)$ from the left.*
- *M is preinjective iff the points $\rho(M_i)$ approach $\vec{\rho}(M)$ from the right.*
- *M is regular iff the points $\rho(M_i)$ approach $\vec{\rho}(M)$ from below.*

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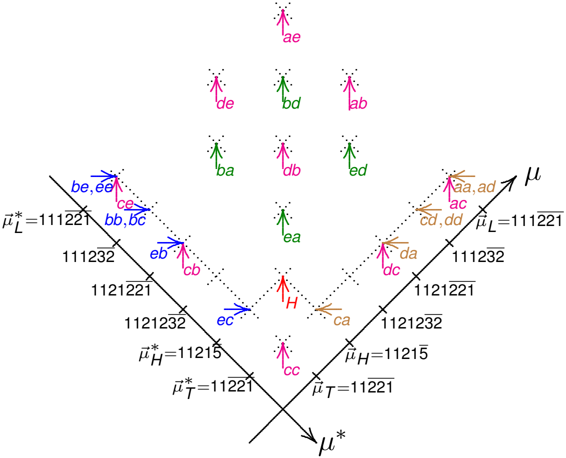
A Rhombic Picture



A General Result

We present a general result, lifting the assumption that the algebra is $k\widetilde{\mathbb{A}}_n$.

Theorem

Let $0 \to A \to B_1 \oplus B_2 \to C \to 0$ be an Auslander-Reiten sequence in a stable tube such that the middle term consists of two indecomposable summands. Then the rhombic limits of A, B_1, B_2, C form a (possibly degenerate) parallelogram in the rhombic picture. Moreover, the nondegenerate sides of the parallelogram are parallel to the μ and μ^* axes.

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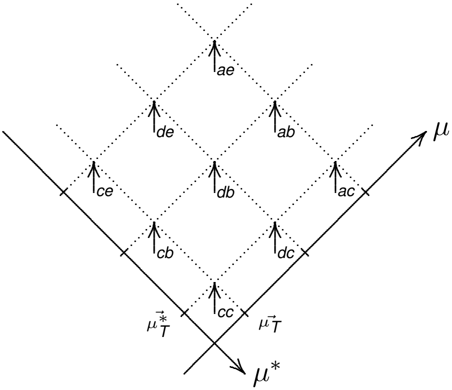
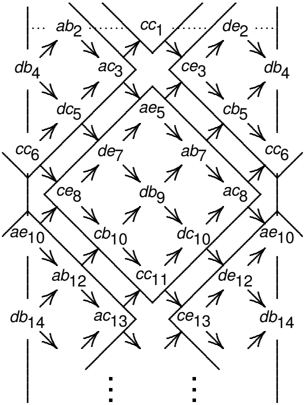
Side by Side


ae
de
ab
ce
db
ac
cb
dc
cc
μ
μ^*
$\mu_T^{\vec{*}}$
$\mu_T^{\vec{}}$


ab_2
cc_1
de_2
db_4
ac_3
ce_3
db_4
dc_5
ae_5
cb_5
cc_6
de_7
ab_7
cc_6
ce_8
db_9
ac_8
ae_{10}
cb_{10}
dc_{10}
ae_{10}
ab_{12}
cc_{11}
de_{12}
db_{14}
ac_{13}
ce_{13}
db_{14}