ALGEBRAIC EXTENSIONS OF GRADED AND VALUED FIELDS

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If F is a field with a (Krull) valuation, then the filtration of F induced by the valuation yields an associated graded ring, which is a graded field. Conversely, if R is a graded field with totally ordered grade group, then R is an integral domain and there is a canonically associated valuation on the quotient field of R. The processes of passing from valued field to graded field and vice versa are not quite inverses of each other, but many properties in one setting are well-reflected in the other.

The goal of this paper is to describe an algebraic extension theory for graded fields analogous to what is known for valued fields, and then to spell out the correspondence between tame extensions of graded fields and Henselian valued fields. This has the benefit that graded fields are easier to work with for many purposes

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than valued fields. But beyond this, there is a similar correspondence between graded division rings and valued division rings, where the graded objects seem to be significantly easier to work with than the valued objects. We first learned of this correspondence from a paper by M. Boulagouaz [B_2]. The correspondence for division rings is actually far more extensive than what was described by Boulagouaz, and we pursue that subject in a sequel to this paper [HW]. The choice of topics to treat here was influenced by the needs of the study of division rings. But, we feel that the commutative theory presented here is of interest in its own right.

§1 Graded fields (with totally ordered grade group)

Let $(\Gamma, +)$ be an abelian group and let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a graded ring with respect to Γ (i.e., R is a ring with 1, such that $R_\gamma \cdot R_\delta \subseteq R_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$). We set

$$\Gamma_R = supp(R) = \{\gamma \in \Gamma \mid R_\gamma \neq (0)\}.$$

Also, let $R^h = \bigcup_{\gamma\in\Gamma} R_\gamma$, the set of homogeneous elements of R. If $r \in R_\gamma$, $r \neq 0$, we write $deg(r) = \gamma$.

The graded ring R is said to be a *graded field* if R is commutative with $1 \neq 0$ and every nonzero homogeneous element of R is a unit. When this occurs, Γ_R is a subgroup of Γ, and we call it the *grade group* of R. We will be interested exclusively in the case where Γ_R is totally ordered. We adopt as a standing hypothesis throughout the paper that all the graded fields R we consider are equipped with a total ordering on Γ_R. (But, note that any torsion-free abelian group admits a total ordering. Therefore, all the results in §§1-3 below on extensions of graded fields hold if we merely assume that Γ_R is torsion-free abelian. Only when we wish to build a valuation ring on the quotient field of R will we need to specify an ordering on Γ_R.) A graded isomorphism $R \to R'$ of graded fields consists of an order-preserving group isomorphism $\alpha\colon \Gamma_R \to \Gamma_{R'}$ and a ring isomorphism $\beta\colon R \to R'$ such that $\beta(R_\gamma) = R'_{\alpha(\gamma)}$ for all $\gamma \in \Gamma_R$. When such a graded isomorphism exists, we write $R \cong_g R'$.

It follows from the total ordering of Γ_R that a graded field R is an integral domain and that its group of units $R^* = R^h - \{0\}$. Let

$$QR = \text{quotient field of } R.$$

Since Γ_R is totally ordered, it is a torsion-free abelian group, and we set

$$\Delta_R = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_R, \text{ the divisible hull of } \Gamma_R.$$

We identify Γ_R with its isomorphic image in Δ_R ($\gamma \leftrightarrow 1 \otimes \gamma$). The total ordering on Γ_R extends uniquely to a total ordering on Δ_R. Moreover, if Λ is any torsion-free abelian group containing Γ_R as a subgroup with Λ/Γ_R torsion, then there is a unique monomorphism $\Lambda \to \Delta_R$ extending the embedding $\Gamma_R \hookrightarrow \Delta_R$. We will thus routinely view any such Λ as a subgroup of Δ_R. The ordering on Γ_R extends uniquely to an ordering on Λ.

Since R is a graded field, it is clear that R_0 is a field and each R_γ is a 1-dimensional vector space over R_0. For each $\gamma \in \Gamma_R$, fix some nonzero $t_\gamma \in R_\gamma$. Then for each $\gamma, \delta \in \Gamma_R$ there is $c_{\gamma,\delta} \in R_0^*$, such that $t_\gamma t_\delta = c_{\gamma,\delta} t_{\gamma+\delta}$. We call $\{c_{\gamma,\delta} \mid \gamma, \delta \in \Gamma_R\}$ a family of structure constants of R. The commutativity and associativity of R imply

$$c_{\gamma,\delta} = c_{\delta,\gamma} \quad \text{and} \quad c_{\gamma,\delta} c_{\gamma+\delta,\rho} = c_{\delta,\rho} c_{\gamma,\delta+\rho} \quad \text{for all } \gamma, \delta, \rho \in \Gamma_R. \tag{1.1}$$

Conversely, it is clear that given any field F, any totally ordered abelian group Γ, and any function $\Gamma \times \Gamma \to F^*$ ($(\gamma, \delta) \mapsto c_{\gamma,\delta}$) satisfying the conditions in (1.1), there is a graded field R with $R_0 = F$, $\Gamma_R = \Gamma$, and structure constants given by the specified function.

We say that a graded field R is of *group-ring type* if there is a family of structure constants of R with $c_{\gamma,\delta} = 1$ for all $\gamma, \delta \in \Gamma_R$. Clearly, R is of group-ring type iff R is isomorphic (as a graded ring) to the group ring $R_0[\Gamma]$.

Proposition 1.1. *A graded field R is of group-ring type iff the canonical short exact sequence of abelian groups*

$$0 \to R_0^* \to R^* \to \Gamma_R \to 0$$

is split exact.

Proof. Clearly, the exact sequence splits iff there is a family $\{t_\gamma \mid \gamma \in \Gamma_R\}$ with each $t_\gamma \in R_\gamma - \{0\}$ and $t_\gamma t_\delta = t_{\gamma+\delta}$. This is exactly what is needed so that the structure constants relative to $\{t_\gamma\}$ will all be 1. $\square$

Prop. 1.1 shows that if $\Gamma_R \cong \mathbb{Z}$, or Γ_R is any free abelian group, then the graded field R is of group-ring type (and hence determined up to isomorphism by R_0 and Γ_R). But, not every graded field is of group-ring type, as the following example shows.

Example 1.2. *A graded field R with $R_0 = \mathbb{Q}$ and $\Gamma_R = \mathbb{Z}\left[\frac{1}{2}\right]$, and R not of group-ring type.* Let t be an indeterminate, and let $A = \mathbb{Q}[t, t^{-1}]$, the Laurent

polynomial ring in t over the rational numbers $\mathbb{Q}$. This A is the group ring $\mathbb{Q}[\mathbb{Z}]$, so a graded field, with $\Gamma_A = \mathbb{Z}$ and $A_i = \mathbb{Q}t^i$, $i \in \mathbb{Z}$. Let p be any prime number. In an algebraic closure of the quotient field of A, there are $y_1, y_2, \ldots$ satisfying $y_1^2 = pt$, $y_2^2 = p^2 y_1$, $y_3^2 = p^4 y_2, \ldots$, $y_i^2 = p^{2^{i-1}} y_{i-1}, \ldots$. Let $B_1 = A[y_1]$, $B_2 = B_1[y_2], \ldots, B_i = B_{i-1}[y_i], \ldots$, and let $R = \bigcup_{i=1}^{\infty} B_i$. Note that, as an A-module, $B_1 = A \oplus y_1 A$, and B_1 is a graded field with $(B_1)_0 = \mathbb{Q}$, $\Gamma_{B_1} = \frac{1}{2}\mathbb{Z}$, and for $j \in \mathbb{Z}$, $(B_1)_{j/2} = \mathbb{Q}y_1^j$. Also, the gradings on A and B_1 are compatible, so A is a graded subfield of B_1. Proceeding inductively, we have for each i, $B_i = B_{i-1} \oplus B_{i-1}y_i$, B_i is a graded field with $(B_i)_0 = \mathbb{Q}$, $\Gamma_{B_i} = 2^{-i}\mathbb{Z}$, $(B_i)_{j/2^i} = \mathbb{Q}y_i^j$ for all $j \in \mathbb{Z}$, and B_{i-1} is a graded subfield of B_i. Hence, R is a graded field with $R_0 = \mathbb{Q}$ and $\Gamma_R = \mathbb{Z}\left[\frac{1}{2}\right]$ (the additive group of the ring $\mathbb{Z}\left[\frac{1}{2}\right]$, with the usual ordering). If R were of group-ring type, then there would exist nonzero $s_1 \in R_1 = \mathbb{Q}t$ and $s_{2^{-i}} \in R_{2^{-i}} = \mathbb{Q}y_i$ such that $(s_{2^{-i}})^{2^i} = s_1$, for all integers $i > 0$. However, observe that $y_i^{2^i} = p^{(4^i-1)/3}t$. Hence, $s_1 \in \bigcap_{i=1}^{\infty} (R_{2^{-i}})^{2^i} = \bigcap_{i=1}^{\infty} \mathbb{Q}^{2^i} p^{(4^i-1)/3} t = (0)$, a contradiction. Hence, R is not of group-ring type.

