WITT KERNELS OF BILINEAR FORMS FOR ALGEBRAIC EXTENSIONS IN CHARACTERISTIC 2

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In memory of Professor Martin Kneser

Abstract. Let F be a field of characteristic 2 and let K/F be a purely inseparable extension of exponent 1. We determine the kernel $W(K/F)$ of the natural restriction map $WF \to WK$ between the Witt rings of bilinear forms of F and K, respectively. This complements a result by Laghribi who computed the kernel for the Witt groups of quadratic forms for such an extension K/F. Based on this result, we will determine $W(K/F)$ for a wide class of finite extensions which are not necessarily purely inseparable.

1. Introduction

Throughout this note, we will only consider fields of characteristic 2. Let K be a finite purely inseparable multiquadratic extension of a field F, in other words, $K = F(\sqrt{a_1}, \cdots, \sqrt{a_n})$. In [7], Laghribi computed the kernel $W_q(K/F)$ of the natural map (induced by scalar extension) $W_qF \to W_qK$ between the Witt groups of nonsingular quadratic forms over F and K, respectively. He shows that

$$W_q(K/F) = \sum_{i=1}^{n} \langle 1, a_i \rangle \otimes W_qF \ ,$$

where $\langle 1, a_i \rangle$ is understood to be a binary bilinear form in its diagonal notation, and W_qF is considered as a WF-module. The proof, although elementary as such, uses as nontrivial ingredients the analogues in characteristic 2 of the Arason-Pfister Hauptsatz due to Baeza [1, Satz 4.1], and of the Milnor conjecture due to Kato [6]. For multiquadratic extensions which are not purely inseparable, results seem to be much harder to obtain, and the precise kernels are only known in the quadratic and biquadratic case. For more details, we refer the reader to Laghribi's article and the references there.

Here, we will determine the kernel $W(K/F)$ of such a purely inseparable multiquadratic extension K/F for the restriction map $WF \to WK$ between the Witt rings of bilinear forms. It turns out that the behaviour of bilinear forms under such extensions can be studied in a truly elementary way, without using any deep results which become necessary when dealing with quadratic forms.

Our main result is the following:

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Theorem 1.1. *Let K/F be a field extension such that $K^2 \subset F$. Then $W(K/F)$ is the ideal in WF generated by $\{\langle 1, t\rangle \mid t \in K^{*2}\}$.*

Note that a field extension K/F with $K^{2^n} \subset F$ is said to have *exponent* $\leq n$. So our result is a statement concerning field extensions of exponent 1. Such a field extension is obviously algebraic and purely inseparable. It turns out that the inseparable extensions K/F we consider have the property that they are *excellent*, i.e., if φ is a bilinear form over F, then there exists a bilinear form ψ over F such that the anisotropic part of φ_K is isometric to ψ_K. The above theorem and the excellence will be shown in Section 3.

In the case of a finite purely inseparable multiquadratic extension K/F, we will give a description of $W(K/F)$ in terms of the annihilator ideal of a certain bilinear Pfister form naturally attached to such an extension (Section 4), and we will generalize this result in Section 5 to an even wider class of finite extensions which are not necessarily purely inseparable and which may contain purely inseparable subextensions of exponent greater than one. It turns out that all extensions in this class are excellent in the above sense, but we will also give an example of an extension of degree 6 which is not excellent.

2. Terminology and definitions

Before turning to the proofs, let us fix some terminology and recall some basic facts about bilinear forms as can be found, for example, in Chapters 1 and 2 of Pfister's book [8]. By a *form* we will always mean a finite-dimensional regular symmetric bilinear form.

$\alpha \cong \beta$ denotes *isometry* of the two forms α and β. The *orthogonal sum* (resp. the *product*) of α and β is written $\alpha \perp \beta$ (resp. $\alpha \otimes \beta$).

A form β is said to be *isotropic* if it represents 0 nontrivially, *anisotropic* otherwise, and $D_F(\beta) = \{\beta(x, x) \mid x \in V \setminus \{0\}\}$ denotes the set of all values represented nontrivially by β (here, V denotes the underlying F-vector space). Thus, β being isotropic is equivalent to $0 \in D_F(\beta)$. We put $D_F(\beta)^* = D_F(\beta) \cap F^*$ and $D_F(\beta)_0 = D_F(\beta) \cup \{0\}$.