Corollary 1.3. *Let R be a graded field. Then R is integrally closed, and R_0 is algebraically closed in QR.*

Proof. If R were not integrally closed, this would be detected by an equation involving only finitely many elements of R, hence only finitely many homogeneous elements. Thus, there is a finitely generated subgroup Λ of Γ_R such that for the graded subfield $A = R|_\Lambda = \bigoplus_{\lambda \in \Lambda} R_\lambda$ of R, A is not integrally closed. Note that $\Gamma_A = \Lambda$ and $A_0 = R_0$. As Λ is finitely generated and torsion-free, $\Lambda \cong \mathbb{Z}^n$ for some n. So, Prop. 1.1 shows $A \cong A_0[\Lambda] \cong A_0[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, where $x_1, \ldots, x_n$ are independent indeterminates. Since A is a localization of a polynomial ring over a field, A is integrally closed, a contradiction. Hence, R must be integrally closed. That R_0 is algebraically closed in QR is proved in the same way, as $R_0 = A_0$, which is algebraically closed in the rational function field $QA \cong R_0(x_1, \ldots, x_n)$. $\square$

Let R be a graded field. By a *graded R-module*, we mean an R-module M such that M has a direct sum decomposition as abelian groups $M = \bigoplus_{\mu \in \Gamma_M} M_\mu$, where Γ_R acts freely on the set Γ_M, and for $\gamma \in \Gamma_R$, $\mu \in \Gamma_M$, we have $R_\gamma \cdot M_\mu \subseteq M_{\gamma+\mu}$. (Here $\gamma + \mu$ denotes the image of μ under the action of γ.) There is no group structure assumed on Γ_M. That Γ_R acts freely on Γ_M means that for all $\gamma, \delta \in \Gamma_R$, $\mu \in \Gamma_M$, we have $\gamma + \mu \neq \delta + \mu$ whenever $\gamma \neq \delta$. Observe that every graded module M over any graded field R is a free R-module, with a homogeneous base.

Indeed, any maximal R-linearly independent homogeneous subset of M is a base (cf. [B_1, Th. 3]). We write $dim_R(M)$ for the rank of M as a free R-module. Note that if N is any graded R-submodule of M then, as M/N is a graded, hence free, R-module, we have $M \cong N \oplus (M/N)$; so $dim_R(M) = dim_R(N) + dim_R(M/N)$. In particular, if N is a proper graded R-submodule of M and $dim_R(M) < \infty$ then $dim_R(N) < dim_R(M)$.

§2 Graded algebraic extensions of graded fields

Let $R \subseteq S$ be graded fields (i.e., R is a graded subfield of S). We set

$$[S : R] = dim_R(S).$$

Clearly, R_0 is a subfield of S_0, Γ_R is a subgroup of Γ_S, and QR is a subfield of QS. The following proposition is easy but important.

Proposition 2.1. *(cf. [B_2, p. 4278]) Let $R \subseteq S$ be graded fields with $[S : R] < \infty$. Then,*

$$[S : R] = [S_0 : R_0]|\Gamma_S : \Gamma_R| = [QS : QR],$$

and $QS \cong QR \otimes_R S$.

Proof. If $\{s_i\}_{i\in I}$ is an R_0-base of S_0, and $\{t_j\}_{j\in J} \subseteq S^*$ with $\{deg(t_j)\}$ a set of representatives for the cosets of Γ_R in Γ_S. It is easy to check that $\{s_i t_j\}$ is an R-base of S. This gives the first formula for $[S : R]$, and the second follows from $QS \cong QR \otimes_R S$. This isomorphism holds as S is an integral domain finitely generated as an R-module; so S (a torsion-free R-module) embeds in its localization $QR \otimes_R S$ which is a field, since it is an integral domain finite-dimensional over the field QR. $\square$

We will want to consider infinite degree extensions of graded fields which are algebraic in an appropriate sense. For this, we first look at gradings on the polynomial ring $R[x]$ over a graded field R. For any $\delta \in \Delta_R$, there is a unique grading on $R[x]$ extending that on R, such that $x \in R[x]_\delta$. We call this the δ-grading of $R[x]$. An $f \in R[x]$ is said to be *homogenizable* if there is $\delta \in \Delta_R$ such that f is homogeneous with respect to the δ-grading of $R[x]$ (cf. [B_1, §3], [vGvO, p. 274]). The following proposition is essentially in [vGvO, pp. 274–276] for $\Gamma_R \cong \mathbb{Z}$.

Proposition 2.2. *Let R be a graded field which is a subring of the (ungraded) field F. For any $\alpha \in F$, the following are equivalent:*

(i) *α is algebraic over QR, the minimal polynomial of α over QR, denoted $m_{QR,\alpha}$, lies in $R[x]$ and $m_{QR,\alpha}$ is homogenizable.*

(ii) *$R[\alpha]$ is a graded field extension of R, with $\alpha \in R[\alpha]^h$ and $[R[\alpha] : R] < \infty$.*

(iii) *There is a graded field extension S of R, with $R \subseteq S \subseteq F$, $[S : R] < \infty$, and $\alpha \in S^h$.*

PROOF. (i) $\Rightarrow$ (ii) Suppose α is algebraic over QR and that $m_{QR,\alpha}$ lies in $R[x]$ and is homogeneous with respect to the δ-grading on $R[x]$, for some $\delta \in \Delta_R$. Then $(m_{QR,\alpha})$ is a homogeneous ideal $R[x]$, so the ring $R[x]/(m_{QR,\alpha})$ inherits a grading from $R[x]$ which extends the grading on R. The canonical evaluation homomorphism $R[x] \to F$ given by $f \mapsto f(\alpha)$ has image $R[\alpha]$ and kernel $\{f \in R[x] \mid f(\alpha) = 0\} = (m_{QR,\alpha}QR[x]) \cap R[x] = m_{QR,\alpha}R[x]$. (The last equality holds as $m_{QR,\alpha}$ is monic in $R[x]$.) Hence, the isomorphism $R[\alpha] \cong R[x]/(m_{QR,\alpha})$ allows us to define a grading on $R[\alpha]$ extending that on R, with respect to which $\alpha \in R[\alpha]^h$ since α is the image of $x \in R[x]^h$. Also, $[R(\alpha) : R] = [R[x]/(m_{QR,\alpha}) : R] = deg(m_{QR,\alpha}) < \infty$. Because $R[\alpha]$ is a graded integral domain (since it lies in F), with $[R[\alpha] : R] < \infty$, $R[\alpha]$ is actually a graded field. (For, if $\beta \in R[\alpha]^h$, $\beta \neq 0$, then the map $\lambda_\beta : R[\alpha] \to R[\alpha]$ given by $c \mapsto \beta c$ is an injective graded R-module homomorphism. Then, as $im(\lambda_\beta)$ is a graded R-submodule of $R[\alpha]$ with $dim_R(im(\lambda_\beta)) = [R(\alpha) : R] < \infty$, so λ_β is surjective by dimension count. (See the remarks at the end of §1.) Hence $\beta \in R[\alpha]^*$.

(ii) $\Rightarrow$ (iii) Take $S = R[\alpha]$.

(iii) $\Rightarrow$ (i) Assume $\alpha \in S^h$ for a graded field $S \supseteq R$, with $[S : R] < \infty$. Then α is integral over R, since $[S : R] < \infty$. So α is algebraic over QR, and $m_{QR,\alpha} \in R[x]$, as R is integrally closed by Cor. 1.3. Write $m_{QR,\alpha} = x^n + c_{n-1}x^{n-1} + \ldots + c_0$. If we take the $n\, deg(\alpha)$ homogeneous component of the equation $\alpha^n + c_{n-1}\alpha^{n-1} + \ldots + c_0$, we get another monic polynomial in α of degree n with coefficients in R, which equals 0. The uniqueness of $m_{QR,\alpha}$ assures that this polynomial must coincide with $m_{QR,\alpha}$. Hence, each $c_i \in R_{(n-i)deg(\alpha)}$. So $m_{QR,\alpha}$ is homogeneous for the $deg(\alpha)$-grading of $R[x]$, and $deg(\alpha) = \frac{1}{n}deg(c_0) \in \Delta_R$. $\square$

Definition 2.3. An α satisfying the equivalent conditions of Prop. 2.2 is said to be *gr-algebraic* over the graded field R. For a graded field extension S of R, we say S is *gr-algebraic* over R if each $\alpha \in S^h$ is gr-algebraic over R. (In particular, by Prop. 2.2, if $[S : R] < \infty$, then S is gr-algebraic over R.)

Corollary 2.4. *Suppose α is gr-algebraic over a graded field R. Then,*

(a) *$m_{QR,\alpha}$ determines $deg(\alpha)$.*

(b) *$\Gamma_{R[\alpha]} = \langle deg(\alpha) \rangle + \Gamma_R$.*

(c) *$deg(\alpha) = 0$ iff $m_{QR,\alpha} \in R_0[x]$, iff α is algebraic over R_0.*

(d) *$R[\alpha]_0 = R_0$ iff $\alpha^m \in R$, where m is the order of the image of $deg(\alpha)$ in*

Δ_R/Γ_R. *This occurs iff* $m_{QR,\alpha} = x^n - d_0$, *where* $d_0 \in R^h$ *and* $deg(d_0)$ *has order* n *in* $\Gamma_R/n\Gamma_R$. *Then* $n = m = [R[\alpha] : R]$.

PROOF. (a) The proof of Prop. 2.2 shows $deg(\alpha) = \frac{1}{n}deg(c_0)$, where $n = deg(m_{QR,\alpha})$ and c_0 is the constant term of $m_{QR,\alpha}$. Part (b) holds since $\alpha \in R[\alpha]^h$. For (c), if α is algebraic over R_0, then $\alpha \in R[\alpha]_0$ by Cor. 1.3; the rest of (c) is clear from the proof of Prop. 2.2.

(d) By (b), $m = |\Gamma_{R[\alpha]}/\Gamma_R|$. If $R[\alpha]_0 = R_0$, then $R = \bigoplus_{\gamma\in\Gamma_R} R[\alpha]_\gamma$, by Prop. 2.1, since these graded fields have the same grade group and the same degree 0 component. Since $deg(\alpha^m) \in \Gamma_R$, this shows $\alpha^m \in R$. Now, suppose we just know $\alpha^m \in R$. Let $c_0 = \alpha^m$, so $m_{QR,\alpha} \mid (x^m - c_0)$. But, using Prop. 2.1, $deg(x^m - c_0) = m = |\Gamma_{R[\alpha]}/\Gamma_R| \leq [QR(\alpha) : QR] = deg(m_{QR,\alpha})$. Hence, $m_{QR,\alpha} = x^m - c_0$. Since $deg(\alpha)$ has order m in Δ_R/Γ_R, we have $deg(c_0) = m\, deg(\alpha)$ has order m in $\Gamma_R/m\Gamma_R$. Now, suppose instead that $m_{QR,\alpha} = x^n - d_0$, where $d_0 \in R^h$ and $deg(d_0)$ has order n in $\Gamma_R/n\Gamma_R$. Then $\frac{1}{n}deg(d_0)$ has order n in Δ_R/Γ_R. But $deg(\alpha) = \frac{1}{n}deg(d_0)$. Hence, $m = n$. Since $m = |\Gamma_{R[\alpha]}/\Gamma_R|$ and $n = [R[\alpha] : R]$, the equality $m = n$ forces $R[\alpha]_0 = R_0$, by Prop. 2.1. □

Corollary 2.5. *Let* $R \subseteq S$ *be graded fields with* S *gr-algebraic over* R. *Then,*

(a) S *is the integral closure of* R *in* QS.
(b) $QR \cap S = R$.
(c) *If* $\alpha \in S$ *and* a *is gr-algebraic over* R, *then* $\alpha \in S^h$.
(d) *If* τ *is any* QR*-automorphism of* QS, *then* $\tau(S) = S$ *and* $\tau|_S : S \to S$ *is a graded field isomorphism.*

PROOF. (a) S is integral over R by Prop. 2.2, and is integrally closed by Cor. 1.3. (b) is immediate from (a), as R is integrally closed.

(c) By Prop. 2.2 there is a grading on $R[x]$ with $\Gamma_{R[x]} \subseteq \Delta_R$ and $x \in R[x]^h$, such that $m_{QR,\alpha} \in R[x]^h$. This grading extends (uniquely) to a grading on $S[x]$ which extends the grading on S. Since $\Gamma_{S[x]} \subseteq \Delta_S$, $\Gamma_{S[x]}$ is totally ordered. Hence, the factor $x - \alpha$ of $m_{QR,\alpha}$ in $S[x]$ must be homogeneous, as $m_{QR,\alpha} \in S[x]^h$. So, $\alpha \in S^h$.

(d) For any QR-automorphism τ of QS, we have $\tau(S) = S$ as S is the integral closure of R in QS. Take any $\alpha \in S^h$. Then $m_{QR,\tau(\alpha)} = m_{QR,\alpha}$. Since α is gr-algebraic over R, Prop. 2.2 shows $\tau(\alpha)$ is also gr-algebraic over R. Then $\tau(\alpha) \in S^h$ by (c) above, and $deg(\tau(\alpha)) = deg(\alpha)$ by Cor. 2.4(a). So the ring isomorphism $\tau|_S$ maps each homogeneous component of S onto itself. □

Corollary 2.6. *Let $R \subseteq S$ be graded fields, and let K be a field containing S. If $\alpha \in K$ is gr-algebraic over R, then α is gr-algebraic over S.*

PROOF. There is a grading on $R[x]$ extending that on R with $x \in R[x]^h$, such that $m_{QR,\alpha} \in R[x]^h$. This grading extends (uniquely) to a grading on $S[x]$ extending that on S, with $\Gamma_{S[x]} \subseteq \Delta_S$. Since α is integral over S, we have $m_{QS,\alpha} \in S[x]$; then $m_{QS,\alpha} \mid m_{QR,\alpha}$ in $S[x]$, since this divisibility holds in $QS[x]$, and $m_{QS,\alpha}$ is monic. Since $m_{QR,\alpha} \in S[x]^h$ and $\Gamma_{S[x]}$ is totally ordered, we must have $m_{QS,\alpha} \in S[x]^h$. Hence, α is gr-algebraic over S. $\square$

The following corollary can now be proved just as in the ungraded case:

Corollary 2.7. *Let R be a graded field which is a subring of the ungraded field F. Then,*

(a) *If $\alpha_1, \ldots, \alpha_n \in F$ and each α_i is gr-algebraic over F, then $R[\alpha_1, \ldots, \alpha_n]$ is a graded field graded algebraic over R, with*

$$[R[\alpha_1, \ldots, \alpha_n] : R] \leq \prod_{i=1}^{n} [R[\alpha_i] : R] < \infty\,.$$

(b) *If S is a graded field, $R \subseteq S \subseteq F$, with S gr-algebraic over R, and if $\alpha \in F$ is gr-algebraic over S, then α is gr-algebraic over R.*

(c) *Let $A = R[\{\alpha \in F \mid \alpha \text{ is gr-algebraic over } R\}]$. Then A is a graded field which is gr-algebraic over R, and for every graded field S with $R \subseteq S \subseteq F$ and S gr-algebraic over R, S is a graded subfield of A.*

Definition 2.8. We call the A of Cor. 2.7(c) the *graded algebraic closure* of R in F.