A 2-dimensional isotropic form is called a *metabolic plane*, in which case one can always find a basis such that the Gram matrix with respect to that basis is of the shape

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

for some $a \in F$. If $a = 0$, this is called a *hyperbolic plane* and denoted by $\mathbb{H}$. A form β is said to be *metabolic* (resp. *hyperbolic*) if it is the orthogonal sum of metabolic (resp. hyperbolic) planes. It is not difficult to see that a form β is hyperbolic iff $D_F(\beta) = \{0\}$.

If the Gram matrix of a form β with respect to a certain basis is a diagonal matrix with entries a_i, $1 \leq i \leq n = \dim(\beta)$, then we write $\beta \cong \langle a_1, \cdots, a_n\rangle$. Such a diagonalization exists iff β is not hyperbolic, by virtue of the relations

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \cong \langle a, a\rangle \quad \text{and} \quad \langle a\rangle \perp \mathbb{H} \cong \langle a, a, a\rangle.$$

A form β can be decomposed as $\beta \cong \beta_{\text{an}} \perp \beta_m$ with β_{an} anisotropic and β_m metabolic. β_{an} is uniquely determined up to isometry, but generally not β_m as follows easily from the above relations.

Two forms β and β' are called *Witt equivalent* if there are metabolic spaces M and M' such that $\beta \perp M \cong \beta' \perp M'$. The equivalence classes together with addition induced by $\perp$ and multiplication induced by $\otimes$ define the Witt ring of F denoted by WF. By abuse of notation, if β is a form over F, we will denote its class in WF also by β. Every class can be represented by an anisotropic form which is uniquely determined up to isometry.

An *n-fold bilinear Pfister form* (or *n-Pfister* for short) is a form of type

$$\langle\langle a_1, \cdots, a_n \rangle\rangle := \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$$

for some $a_i \in F^*$. A bilinear Pfister form π is *round*, i.e., $\pi \cong x\pi$ for all $x \in D_F(\pi)^*$ (cf. [2, IV.2.8]), and as an easy consequence one has that π is isotropic iff π is metabolic.

If β is a form over F and if K/F is a field extension, then we write β_K for the form obtained by scalar extension. This induces a natural ring homomorphism $WF \to WK$ whose kernel will be denoted by $W(K/F)$. In analogy to the definition of excellence of field extensions in the theory of quadratic forms in characteristic $\neq 2$ as defined in [3], we say that K/F is *excellent* if for any form β over F there exists a form γ over F with $(\beta_K)_{\mathrm{an}} \cong \gamma$, in other words, the anisotropic part of β over K is defined over F.

The *annihilator ideal* of a form β over F given by

$$\mathrm{ann}_F(\beta) = \{\psi \in WF \mid \beta \otimes \psi = 0 \in WF\}.$$

3. Witt kernels and excellence for extensions of exponent one

Let us now turn to the proof of the main theorem. Throughout this section, K/F will be a field extension with $K^2 \subset F$. We define

$$J_{K/F} \;=\; \text{the ideal in } WF \text{ generated by the forms } \langle 1, t \rangle,\ t \in K^{*2}.$$

Lemma 3.1. *$J_{K/F} \subset W(K/F)$.*

Proof. If $t \in K^{*2}$, then clearly $\langle 1, t \rangle_K \cong \langle 1, 1 \rangle_K$ is metabolic and hence $\langle 1, t \rangle_K = 0 \in WK$. □

Proposition 3.2. *Let φ be an anisotropic form over F. Then φ_K is isotropic iff there exist forms $\mu \in J_{K/F}$ and ψ over F with $\dim \psi < \dim \varphi$ such that*

$$\varphi = \psi + \mu \in WF\ .$$

Proof. If $\varphi = \psi + \mu \in WF$ as in the statement of the proposition, then $\varphi_K = \psi_K \in WK$ by the previous Lemma, hence $(\varphi_K)_{\mathrm{an}} \cong (\psi_K)_{\mathrm{an}}$, implying the isotropy of φ_K by comparing dimensions.

Conversely, let φ be an anisotropic form over F with φ_K isotropic. Then we may assume that $\varphi \cong \langle a_0, \cdots, a_n \rangle$ with $a_i \in F^*$ and $n \geq 1$. For $\emptyset \neq I \subset \{0, \cdots, n\}$, let

$$\varphi_I := \bigsqcup_{i \in I} \langle a_i \rangle\ .$$

We may assume that the form φ_I for any $I \neq \{0, \cdots, n\}$ is anisotropic over K, for otherwise it clearly would suffice to prove the statement in the proposition with φ replaced by φ_I which is of smaller dimension. Let us call this assumption (X).

Now the isotropy of φ_K implies that there exist $t_i = x_i^2 \in K^2 \subset F$, $0 \leq i \leq n$, not all t_i equal to 0, such that

$$\sum_{i=1}^{n} a_i t_i = 0, \tag{3.1}$$

and by assumption (X), we have $t_0 \neq 0$. In particular, after dividing by t_0, we may assume that $t_0 = 1$. By putting $A_0 = 0$ and $A_k = \sum_{i=k}^{n} a_i t_i$ for $1 \leq k \leq n$, we have $a_0 = A_1$ by equation 3.1, and it follows from assumption (X) that

$$A_k = a_{k-1}t_{k-1} + A_{k-1} \neq 0 \quad \text{for } 1 \leq k \leq n. \tag{3.2}$$

Now if $u, v \in F^*$ such that $u+v \in F^*$ (i.e. $u \neq v$ as $\mathrm{char}(F) = 2$), then one verifies easily that $\langle u, v\rangle \cong \langle u+v, (u+v)uv\rangle$, or

$$\langle u+v\rangle = \langle u, v, (u+v)uv\rangle \in WF. \tag{3.3}$$

Now define

$$\alpha_k := a_{k-1}t_{k-1}A_{k-1}A_k = (a_{k-1}t_{k-1} + A_k)a_{k-1}t_{k-1}A_k \quad \text{for } 2 \leq k \leq n.$$

Equations 3.2 and 3.3 yield in WF:

$$\begin{aligned}
\varphi &= \langle a_0, a_1, \cdots, a_n\rangle \\
&= \langle A_1, a_1, \cdots, a_n\rangle \\
&= \langle a_1t_1 + A_2, a_1, a_2, \cdots, a_n\rangle \\
&= \langle a_1t_1, A_2, \alpha_2, a_1, a_2, \cdots, a_n\rangle \\
&= a_1\langle 1, t_1\rangle + \langle A_2, a_2, \cdots, a_n\rangle + \langle \alpha_2\rangle\ .
\end{aligned}$$

Repeating this for $\langle A_2, a_2, \cdots, a_n\rangle$ and so on, and using in the last step that $\langle A_n, a_n\rangle \cong a_n\langle 1, t_n\rangle$, we get

$$\begin{aligned}
\varphi &= a_1\langle 1, t_1\rangle + a_2\langle 1, t_2\rangle + \langle A_3, a_3, \cdots, a_n\rangle + \langle \alpha_2, \alpha_3\rangle \\
&\ \ \vdots \\
&= \textstyle\sum_{i=1}^{n} a_i\langle 1, t_i\rangle + \langle \alpha_2, \cdots, \alpha_n\rangle\ .
\end{aligned}$$

The proof is now finished by putting $\psi := \langle \alpha_2, \cdots, \alpha_n\rangle$ and $\mu = \perp_{i=1}^{n} a_i\langle 1, t_i\rangle$. □

Corollary 3.3. *Let φ be a form over F. Then there exist forms ψ and μ over F with ψ_K anisotropic (possibly $\dim\psi = 0$) and $\mu \in J_{K/F}$, such that $\varphi = \psi + \mu \in WF$. In particular, $(\varphi_K)_{\mathrm{an}} \cong \psi_K$.*

Proof. If φ_K is anisotropic, the we can put $\psi := \varphi$ and $\mu = 0$.

If φ_K is isotropic, then we apply the above proposition to find forms $\mu' \in J_{K/F}$ and ψ' over F with $\dim\psi' < \dim\varphi$ such that $\varphi = \psi' + \mu' \in WF$. If ψ'_K is anisotropic, we are done, otherwise we conclude by induction on the dimension applied to $\dim\psi'$.