Note that even if F is algebraic over QR, the field QA may be a proper subfield of F. The following example is given in [vGvO, Ex. 3.10.2]: Let $R = K[t, t^{-1}]$, where K is any field of characteristic not 2, t is transcendental over K, and R is graded with $R_0 = K$, $\Gamma_R = \mathbb{Z}$, $R_i = Kt^i$; so $QR = K(t)$. Let $F = K(t)(\sqrt{t+1})$. For $\alpha \in F - QR$ write $\alpha = r + s\sqrt{t+1}$, with $r, s \in QR$, $s \neq 0$; so, $m_{QR,\alpha} = x^2 - 2rx + (r^2 - s^2(t+1))$. If α is integral over R, then $r, s \in R$, but then $m_{QR,\alpha}$ is not homogenizable, since its constant term is not homogeneous in R. The gr-algebraic closure of R in F is in this case R itself.

Now, for any graded field R, let QR_{alg} denote the algebraic closure of QR. Let A be the graded algebraic closure of R in QR_{alg}. It is clear that for any gr-algebraic extension field S of R, S is graded R-isomorphic to a graded subfield of A. Also,

from Cor. 2.7(b) it follows easily that $A_0 \cong R_{0alg}$ and $\Gamma_A = \Delta_R$. Also, QA is normal over QR, since for each $\alpha \in A^h$, Prop. 2.2 and Cor. 2.4 show that each root of $m_{QR,\alpha}$ in QR_{alg} actually lies in A^h. However, QA need not be Galois over QR. We call A the *gr-algebraic closure* of R, denoted $R_{gr\text{-}alg}$.

§3. Unramified, totally ramified, and tame graded field extensions

We now look at some specific types of gr-algebraic extensions of graded fields. Let $R \subseteq S$ be graded fields with S gr-algebraic over R. The torsion group Γ_S/Γ_R is called the *ramification group* of S over R, and $|\Gamma_S/\Gamma_R|$ the *ramification index* of S over R. We say S is *totally ramified* over R if $S_0 = R_0$. At the other extreme, we say S is *unramified* over R if $\Gamma_S = \Gamma_R$ and S_0 is separable over R_0. (This is analogous to the terminology for extensions of valued fields.) Note that every graded extension of graded fields $R \subseteq S$ has a unique subextension A with $\Gamma_A = \Gamma_R$ and S totally ramified over A, namely $A = S_0 \otimes_{R_0} R$.

Remark 3.1. If $\Gamma_S = \Gamma_R$, then clearly $S \cong_g S_0 \otimes_{R_0} R$, and $QS \cong S_0 \otimes_{R_0} QR$. For any intermediate graded field A, $R \subseteq A \subseteq S$, necessarily $\Gamma_A = \Gamma_R$, so $A \cong_g A_0 \otimes_{R_0} R$. Thus, there is a one-to-one correspondence between intermediate graded fields A, $R \subseteq A \subseteq S$, and intermediate fields A_0, $R_0 \subseteq A_0 \subseteq S_0$. Further QS is separable (resp. purely inseparable, resp. Galois) over QR iff S_0 is separable (resp. purely inseparable, resp. Galois) over R_0. For the Galois case, use Cor. 2.5; note that then $\mathcal{G}(QS/QR) \cong \mathcal{G}(S_0/R_0)$.

For any graded field R and any subgroup Λ of Γ_R, we write $R|_\Lambda$ for $\bigoplus_{\gamma \in \Lambda} R_\gamma$, a graded subfield of R.

Proposition 3.2. *Let $R \subseteq S$ be graded fields with S gr-algebraic over R. Suppose S is totally ramified over R. Then,*

(a) $R = S|_{\Gamma_R}$.

(b) *Every intermediate graded field A, $R \subseteq A \subseteq S$ has the form $A = S|_\Lambda$, where Λ is a group, $\Gamma_R \subseteq \Lambda \subseteq \Gamma_S$. Thus, there is a one-to-one correspondence between subgroups of Γ_S/Γ_R and intermediate graded fields.*

(c) *Suppose $[S : R] = n < \infty$, so $\Gamma_S/\Gamma_R \cong \mathbb{Z}/t_1\mathbb{Z} \times \ldots \times \mathbb{Z}/t_k\mathbb{Z}$ with $t_1 \ldots t_k = n$. Pick any $s_i \in S^h - \{0\}$ with $\deg(s_i)$ mapping to a generator of the i-th component in the given cyclic decomposition of Γ_S/Γ_R, $1 \le i \le k$. Then, $s_i^{t_i} \in R^h$, $[R[s_i] : R] = t_i$, and $S \cong_g R[s_1] \otimes_R \ldots \otimes_R R[s_k]$. Likewise, $QS \cong QR(s_1) \otimes_{QR} \ldots \otimes_{QR} QR(s_k)$, with $[QR(s_i) : QR] = t_i$ and $s_i^{t_i} \in QR$.*

Proof. (a) is clear since $(S|_{\Gamma_R})_0 = S_0 = R_0$ and also $S|_{\Gamma_R}$ and R have the same grade group. (b) is immediate from (a), since for $R \subseteq A \subseteq S$, S is totally ramified over A, since it is totally ramified over R. For (c), take any s_i as described in (c). Since $t_i deg(s_i) \in \Gamma_R$, we have $s_i^{t_i} \in S_{t_i deg(s_i)} = R_{t_i deg(s_i)}$ (see (a)). Then $[R[s_i] : R] = t_i$ by Cor. 2.4(d). The graded ring homomorphism $\rho\colon R[s_i] \otimes_R \ldots \otimes_R R[s_k] \to S$ has image a graded subfield of S with $\Gamma_{im(\rho)} = \Gamma_S$. Hence $im(\rho) = S$ by (b). Because $[S : R] = t_i \ldots t_n = dim_R(R[s_1] \otimes_R \ldots \otimes_R R[s_k])$, ρ must be an isomorphism. The isomorphism for QS follows from this by Prop. 2.1. $\square$

The Galois case deserves special attention:

Proposition 3.3. *Let $R \subseteq S$ be graded fields with $[S : R] < \infty$ and S totally ramified over R. Let ℓ be the exponent of Γ_S/Γ_R, and let μ_ℓ denote the group of all ℓ-th roots of unity in QR_{alg}. Then,*

(a) *QS is Galois over QR iff $char(R_0) \nmid [S : R]$ and $\mu_\ell \subseteq R_0$.*

(b) *When QS is Galois over QR, there is a (well-defined) perfect pairing*

$$\Gamma_S/\Gamma_R \times \mathcal{G}(QS/QR) \to \mu_\ell$$

given by $(\gamma + \Gamma_R, \tau) \mapsto a/\tau(a)$, for any $a \in S^h$, $a \neq 0$, with $deg(a) = \gamma$. Furthermore, QS is an ℓ-Kummer extension of QR, and if $B = \{b \in QS^* \mid b^\ell \in QR^*\}$, then $B/QR^* \cong \Gamma_S/\Gamma_R$ and there is a commutative diagram

$$\begin{array}{ccc} \Gamma_S/\Gamma_R \times \mathcal{G}(QS/QR) & \longrightarrow & \mu_\ell \\ \cong \downarrow & & \| \\ B/QR^* \times \mathcal{G}(QS/QR) & \longrightarrow & \mu_\ell \end{array}$$

with the lower row the perfect Kummer pairing, given by $(b\,QR^*, \tau) \mapsto b/\tau(b)$.

Proof. (a) From the description of QS in Prop. 3.2(c) as a radical extension of QR, it is clear that QS is separable over QR iff $char(R_0) \nmid [S : R]$; further, as $\ell = lcm(t, \ldots, t_k)$ for the t_i of Prop. 3.2(c), QS is Galois over QR iff additionally QR contains the required roots of unity, i.e., $\mu_\ell \subseteq QR$. When this occurs, $\mu_\ell \subseteq R_0$ by Cor. 1.3, since μ_ℓ is algebraic over R_0. Also, QS is clearly an ℓ-Kummer extension of QR.