Now by Lemma 3.1, we have $\mu_K = 0$, hence $\varphi_K = \psi_K \in WK$. The anisotropy of ψ_K yields $(\varphi_K)_{\mathrm{an}} \cong \psi_K$. □

Our main theorem from the introduction and the excellence of K/F are now immediate consequences of the preceding corollary and Lemma 3.1:

Theorem 3.4. (i) $W(K/F) = J_{K/F}$.

(ii) *K/F is excellent.*

4. Annihilator ideals for Pfister forms

Consider an n-Pfister

$$\pi = \langle\langle a_1, \cdots, a_n \rangle\rangle = \bigperp_{\emptyset \subset I \subset \{1,\cdots,n\}} \langle a_I \rangle$$

where $a_I = \prod_{i \in I} a_i$. We associate to this Pfister form the field $K = F(\sqrt{a_1}, \cdots, \sqrt{a_n})$ which is purely inseparable of exponent 1, and we have $K^2 = F^2(a_1, \cdots, a_n)$. It is clear that

$$\sum_{\emptyset \subset I \subset \{1,\cdots,n\}} a_I F^2 = D_F(\pi)_0 = K^2.$$

We call K the *root field of* π (as it consists exactly of all square roots of elements in $D_F(\pi)_0$). Note that π is anisotropic iff the a_I are F^2-linearly independent, which in turn is equivalent to saying that $[K : F] = [K^2 : F^2] = 2^n$.

Conversely, if K/F is a purely inseparable extension of exponent 1 and degree 2^n, then we can find $a_1, \cdots, a_n \in F^*$ such that $K = F(\sqrt{a_1}, \cdots, \sqrt{a_n})$. We then call $\pi = \langle\langle a_1, \cdots, a_n \rangle\rangle$ a *Pfister form associated to* K/F, and we have $D_F(\pi)_0 = K^2$. Note that this Pfister form is generally not unique up to isometry. For if we choose different elements $b_1, \cdots, b_n \in F$ with $K = F(\sqrt{b_1}, \cdots, \sqrt{b_n})$, then generally $\langle\langle b_1, \cdots, b_n \rangle\rangle \not\cong \langle\langle a_1, \cdots, a_n \rangle\rangle$.

For example, if $F = \mathbb{F}_2(T)$ is the rational function field in one variable T over the field with two elements, and if $K = F(\sqrt{T}) = F(\sqrt{1+T})$, then $\langle\langle T \rangle\rangle \not\cong \langle\langle 1+T \rangle\rangle$ by comparing determinants.

We will focus on annihilator ideals for Pfister forms.

For the remainder of this section, π will be an anisotropic n-Pfister, and K will be the root field of π.

Lemma 4.1. $J_{K/F} \subset \operatorname{ann}_F(\pi)$.

Proof. $J_{K/F}$ is generated by elements of type $\langle 1, t \rangle$ with $t \in K^{*2} = D_F(\pi)^*$. Since π is round, we have $t\pi \cong \pi$, i.e. $\langle 1, t \rangle \otimes \pi = 0 \in WF$ for all $t \in D_F(\pi)^*$. This clearly implies $J_{K/F} \subset \operatorname{ann}_F(\pi)$. □

Proposition 4.2. *Let $\varphi = \langle a_0, \cdots, a_n \rangle$ be an anisotropic form over F. Then $\pi \otimes \varphi$ is isotropic iff φ_K is isotropic.*

Proof. $\pi \otimes \varphi \cong \perp_{i=0}^n a_i \pi$ is isotropic iff there exist $t_i \in D_F(\pi)_0 = K^2$, $0 \le i \le n$, not all t_i equal to 0 (here, we use the anisotropy of π), such that $\sum_{i=0}^n a_i t_i = 0$. Since $t_i = x_i^2$ for a uniquely determined $x_i \in K$, this immediately shows that the isotropy of $\pi \otimes \varphi$ is equivalent to the isotropy of φ_K. □

Corollary 4.3. *Let φ be a form over F. Then there exist forms ψ and μ over F with $\pi \otimes \psi$ anisotropic (possibly $\dim \psi = 0$) and $\mu \in J_{K/F}$, such that $\varphi = \psi + \mu \in WF$. In particular, $(\pi \otimes \varphi)_{\mathrm{an}} \cong \pi \otimes \psi$.*

Proof. By Corollary 3.3, there exist forms ψ and μ over F with ψ_K anisotropic (possibly $\dim \psi = 0$) and $\mu \in J_{K/F}$, such that $\varphi = \psi + \mu \in WF$. Now $\pi \otimes \mu = 0 \in WF$ by Lemma 4.1, and $\pi \otimes \psi$ is anisotropic by Proposition 4.2. Hence $\pi \otimes \varphi = \pi \otimes \psi \in WF$ and thus $(\pi \otimes \varphi)_{\mathrm{an}} \cong \pi \otimes \psi$. □

This corollary together with Lemma 4.1 and Theorem 3.4(i) immediately implies

Theorem 4.4. $\operatorname{ann}_F(\pi) = J_{K/F} = W(K/F)$.

Corollary 4.5. *Let E/F be purely inseparable of exponent 1. For any finite subextension L of F inside E, let $\pi^{(L)}$ be the Pfister form associated to L/F. Then*

$$W(E/F) = \bigcup_{\substack{F\subset L\subset E\\ L/F \text{ finite}}} \mathrm{ann}_F(\pi^{(L)}).$$

Proof. Every form φ over F which becomes metabolic over E will become so already over some finite subextension of E/F. Hence, $W(E/F) = \bigcup_L W(L/F)$ where L ranges over all finite subextensions of F inside E. The claim now follows by applying Theorem 4.4. □

5. A generalization

Our results on isotropy over purely inseparable (finite) extensions can be extended to a wider class of algebraic extensions, not necessarily purely inseparable. We will often refer to [4] where diagonal forms of degree p over fields of characteristic $p > 0$ are studied. In our situation, the characteristic is 2, and diagonal forms of degree 2 are then nothing else but totally singular quadratic forms. The results in [4] can now be applied by noting that the isotropy of a bilinear form β is by definition the same as the isotropy of the associated totally singular quadratic form q given by $q(x) = \beta(x, x)$.

The case of separable extensions can be disposed of immediately.

Lemma 5.1. *Let φ be an anisotropic form over F and let K be a separable extension of F. Then φ_K is anisotropic.*

Proof. [4, Prop. 5.3]. □

This result applies to separable extensions in the widest sense, namely extensions K/F which can be written as a purely transcendental extension (of arbitrary transcendence degree) followed by a separable algebraic extension (not necessarily finite).

Let now K/F be an algebraic extension. We still assume $\mathrm{char}(F) = 2$ although all the field theoretic definitions and results in the next few paragraphs make sense in arbitrary (positive) characteristic.

It is well-known that K/F can be written as $K/L/F$ with L/F separable and K/L purely inseparable. If it can also be written as $K/M/F$ with M/F purely inseparable and K/M separable, then the field extension K/F is said to be *balanced*. Now generally, finite extensions need not be balanced. But if K/F is balanced, then the M above is necessarily the maximal purely inseparable extension of F in K.

More precisely, if L (resp. M) is the maximal separable (resp. purely inseparable) extension of F inside K, then K/F is balanced iff K/M is separable iff $K = LM$. For example, every finite normal extension K/F is balanced. (see, e.g., [5, Th. 46]).

Recall that elements $b_1, \cdots, b_m \in F$ are called 2-*independent* if they are F^2-linearly independent as elements in the F^2-vector space F. This is equivalent to saying that $[F(\sqrt{b_1}, \cdots, \sqrt{b_m}) : F] = 2^m$.

Now suppose that K/F is a finite purely inseparable extension of degree $2^r > 1$. Then there exist elements $a_1, \cdots, a_n \in F$, $\alpha_1, \cdots, \alpha_n \in K$ and integers $r_1, \cdots, r_n \geq 1$ such that $a_i = \alpha_i^{2^{r_i}}$, and $K = F(\alpha_1, \cdots, \alpha_n)$. One can show that

$$2^r = [K : F] = 2^{r_1+\cdots+r_n}$$

if and only if $[F(\sqrt{a_1},\cdots,\sqrt{a_n}):F]=2^n$, i.e., the a_i are 2-independent (cf. [4, Prop. 5.7]). If we are in this situation, then we have

$$K=\prod_{i=1}^{n}F(\alpha_i)\cong F(\alpha_1)\otimes_F\cdots\otimes_F F(\alpha_n),$$

and we say that K/F admits a higher 2-basis $\{\alpha_1,\cdots,\alpha_n\}$. Generally, finite purely inseparable extensions do not admit higher 2-bases (cf. [9]).