For (b) assume now that QS is Galois over QR. It is easy to check that the map $\Gamma_S/\Gamma_R \times \mathcal{G}(QS/QR) \to \mu_\ell$ is a well-defined pairing of finite abelian groups.

Also, there is a well-defined group homomorphism $\beta\colon \Gamma_S/\Gamma_R \to B/QR^*$ given by $\gamma + \Gamma_R \mapsto a\,QR^*$ for any $a \in S_\gamma$, $a \neq 0$. This β is clearly injective, and is an isomorphism since $|\Gamma_S/\Gamma_R| = [S : R] = [QS : QR] = |B/QR^*|$, the last equality given by Kummer theory. The diagram of the proposition is clearly commutative, and its vertical lines are isomorphisms. Since the bottom row of the diagram is the perfect pairing of Kummer theory, the top row must also be a perfect pairing. □

The graded analogue to a tamely ramified extension of a valued field is of particular interest here.

Definition 3.4. Let $R \subseteq S$ be graded fields with S gr-algebraic over R. Then S is said to be *tame* over R if $char(R_0) = 0$ or $char(R_0) = p \neq 0$ and S_0 is separable over R_0 and Γ_S/Γ_R has no p-torsion.

Clearly, for graded fields $R \subseteq A \subseteq S$, S is tame over R iff S is tame over A and A is tame over R.

Proposition 3.5. *(cf. [B_1, Th. 4]) Let $R \subseteq S$ be a gr-algebraic extension of graded fields. Then S is tame over R iff QS is separable over QR.*

Proof. Since this is clear if $char(R_0) = 0$, assume $char(R_0) = p \neq 0$. Let $I = S_0 \otimes_{R_0} R$. Since $QI \cong S_0 \otimes_{R_0} QR$, $I_0 = S_0$, and $\Gamma_I = \Gamma_R$, we have I is tame over R iff I_0 is separable over R_0 iff QI is separable over QR. Since S is totally ramified over I, Prop. 3.2 shows that for every finite degree subextension S' of S, QS' is a radical extension of QI obtained by adjoining t_i-th roots of homogeneous elements of I, where $lcm(\{t_i\}) = exp(\Gamma_{S'}/\Gamma_I)$. Thus, QS is separable over QI iff each QS' is separable over QI iff each $\Gamma_{S'}/\Gamma_I$ has no p-torsion iff Γ_S/Γ_I has no p-torsion iff S is tame over I. The transitivity of tameness and of separable field extensions completes the proof. □

Lemma 3.6. *Let $R \subseteq S$ be a graded algebraic extension of graded fields with $char(R_0) = p$. There is no proper tame extension of R in S iff S_0 is purely inseparable over R_0 and Γ_S/Γ_R is a p-primary torsion group, iff QS is purely inseparable over QR.*

Proof. In the first equivalence, $\Leftarrow$ is clear. For $\Rightarrow$, suppose R has no proper tame extension in S. Let A_0 be the separable closure of R_0 in S_0. Since $A_0 \otimes_{R_0} R$ is a tame extension of R in S, we must have $A_0 = R_0$, so S_0 is purely inseparable over R_0. Suppose Γ_S/Γ_R is not p-primary. Then, there is $\gamma \in \Gamma_S - \Gamma_R$ with image of order n in Γ_S/Γ_R, where $p \nmid n$. Take any $s \in S_\gamma$, $s \neq 0$. Then $deg(s^n) \in \Gamma_R$, so if

we take any $r \in R_{n\gamma}$, $r \neq 0$, we have $s^n/r \in S_0$. Because S_0 is purely inseparable over R_0, $(s^n/r)^{p^k} \in R_0$ for some k. So, $(s^{p^k})^n = (s^n/r)^{p^k} r^{p^k} \in R_{p^k n\gamma}$. Since $deg(s^{p^k}) = p^k\gamma$, which has order n in Γ_S/Γ_R, $R[s^{p^k}]$ is a proper tame and totally ramified extension of R, by Cor. 2.4(b) and (d), a contradiction. Hence, Γ_S/Γ_R must be p-primary. This proves the first equivalence.

Let $A = S_0 \otimes_{R_0} R$. Then, QA is purely inseparable over QR iff S_0 is purely inseparable over R_0 (see Remark 3.1). Also, as S is totally ramified over A and $\Gamma_A = \Gamma_R$, QS is purely inseparable over QA iff Γ_S/Γ_R is p-primary torsion, as QS is built from QA by radical extensions of degrees the orders of elements in Γ_S/Γ_R, by Prop. 3.2(a). $\square$

Proposition 3.7. *Let $R \subseteq S$ be a gr-algebraic extension of graded fields. Let $T = L \cap S$, where L is the separable closure of QR in QS. Then T is a graded field gr-algebraic and tame over R. Also, for any graded field A with $R \subseteq A \subseteq S$, A is tame over R iff $A \subseteq T$.*

Proof. Let U be a maximal tame graded field extension of R in S. Such a U exists by Zorn's Lemma, using the transitivity of the property of tameness. Then QU is separable over QR, by Prop. 3.5, and QS is purely inseparable over QU by Lemma 3.6. Hence, $QU = L$, so that by Cor. 2.5(b), $U = QU \cap S = L \cap S = T$. This shows T is a graded field tame over R. Also, $QT = QU = L$. If A is any graded field, $R \subseteq A \subseteq S$ and A is tame over R, then QA is separable over QR by Prop. 3.5. Hence, by Cor. 2.5(b) again, $A = QA \cap S \subseteq L \cap S = T$. Conversely, if $A \subseteq T$, then A is tame over R, since T is tame over R. $\square$

Definition 3.8. The T of Prop. 3.7 is called the *tame closure* of R in S. Note that by Lemma 3.6, T_0 is the separable closure of R_0 in S_0, and (if $char(R_0) = p \neq 0$) Γ_T/Γ_R is the prime-to-p part of Γ_S/Γ_R. Also, QT is the separable closure of QR in QS, as we saw in the proof of Prop. 3.7.

Theorem 3.9. *Let R be a graded field. Let T be the tame closure of R in $R_{gr\text{-}alg}$. Then,*

(a) *QT is the separable closure of QR in $Q(R_{gr\text{-}alg})$, so QT is Galois over QR.*
(b) *There is a one-to-one correspondence between the graded fields A such that $R \subseteq A \subseteq R_{gr\text{-}alg}$ with A tame over R and the fields L with $QR \subseteq L \subseteq QT$. (The correspondence is given by $A \mapsto QA$ and $L \mapsto T \cap L$.)*

Proof. (a) Since $Q(R_{gr\text{-}alg})$ is normal over QR, as noted after Definition 2.8, and QT is the separable closure of QR in $Q(R_{gr\text{-}alg})$, QT is Galois over QR (typically

of infinite degree).

(b) If A is a graded field with $R \subseteq A \subseteq R_{gr\text{-}alg}$ and A tame over R, then $A \subseteq T$ by Prop. 3.7, so $QA \subseteq QT$ and $A = T \cap QA$, by Cor. 2.5(b). On the other hand, let L be any field with $QR \subseteq L \subseteq QT$. So QT is Galois over L. Let $H = \mathcal{G}(QT/L)$. By Cor. 2.5(d), each $\tau \in H$ maps T to T by a graded field isomorphism. Hence, as $T \cap L = T^H$, the H-fixed points of T, $T \cap L$ is a graded integral domain, which is a graded field, since it is gr-algebraic over R. For any $\ell \in L$, since ℓ is algebraic over QR, there is $r \in R$, $r \neq 0$ with $r\ell$ integral over R. Then, as T is the integral closure of R in QT, $r\ell \in T \cap L$. This shows $L = Q(T \cap L)$, which proves the one-to-one correspondence asserted in the Theorem. $\square$

Note one somewhat surprising consequence of the theorem just proved: If $t \in T$, then $QR(t) \cap T$ is a graded field containing t, so all the homogeneous components of t lie in $QR(t)$. This requires the tameness of $R[t]$ over R, as the next example illustrates:

Example 3.10. Let R_0 be a field, $char(R_0) = p \neq 0$, and let $R = R_0[s, t, s^{-1}, t^{-1}]$, where s and t are algebraically independent over R_0. Then R is a graded field with $deg(s) = (1, 0)$ and $deg(t) = (0, 1)$ in $\Gamma_R = \mathbb{Z} \times \mathbb{Z}$, and QR is the rational function field $R_0(s, t)$. Let $S = R[\sqrt[p]{s}, \sqrt[p]{t}]$, which is a totally ramified but not a tame gr-algebraic graded field extension of R. Let $u = \sqrt[p]{s} + \sqrt[p]{t} = \sqrt[p]{s+t} \in S$. Then, $\sqrt[p]{s}, \sqrt[p]{t} \notin QR(u)$, since otherwise $QR(u) = QS$. This cannot occur, as $[QR(u) : QR] = p$ while $[QS : QR] = p^2$. Here, $QR(u) \cap S$ is not a graded subring of S since it contains u but not its homogeneous components $\sqrt[p]{s}$ and $\sqrt[p]{t}$. In fact, the kind of one-to-one correspondence described in the theorem for the tame case fails dramatically here, since there are infinitely many fields L with $QR \subseteq L \subseteq QS$, but by Prop. 3.2(b) there are only finitely many graded fields A with $R \subseteq A \subseteq S$.

Theorem 3.11. *Let $R \subseteq S$ be graded fields with $[S : R] < \infty$. Then,*

(a) *S is separable over R iff S is tame over R iff QS is separable over QR.*
(b) *S is Galois over R iff QS is Galois over QR. When this occurs, $\mathcal{G}(S/R) \cong \mathcal{G}(QS/QR)$.*
(c) *If S is separable over R, there is a graded field $A \supseteq S$ with $[A : S] < \infty$ and A Galois over R.*

Proof. (b) If S is Galois over R, with group N, then by base extension (cf. [G, p. 5, Lemma 1.11]) QS is Galois over QR, as $QS \cong QR \otimes_R S$ by Prop. 2.1. Also, the general base extension result shows $\mathcal{G}(QS/QR) \cong N$. Conversely, suppose QS is Galois over QR. Let $G = \mathcal{G}(QS/QR)$. By Cor. 2.5, each $\tau \in G$ restricts to

a graded automorphism of S. Furthermore, $S^G = S \cap QR = R$ by Cor. 2.5(b). For $\tau \in G$, $\tau \neq id_{QS}$ there is $t \in S^h$, $t \neq 0$ with $\tau(t) \neq t$. Since τ preserves degrees $\tau(t) - t \in S^h - \{0\} = S^*$. So, there is no maximal ideal M of S containing $\{\tau(s) - s \mid s \in S\}$. This together with $S^G = R$ shows that S is Galois over R with respect to G, by [CHR, Th. 1.3] or [G, pp. 2–3, Th. 1.6].

(a) and (c) Now, assume QS is separable over QR. Then S is tame over R by Prop. 3.5, so we can assume S lies in the tame closure T of R in $R_{gr\text{-}alg}$. Since QT is Galois over QR by Th. 3.9, the normal closure L of QS over QR lies in QT. Furthermore, L is Galois over QR and $[L : QR] < \infty$. Let $A = T \cap L$, which is a graded subfield of T with $QA = L$, by Th. 3.9, and $[A : R] = [L : QR] < \infty$ by Prop. 2.1. So, A is Galois over R with respect to (the restriction to A of) $\mathcal{G}(L/QR)$, as we just proved. Let $H = \mathcal{G}(L/QS) \subseteq \mathcal{G}(L/QR)$. Since $A^H = A \cap QS = S$ by Cor. 2.5(b), S is separable over R by [CHR, Th. 2.2] or [G, p. 7, Th. 2.2]. On the other hand, if S is separable over R, then $QS \cong QR \otimes_R S$ is separable over QR as separability is preserved under base extension. By Prop. 3.5 this is equivalent to S being tame over R. This yields (a), and also (c) along the way. $\square$

§4. Tame extensions of valued fields

We now recall some facts about tame extensions of valued fields, which will be needed for our comparison of valued fields and graded fields in the next section. Everything we mention in this section is known, but a concise summary seems worthwhile.

Let F be a field, Γ a totally ordered abelian group, and $v\colon F^* \to \Gamma$ a (Krull) valuation on F. (That is, (i) $v(\alpha\beta) = v(\alpha)+v(\beta)$, and (ii) $v(\alpha+\beta) \geq \min(v(\alpha), v(\beta))$ for all $\alpha, \beta \in F^*$ (with $\beta \neq -\alpha$ in (ii)).) Let $\Gamma_F = im(v)$, the value group of v; let V_F be the valuation ring of v; M_F the unique maximal ideal of V_F; and $\overline{F} = V_F/M_F$, the residue field of v. The indexing by F will cause no confusion, because we will never consider more than one valuation at a time on any given field. Let $\overline{p} = char(\overline{F})$.

Now, let $F \subseteq K$ be fields with K algebraic over F, and let v be a valuation on F which has a unique extension (also called v) to K. If $[K : F] < \infty$, we say K is *tame* over F (with respect to v) if $\overline{K}$ is separable over $\overline{F}$, $\overline{p} \nmid |\Gamma_K : \Gamma_F|$, and $[K : F] = [\overline{K} : \overline{F}]|\Gamma_K : \Gamma_F|$. In the terminology of Endler [E, pp. 178-180], K is tamely ramified and defectless over F. If $[K : F] = \infty$, we say that K is tame over F if for every field L with $F \subseteq L \subseteq K$ and $[L : F] < \infty$ we have L is tame over F (with respect to the restriction of v from K to L). Recall that if K is tame over F,

then K is separable over F. This follows at once from the fact that if L is purely inseparable over F, then $\overline{L}$ is purely inseparable over $\overline{F}$ and Γ_L/Γ_F is a $\overline{p}$-group (cf. [E, Ex. III-6, pp. 228-229]). Note also that if $[K:F]<\infty$ and N is any field with $F\subseteq N\subseteq K$, then K is tame over F iff N is tame over F and K is tame over N. This is immediate from the fundamental inequality [E, (13.10)].

Still assuming v has a unique extension from F to K, suppose $[K:F]<\infty$ and K is Galois over F. Let G be the Galois group $\mathcal{G}(K/F)$. Let $G^V=\{\sigma\in G \mid \sigma(c)/c\in M_K$ for all $c\in K^*\}$ the *ramification subgroup* (= Verzweigungsgruppe) of v, and let F^V be the fixed field K^{G^V}, which is called the *ramification field* of K over F. Then it is known (cf. [E, (20.11), (19.12), (20.20), (19.14)]) that G^V is a $\overline{p}$-group (if $\overline{p}=0$ this means $|G^V|=1$), G^V is a normal subgroup of G, so F^V is Galois over F, and that F^V is tame over F (since F^V is tame over the inertia field F^T and F^T is tame over F), and that $\overline{K}$ purely inseparable over $\overline{F^V}$ and Γ_K/Γ_{F^V} is a $\overline{p}$-group (so $\Gamma_{F^V}=\Gamma_K$ if $\overline{p}=0$). Hence, F^V can have no proper tame extension in K, so F^V is a maximal tame extension of F in K. For any field L with $F\subseteq L\subseteq K$, let H be the corresponding group $\mathcal{G}(K/L)$. Since $H^V=G^V\cap H$, we have $L^V=L\cdot F^V$. If L is tame over F, then L^V is also tame over F, by the transitivity of tameness, so $L^V=F^V$, and hence $L\subseteq F^V$. Thus, F^V is the unique maximal tame extension of F, and an intermediate field L is tame over F iff $L\subseteq F^V$. It follows that if intermediate fields L_1,L_2 are each tame over F, then $L_1\cdot L_2$ is tame over F. Thus, if N is any intermediate field, then there is a unique maximal tame extension N_0 of F inside N, and we have $N_0=F^V\cap N$. We call N_0 the *tame closure* of F in N.

All of the preceding paragraph extends readily to the case of infinite degree algebraic extensions (see [E, (20.12), (20.17), (20.18)]) and leads to the following in the Henselian case:

Suppose F is a field with a Henselian valuation v. We work in some fixed algebraic closure F_{alg} of F. That v is Henselian means that there is a unique extension of v to F_{alg}.