We now have all the ingredients to generalize the results from the previous sections to a much wider class of field extensions.

Theorem 5.2. *Let K/F be a finite balanced extension and let M be the maximal purely inseparable extension of F inside K. Suppose that M/F admits a higher 2-basis $\{\alpha_1,\cdots,\alpha_n\}\subset M$, and let $\{a_1,\cdots,a_n\}\subset F$ be 2-independent with $a_i=\alpha^{2^{r_i}}$ with integers $r_1,\cdots,r_n\geq 1$. Let $L=F(\sqrt{a_1},\cdots,\sqrt{a_n})$.*

Then $\pi=\langle\langle a_1,\cdots,a_n\rangle\rangle$ is anisotropic, and for an anisotropic form φ over F, the following are equivalent:

(i) *φ_K is isotropic;*
(ii) *φ_M is isotropic;*
(iii) *φ_L is isotropic;*
(iv) *$\pi\otimes\varphi$ is isotropic.*

In particular, K/F is excellent and $W(K/F)=W(M/F)=W(L/F)=\mathrm{ann}_F(\pi)$.

Proof. The anisotropy of π is clear (see the introductory remarks in Section 4).

The equivalence of (i) and (ii) follows from the assumption that K/F is balanced which in turn means that K/M is separable, implying that a form (defined over M) is anisotropic over K iff it is so over M by Lemma 5.1.

The equivalence of (ii) and (iii) has been proved in [4, Th. 5.9], and Proposition 4.2 yields the equivalence of (iii) and (iv).

The excellence follows now readily from Corollary 3.3. The equality of the Witt kernels is also immediate, and the proof is finished by invoking Theorem 4.4. □

We conclude by giving an example of a finite algebraic extension K/F which is *not* excellent.

Example 5.3. Let $F=\mathbb{F}_2(a,b,c)$ be the rational function field in the variables a,b,c over $\mathbb{F}_2$. One readily checks that $X^3+aX^2+bX+c\in F[X]$ is irreducible over F. Hence,

$$P(X)=X^6+aX^4+bX^2+c\in F[X]$$

is irreducible over F. Let α be a root of $P(X)$ is some algebraic closure of F, and put $K=F(\alpha)$. We have $[K:F]=6$ and clearly K/F is neither separable nor purely inseparable.

Now let

$$\varphi=\langle 1,a,b,c\rangle\quad\text{and}\quad\pi=\langle\langle a,b,c\rangle\rangle.$$

One easily sees that φ and π are anisotropic as a,b,c are 2-independent. Furthermore, φ_K is isotropic because we have

$$(\alpha^3)^2+a(\alpha^2)^2+b\alpha^2+c.1^2=0.$$

Suppose that there exists a form ψ over F with $(\varphi_K)_{\mathrm{an}}\cong\psi_K$. Then $\dim\psi\in\{0,2\}$. Let

$$\eta\cong(\varphi\perp\psi)_{\mathrm{an}}.$$

By comparing dimensions, we have $2 \leq \dim \eta \leq 6$. Furthermore, by construction, η_K is metabolic. Let $\widehat{\eta}$ (resp. $\widehat{\pi}$) be the anisotropic totally singular quadratic form given by $\widehat{\eta}(x) = \eta(x,x)$ (resp. $\widehat{\pi}(x) = \pi(x,x)$). Since η_K is metabolic, we have that $\widehat{\eta}_K$ is 2-split in the terminology of [4, Def. 2.11].

Now the coefficients of $P(X)$ are $1, a, b, c$, and we have $[F^2(a,b,c) : F^2] = 8$ and $K \cong F[X]/P(X)$. We then can invoke [4, Th. 6.10] to get that $\widehat{\eta} \cong \widehat{\pi} \otimes \widehat{\gamma}$ for some totally singular quadratic form $\widehat{\gamma}$. In particular, $\dim \eta = \dim \widehat{\eta}$ is divisible by $\dim \pi = \dim \widehat{\pi} = 8$, a contradiction. Thus, $(\varphi_K)_{\mathrm{an}}$ is not defined over F.

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