Proposition 4.1. *Suppose a field F has a Henselian valuation v. Then,*

(a) There is a unique maximal tame extension F_t of F in F_{alg}, and F_t is the compositum of all the finite degree tame extensions of F in F_{alg}.

(b) $\overline{F_t}=(\overline{F})_{sep}$, $\Gamma_{F_t}=\Gamma_F\otimes_{\mathbb{Z}}\mathbb{Z}_{(\overline{p})}$ (if $\overline{p}=0$, then $\Gamma_{F_t}=\Gamma_F\otimes_{\mathbb{Z}}\mathbb{Q}$).

(c) For any field L, $F\subseteq L\subseteq F_{alg}$, we have L is tame over F iff $L\subseteq F_t$.

(d) For any field N, $F\subseteq N\subseteq F_{alg}$, we have $N_t=F_t\cdot N$, and $N\cap F_t$ is the unique maximal tame extension of F in N.

Proof. Since any tame extension of F is separable over F, this follows easily from the preceding discussion, when we take $F_t = F^V$, the ramification field with respect to the (usually infinite degree) Galois extension F_{sep} over F, where F_{sep} is the separable closure of F in F_{alg}. $\square$

Definition 4.2. We call the F_t of Prop. 4.1 the *tame closure* of F and $N \cap F_t$ the tame closure of F in N. Note that $\overline{N \cap F_t}$ is the separable closure of $\overline{F}$ in $\overline{N}$, and $\Gamma_{N\cap F_t}/\Gamma_F$ is the prime-to-$\overline{p}$ part of Γ_N/Γ_F. Also, $[N : N \cap F_t]$ is a power of $\overline{p}$ ($N = N \cap F_t$ if $\overline{p} = 0$).

§5 Correspondence between graded fields and valued fields

We now recall how valued fields can be obtained from graded fields, and vice versa. We will then prove correspondences for the tame extensions of each kind of field.

Let $R = \bigoplus_{\gamma\in\Gamma_R} R_\gamma$ be a graded field (with Γ_R totally ordered, as we are always assuming). Define a function $v\colon R - \{0\} \to \Gamma_R$ by

$$v\big(\sum_\gamma r_\gamma\big) = \text{the least } \delta \text{ such that } r_\delta \neq 0.$$

So, for $a, a' \in R - \{0\}$, we clearly have

(i) $v(aa') = v(a) + v(a')$;
(ii) $v(a + a') \geq min(v(a), v(a'))$, if $a' \neq -a$.

This function v extends canonically to a function $v\colon QR - \{0\} \to \Gamma_R$ by $v(rs^{-1}) = v(r) - v(s)$, for $r, s \in R - \{0\}$. Property (i) shows that v is well-defined on $QR - \{0\}$. Properties (i) and (ii) hold for all $a, a' \in QR - \{0\}$. Thus v is a valuation on QR, for which clearly $\Gamma_{QR} = \Gamma_R$. Also, for the residue field, $\overline{QR} = R_0$. For, the canonical injection $R_0 \to \overline{QR}$ is onto, since if $v(rs^{-1}) = 0$, then $v(r) = v(s)$, so $r = r_\gamma +$ higher degree terms, $s = s_\gamma +$ higher degree terms, for $\gamma = v(r) = v(s)$), yielding $\overline{rs^{-1}} = \overline{r_\gamma s_\gamma^{-1}} \in im(R_0)$.

Let HR denote the Henselization of QR with respect to v (see, e.g. [E, p. 131] for the Henselization of a valued field). If S is a gr-algebraic extension graded field of R, then $QS \supseteq QR$ canonically, and while HS is determined only up to isomorphism in QS_{alg}, we will assume HS has been chosen to be $QS \cdot HR$.

Now, suppose instead we start with a field F with a valuation v on F. For each

$\gamma \in \Gamma_F$, define

$$W^\gamma = \{c \in F^* \mid v(c) \geq \gamma\} \cup \{0\}$$
$$W^{>\gamma} = \{c \in F^* \mid v(c) > \gamma\} \cup \{0\}.$$

Then set $R_\gamma = W^\gamma / W^{>\gamma}$. Define multiplication $R_\gamma \times R_\delta \to R_{\gamma+\delta}$ by, for $a \in W^\gamma$ and $b \in W^\delta$: $(a + W^{>\gamma}) \cdot (b + W^{>\delta}) = ab + W^{>\gamma+\delta}$. This is well-defined, and extends to a multiplication on all of $\bigoplus_{\gamma \in \Gamma_F} R_\gamma$, making it a graded field. We denote this graded field by GF. Clearly, $\Gamma_{GF} = \Gamma_F$ and $GF_0 = \overline{F}$. Note that if K is a field with a valuation extending v on F, then GK is a graded field extension of GF.

If we start with a graded field R, and build the valuation on QR as described above, then form the associated graded field GQR, then $GQR \cong_g R$, canonically. For, each R_γ maps bijectively onto $(GQR)_\gamma$. Likewise, $GHR \cong_g R$ canonically. On the other hand, if we start with a valued field F, we need not have $QGF \cong F$, nor $HGF \cong F$, even if the valuation on F is Henselian. (For example, $char(QGF) = char(\overline{F})$, which need not equal $char(F)$. Also, if $\overline{F}$ and Γ_F are countable, then QGF is countable, though F might be uncountable.)

Proposition 5.1. *Let R be a graded field, and let S be the tame closure of R in $R_{gr\text{-}alg}$. Then HS is the maximal tame extension of HR, and $\mathcal{G}(HS/HR) \cong \mathcal{G}(QS/QR)$. Hence, there are one-to-one correspondences*

tame gr-algebraic graded field extensions of R in $R_{gr\text{-}alg}$
$\leftrightarrow$ *field extensions of QR in QS*
$\leftrightarrow$ *tame field extensions of HR in $(HR)_{alg}$.*

Proof. Let T be a graded field with $R \subseteq T \subseteq S$ and $[T : R] < \infty$. So, T is tame over R by Prop. 3.7. Then, $\overline{QT} = T_0$ is separable over $\overline{QR} = R_0$, and

$$[QT : QR] = [T : R] = [T_0 : R_0]|\Gamma_T : \Gamma_R| = [\overline{QT} : \overline{QR}]|\Gamma_{QT} : \Gamma_{QR}|\,;$$

hence QT is a tame valued field extension of QR. Since by convention $HT = QT \cdot HR$, we have

$$\begin{aligned}[HT : HR] \;\leq\; [QT : QR] \;&=\; [\overline{QT} : \overline{QR}]|\Gamma_{QT} : \Gamma_{QR}| \\ &=\; [\overline{HT} : \overline{HR}]|\Gamma_{HT} : \Gamma_{HR}| \;\leq\; [HT : HR]\,;\end{aligned}$$

the last inequality is the fundamental inequality for valued field extensions [E, (13.10)]. So, equality holds throughout. Since $\overline{HT} = T_0$ is separable over $\overline{HR} = R_0$,

this shows HT is tame over HR. Also, the equality just proved shows QT and HR are linearly disjoint over QR. Since S is the union of such finite-degree extensions as T, HS is tame over HR, and QS and HR are linearly disjoint over QR. Furthermore, as $\overline{HS} = S_0$ is separably closed and $(\Gamma_{HS} \otimes_{\mathbb{Z}} \mathbb{Q})/\Gamma_{HS} = \Delta_R/\Gamma_S$ is p-primary where $p = char(R_0) = char(\overline{HS})$, HS can have no proper tame field extensions. Hence, HS is the maximal tame extension of the valued field HR. From the linear disjointness, we have $\mathcal{G}(HS/HR) = \mathcal{G}(QS \cdot HR/HR) \cong \mathcal{G}(QS/QR)$.

We have seen in Th. 3.9(b) the one-to-one correspondence between tame gr-algebraic graded field extensions of R and field extensions of QR in QS. The isomorphism of Galois groups gives the one-to-one correspondence between field extensions of QR in QS and the field extensions of HR in HS. The latter are precisely the tame field extensions of HR in $(HR)_{alg}$. $\square$

Theorem 5.2. *Let F be a field with Henselian valuation v, and let K be the maximal tame extension of F in F_{alg} re v. Then GK is the tame closure of GF in $GF_{gr\text{-}alg}$, and $\mathcal{G}(QGK/QGF) \cong \mathcal{G}(K/F)$. There are one-to-one correspondences:*

tame field extensions of F in F_{alg} re v

$\leftrightarrow$ *tame graded field extensions of GF in $(GF)_{gr\text{-}alg}$*

$\leftrightarrow$ *field extensions of QGF in QGK.*

Proof. Let $\overline{p} = char(\overline{F}) = char(GF)$. Since $GK_0 = \overline{K} = \overline{F}_{sep} = (GF_0)_{sep}$ and $\Gamma_{GK}/\Gamma_{GF} = \Gamma_K/\Gamma_F$, which is the prime-to-$\overline{p}$ part of Δ_{GF}/Γ_{GF}, GK is tame over GF and has no proper tame extensions. Hence, GK is the tame closure of GF.

We define a homomorphism $\alpha\colon \mathcal{G}(K/F) \to \mathcal{G}(QGK/QGF)$ as follows: For $\sigma \in \mathcal{G}(K/F)$, $\sigma(V_K) = V_K$ since V_K is the unique extension of the Henselian valuation ring V_F to K. Moreover, as Γ_K/Γ_F is a torsion group, σ must induce the identity automorphism on Γ_K. Hence, $\sigma(W_K^\gamma) = W_K^\gamma$ and $\sigma(W_K^{>\gamma}) = W_K^{>\gamma}$ for each $\gamma \in \Gamma_K$. Thus, σ induces a graded ring isomorphism $\sigma'\colon GK \to GK$ which is the identity on GF. This isomorphism extends to an isomorphism $\widetilde{\sigma}\colon QGK \to QGK$ of the quotient field. Define $\alpha(\sigma) = \widetilde{\sigma} \in \mathcal{G}(QGK/QGF)$.

To show that α is an isomorphism, we proceed in stages. First, let I be the maximal unramified extension of F in K, re v. Then $\overline{I} = \overline{F}_{sep}$ and $\Gamma_I = \Gamma_F$, and I is Galois over F with $\mathcal{G}(I/F) \cong \mathcal{G}(\overline{I}/\overline{F})$ canonically. So, GI is the maximal unramified graded field extension of GF, since $GI_0 = \overline{I} = (\overline{F})_{sep} = (GF_0)_{sep}$ and $\Gamma_{GI} = \Gamma_I = \Gamma_F = \Gamma_{GF}$. Hence, $QGI \cong GI_0 \otimes_{GF_0} QGF$, by Remark 3.1, showing that QGI is Galois over QGF, with

$$\mathcal{G}(QGI/QGF) \cong \mathcal{G}(GI_0/GF_0) \cong \mathcal{G}(\overline{F}_{sep}/\overline{F}) \cong \mathcal{G}(I/F)\,.$$

The inverse of this isomorphism corresponds to the mapping induced by α.

The field extension K/I is tame and totally ramified. Let $\Omega = \{\omega \in I_{sep} \mid \omega^n = 1$ for some $n \in \mathbb{N}$ with $\overline{p} \nmid n\}$. Since V_I is Henselian and $\overline{I}$ is separably closed, $\Omega \subseteq I$ and Ω maps injectively to $\overline{I}$, let Ω' be the image of Ω in $\overline{I} = GI_0$. Let L be any finite-degree field extension of I in K. Then, L is tame and totally ramified over I. Because further V_I is Henselian and I contains all ℓ-th roots of unity for $\ell = [L : I]$, L is an ℓ-Kummer extension of I (cf. [S, p. 64, Th. 3] or [TW, Prop. 1.4(iii)]). Likewise, QGL is an ℓ-Kummer extension of QGI, by Prop. 3.3. Moreover, by [E, (20.11)] or [TW, Prop. 1.4(i)] there is a perfect pairing $\Gamma_L/\Gamma_I \times \mathcal{G}(L/I) \to \Omega'$ given by $(\gamma + \Gamma_I, \sigma) \mapsto \overline{a/\sigma(a)}$ for any $a \in L$ with $v(a) = \gamma$. With respect to the canonical isomorphism $\Gamma_L/\Gamma_I \to \Gamma_{GL}/\Gamma_{GI}$ and the map $\mathcal{G}(L/I) \to \mathcal{G}(QGL/QGI)$ induced by α, the following diagram is evidently commutative,

$$\begin{array}{ccc} \Gamma_L/\Gamma_I \times \mathcal{G}(L/I) & \longrightarrow & \Omega' \\ \downarrow & & \| \\ \Gamma_{GL}/\Gamma_{GI} \times \mathcal{G}(QGL/QGI) & \longrightarrow & \Omega' \end{array}$$

where the bottom row is the perfect pairing of Prop. 3.3(b). Because both rows of the diagram are perfect pairings, the map $\mathcal{G}(L/I) \to G(QGL/QGI)$ must be an isomorphism. Since K is the union of fields such as L and QGK is the union of the corresponding fields QGL, the restriction of α mapping $\mathcal{G}(K/I)$ to $\mathcal{G}(QGK/QGI)$ is an isomorphism. Thus, we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{G}(K/I) & \longrightarrow & \mathcal{G}(K/F) & \longrightarrow & \mathcal{G}(I/F) & \longrightarrow & 1 \\ & & \cong\downarrow & & \alpha\downarrow & & \cong\downarrow & & \\ 1 & \longrightarrow & \mathcal{G}(QGK/QGI) & \longrightarrow & \mathcal{G}(QGK/QGF) & \longrightarrow & \mathcal{G}(QGI/QGF) & \longrightarrow & 1 \end{array}$$

The 5-lemma shows that α is an isomorphism. The isomorphism α of Galois groups gives the one-to-one correspondence between tame field extensions of F in F_{alg} (i.e., subfields of K) and field extensions of QGF in QGK. The correspondence between these fields and the tame graded field extensions of GF in $GF_{gr\text{-}alg}$ was given in Th. 3.9. $\square$

In the setting of Th. 5.2, it was shown in [B_1, Th. 5] that if L is a field extension of F with $[L : F] = n < \infty$, then L is tame over F iff QGL is separable over QGF with $[QGL : QGF] = n$.

Corollary 5.3. *Let K be the maximal tame extension of a Henselian valued field F. Then HGK is the maximal tame extension of HGF, and $\mathcal{G}(HGK/HGF) \cong \mathcal{G}(K/F)$.*

Proof. This is immediate from Prop. 5.1 and Th. 5.2. □

References

[B_1] M. Boulagouaz, The graded and tame extensions, pp. 27–40 in *Commutative Ring Theory* (Fès, 1992) (P. J. Cahen, et. al., eds.), Lecture Notes in Pure and Applied Math., No. 153, Marcel Dekker, New York, 1994.

[B_2] M. Boulagouaz, Le graduè d'une algèbre à division valuée, *Comm. Algebra*, **23** (1995), 4275–4300.

[CHR] S. U. Chase, D. K. Harrison, and A. Rosenberg, *Galois Theory and Galois Cohomology of Commutative Rings*, Mem. Amer. Math. Soc., No. 52, Amer. Math. Soc., Providence, R.I., 1965.

[E] O. Endler, *Valuation Theory*, Springer, New York, 1972.

[G] C. Greither, *Cyclic Galois Extensions of Commutative Rings*, Lecture Notes in Math., No. 1534, Springer, Berlin, 1992.

[HW] Y.-S. Hwang and A. R. Wadsworth, *Correspondences between valued division algebras and graded division algebras*, preprint, UCSD, 1997.

[S] O. F. G. Schilling, *The Theory of Valuations*, Math. Surveys, No. 4, Amer. Math. Soc., Providence, R.I., 1950.

[TW] J.-P. Tignol and A. R. Wadsworth, Totally ramified valuations on finite-dimensional division algebras, *Trans. Amer. Math. Soc.*, **302** (1987), 223–249.

[vGvO] J. van Geel and F. van Oystaeyen, About graded fields, *Indag. Math.*, **43** (1981), 273–286.